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Optimal inequalities for the Casorati curvatures of submanifolds of real space forms endowed with semi-symmetric metric connections

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Abstract

In this paper, we prove two optimal inequalities involving the intrinsic scalar curvature and extrinsic Casorati curvature of submanifolds of real space forms endowed with a semi-symmetric metric connection. Moreover, we show that in both cases, the equality at all points characterizes the invariantly quasi-umbilical submanifolds.

MSC: 53C40; 53B05

Keywords: Casorati curvature; real space form; semi-symmetric metric connection

1 Introduction

The idea of a semi-symmetric linear connection on a differentiable manifold was introduced by Friedmann and Schouten in [1]. The notion of a semi-symmetric metric connection on a Riemannian manifold was introduced by Hayden in [2]. Later, Yano in [3] studied some properties of a Riemannian manifold endowed with a semi-symmetric metric connection. In [4, 5], Imai found some properties of a Riemannian manifold and a hypersurface of a Riemannian manifold with a semi-symmetric metric connection. Nakao in [6] studied submanifolds of a Riemannian manifold with semi-symmetric metric connections.

On the other hand, the theory of Chen invariants, initiated by Chen [7] in a seminal paper published in 1993, is presently one of the most interesting research topic in differential geometry of submanifolds. Chen established a sharp inequality for a submanifold in a real space form using the scalar curvature and the sectional curvature, and the squared mean curvature. That is, he established simple relationships between the main intrinsic invariants and the main extrinsic invariants of a submanifold in real space forms with any codimensions in [8]. Many famous results concerned Chen invariants and inequalities for the different classes of submanifolds in various ambient spaces, like complex space forms [9–11]. Recently, in [12, 13], Mihai and Özgür proved Chen inequalities for submanifolds of real, complex, and Sasakian space forms endowed with semi-symmetric metric connections and in [14, 15], Özgür and Murathan gave Chen inequalities for submanifolds of a locally conformal almost cosymplectic manifold and a cosymplectic space form endowed with semi-symmetric metric connections. Moreover, Zhang *et al.* [16] obtained Chen-like

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inequalities for submanifolds of a Riemannian manifold of quasi-constant curvature endowed with a semi-symmetric metric connection by using an algebraic approach.

Instead of concentrating on the sectional curvature with the extrinsic squared mean curvature, the Casorati curvature of a submanifold in a Riemannian manifold was considered as an extrinsic invariant defined as the normalized square of the length of the second fundamental form. The notion of Casorati curvature extends the concept of the principal direction of a hypersurface of a Riemannian manifold. Several geometers in [17–21] found geometrical meaning and the importance of the Casorati curvature. Therefore, it is of great interest to obtain optimal inequalities for the Casorati curvatures of submanifolds in different ambient spaces. Decu *et al.* in [22] obtained some optimal inequalities involving the scalar curvature and the Casorati curvature of a Riemannian submanifold in a real space form and the holomorphic sectional curvature and the Casorati curvature of a Kähler hypersurface in a complex space form. They also proved an inequality in which the scalar curvature is estimated from above by the normalized Casorati curvatures in [23]. Recently, some optimal inequalities involving Casorati curvatures were proved in [24, 25] for slant submanifolds in quaternionic space forms.

As a natural prolongation of our research, in this paper we will study these inequalities for submanifolds in real space forms, endowed with semi-symmetric metric connections.

Theorem 1.1 *Let M^n be a submanifold of a real space form $N^m(c)$ with a semi-symmetric metric connection. Then*:

(i) *The normalized δ-Casorati curvature $\delta_C(n-1)$ satisfies*

$$\rho \leq \delta_C(n-1) + c - \frac{2}{n}\operatorname{trace}(\alpha).$$

Moreover, the equality sign holds if and only if M^n is an invariantly quasi-umbilical submanifold with trivial normal connection in $N^m(c)$, such that with respect to suitable orthonormal tangent frame $\{\xi_1,\ldots,\xi_n\}$ and normal orthonormal frame $\{\xi_{n+1},\ldots,\xi_m\}$, the shape operators $A_r \equiv A_{\xi_r}$, $r \in \{n+1,\ldots,m\}$, take the following forms:

$$A_{n+1} = \begin{pmatrix} a & 0 & 0 & \cdots & 0 & 0 \\ 0 & a & 0 & \cdots & 0 & 0 \\ 0 & 0 & a & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a & 0 \\ 0 & 0 & 0 & \cdots & 0 & 2a \end{pmatrix}, \qquad A_{n+2} = \cdots = A_m = 0.$$

(ii) *The normalized δ-Casorati curvature $\hat{\delta}_C(n-1)$ satisfies*

$$\rho \leq \hat{\delta}_C(n-1) + c - \frac{2}{n}\operatorname{trace}(\alpha).$$

Moreover, the equality sign holds if and only if M^n is an invariantly quasi-umbilical submanifold with trivial normal connection in $N^m(c)$, such that with respect to suitable orthonormal tangent frame $\{\xi_1,\ldots,\xi_n\}$ and normal orthonormal frame

$\{\xi_{n+1}, \ldots, \xi_m\}$, *the shape operators* $A_r \equiv A_{\xi_r}$, $r \in \{n+1, \ldots, m\}$, *take the following forms*:

$$A_{n+1} = \begin{pmatrix} 2a & 0 & 0 & \cdots & 0 & 0 \\ 0 & 2a & 0 & \cdots & 0 & 0 \\ 0 & 0 & 2a & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2a & 0 \\ 0 & 0 & 0 & \cdots & 0 & a \end{pmatrix}, \qquad A_{n+2} = \cdots = A_m = 0.$$

2 Preliminaries

Let N^m be an m-dimensional Riemannian manifold and $\widetilde{\nabla}$ a linear connection on N^m. If the torsion tensor $\widetilde{T}$ of $\widetilde{\nabla}$, defined by

$$\widetilde{T}(\widetilde{X}, \widetilde{Y}) = \widetilde{\nabla}_{\widetilde{X}} \widetilde{Y} - \widetilde{\nabla}_{\widetilde{Y}} \widetilde{X} - [\widetilde{X}, \widetilde{Y}]$$

for any vector field $\widetilde{X}$ and $\widetilde{Y}$ on N^m, satisfies

$$\widetilde{T}(\widetilde{X}, \widetilde{Y}) = \phi(\widetilde{Y}) \widetilde{X} - \phi(\widetilde{X}) \widetilde{Y}$$

for a 1-form ϕ, then the connection $\widetilde{\nabla}$ is called a *semi-symmetric connection*.

Let g be a Riemannian metric on N^m. If $\widetilde{\nabla} g = 0$, then $\widetilde{\nabla}$ is called a *semi-symmetric metric connection* on N^m.

Following [3], a semi-symmetric metric connection $\widetilde{\nabla}$ on N^m is given by

$$\widetilde{\nabla}_{\widetilde{X}} \widetilde{Y} = \overset{\circ}{\widetilde{\nabla}}_{\widetilde{X}} \widetilde{Y} + \phi(\widetilde{Y}) \widetilde{X} - g(\widetilde{X}, \widetilde{Y}) P$$

for any vector fields $\widetilde{X}$ and $\widetilde{Y}$ on N^m, where $\overset{\circ}{\widetilde{\nabla}}$ denotes the Levi-Civita connection with respect to the Riemannian metric g and P is a vector field defined by $g(P, \widetilde{X}) = \phi(\widetilde{X})$, for any vector field $\widetilde{X}$.

We will consider a Riemannian manifold N^m endowed with a semi-symmetric metric connection $\widetilde{\nabla}$ and the Levi-Civita connection denoted by $\overset{\circ}{\widetilde{\nabla}}$.

Let M^n be an n-dimensional submanifold of an m-dimensional Riemannian manifold N^m. On the submanifold M^n, we consider the induced semi-symmetric metric connection, denoted by ∇ and the induced Levi-Civita connection, denoted by $\overset{\circ}{\nabla}$.

Let $\widetilde{R}$ be the curvature tensor of N^m with respect to $\widetilde{\nabla}$ and $\overset{\circ}{\widetilde{R}}$ the curvature tensor of N^m with respect to $\overset{\circ}{\widetilde{\nabla}}$. We also denote by R and $\overset{\circ}{R}$ the curvature tensors of ∇ and $\overset{\circ}{\nabla}$, respectively, on M^n.

The Gauss formulas with respect to ∇ and $\overset{\circ}{\nabla}$, respectively, can be written as

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad X, Y \in \chi(M^n),$$

$$\overset{\circ}{\widetilde{\nabla}}_X Y = \overset{\circ}{\nabla}_X Y + \overset{\circ}{h}(X, Y), \quad X, Y \in \chi(M^n),$$

where $\overset{\circ}{h}$ is the second fundamental form of M^n in N^m and h is a $(0, 2)$-tensor on M^n. According to the formula (7) from [6], h is also symmetric. One denotes by $\overset{\circ}{H}$ the mean

curvature vector of M^n and N^m. Let $N^m(c)$ be a real space form of constant sectional curvature c endowed with a semi-symmetric metric connection $\widetilde{\nabla}$.

The curvature tensor $\overset{\approx}{R}$ with respect to the Levi-Civita connection $\overset{\approx}{\nabla}$ on $N^m(c)$ is expressed by

$$\overset{\approx}{R}(X, Y, Z, W) = c\{g(X, W)g(Y, Z) - g(X, Z)g(Y, W)\}. \tag{2.1}$$

Then the curvature tensor $\widetilde{R}$ with respect to the semi-symmetric metric connection $\widetilde{\nabla}$ on $N^m(c)$ can be written as [5]

$$\begin{aligned}
\widetilde{R}(X, Y, Z, W) = {} & \overset{\approx}{R}(X, Y, Z, W) - \alpha(Y, Z)g(X, W) \\
& + \alpha(X, Z)g(Y, W) - \alpha(X, W)g(Y, Z) \\
& + \alpha(Y, W)g(X, Z)
\end{aligned} \tag{2.2}$$

for any vector fields $X, Y, Z, W \in \chi(M^n)$, where α is a $(0, 2)$-tensor field defined by

$$\alpha(X, Y) = (\overset{\approx}{\nabla}_X \phi)Y - \phi(X)\phi(Y) + \frac{1}{2}\phi(P)g(X, Y), \quad \forall X, Y \in \chi(M^n).$$

From (2.1) and (2.2), it follows that the curvature tensor $\widetilde{R}$ can be expressed as

$$\begin{aligned}
\widetilde{R}(X, Y, Z, W) = {} & c\{g(X, W)g(Y, Z) - g(X, Z)g(Y, W)\} \\
& - \alpha(Y, Z)g(X, W) + \alpha(X, Z)g(Y, W) \\
& - \alpha(X, W)g(Y, Z) + \alpha(Y, W)g(X, Z).
\end{aligned} \tag{2.3}$$

Denote by λ the trace of α.

The Gauss equation for the submanifold M^n in the real space form $N^m(c)$ is

$$\begin{aligned}
\overset{\approx}{R}(X, Y, Z, W) = {} & \overset{\circ}{R}(X, Y, Z, W) + g(\overset{\circ}{h}(X, Z), \overset{\circ}{h}(Y, W)) \\
& - g(\overset{\circ}{h}(X, W), \overset{\circ}{h}(Y, Z)).
\end{aligned} \tag{2.4}$$

Let $\pi \subset T_x M^n$, $x \in M^n$, be a 2-plane section. Denote by $K(\pi)$ the sectional curvature of M^n with respect to the induced semi-symmetric metric connection ∇. For any orthonormal basis $\{e_1, \ldots, e_n\}$ of the tangent space $T_x M^n$ and $\{e_{n+1}, \ldots, e_m\}$ is an orthonormal basis of the normal space $T_x^\perp M$, then the scalar curvature τ at x is defined by

$$\tau(x) = \sum_{1 \le i < j \le n} K(e_i \wedge e_j),$$

and the normalized scalar curvature ρ of M is defined by

$$\rho = \frac{2\tau}{n(n-1)}.$$

We denote by H the mean curvature vector, that is,

$$H(x) = \frac{1}{n} \sum_{i=1}^{n} h(e_i, e_i),$$

and we also set

$$h_{ij}^{\alpha} = g\big(h(e_i, e_j), e_\alpha\big), \quad i,j \in \{1, \dots, n\}, \alpha \in \{n+1, \dots, m\}.$$

Then the squared mean curvature of the submanifold M in N is defined by

$$\|H\|^2 = \frac{1}{n^2} \sum_{\alpha=n+1}^{m} \left(\sum_{i=1}^{n} h_{ii}^{\alpha} \right)^2,$$

and the squared norm of h over dimension n is denoted by $\mathcal{C}$ and is called the *Casorati curvature* of the submanifold M. Therefore, we have

$$\mathcal{C} = \frac{1}{n} \sum_{\alpha=n+1}^{m} \sum_{i,j=1}^{n} \big(h_{ij}^{\alpha}\big)^2.$$

The submanifold M is called *invariantly quasi-umbilical* if there exist $m - n$ mutually orthogonal unit normal vectors $\xi_{n+1}, \dots, \xi_m$ such that the shape operators with respect to all directions ξ_α have an eigenvalue of multiplicity $n-1$ and that for each ξ_α the distinguished eigendirection is the same [26].

Suppose now that L is an r-dimensional subspace of $T_x M$, $r \geq 2$, and $\{e_1, \dots, e_r\}$ be an orthonormal basis of L. Then the scalar curvature $\tau(L)$ of the r-plane section L is given by

$$\tau(L) = \sum_{1 \leq \alpha < \beta \leq r} K(e_\alpha \wedge e_\beta),$$

and the Casorati curvature $\mathcal{C}(L)$ of the subspace L is defined as

$$\mathcal{C}(L) = \frac{1}{r} \sum_{\alpha=n+1}^{m} \sum_{i,j=1}^{r} \big(h_{ij}^{\alpha}\big)^2.$$

The normalized δ-Casorati curvature $\delta_c(n-1)$ and $\hat{\delta}_c(n-1)$ are given by

$$\big[\delta_c(n-1)\big]_x = \frac{1}{2}\mathcal{C}_x + \frac{n+1}{2n} \inf\{\mathcal{C}(L) \mid L\text{: a hyperplane of } T_x M\}$$

and

$$\big[\hat{\delta}_c(n-1)\big]_x = 2\mathcal{C}_x - \frac{2n-1}{2n} \sup\{\mathcal{C}(L) \mid L\text{: a hyperplane of } T_x M\}.$$

3 Proof of the theorem

From [6], the Gauss equation with respect to the semi-symmetric metric connection is

$$\widetilde{R}(X, Y, Z, W) = R(X, Y, Z, W) + g\big(h(X, Z), h(Y, W)\big)$$
$$- g\big(h(Y, Z), h(X, W)\big). \tag{3.1}$$

Let $x \in M^n$ and $\{e_1, e_2, \dots, e_n\}$ and $\{e_{n+1}, \dots, e_m\}$ be orthonormal basis of $T_x M^n$ and $T_x^{\perp} M^n$, respectively. For $X = W = e_i$, $Y = Z = e_j$, $i \neq j$, from (2.3), it follows that

$$\widetilde{R}(e_i, e_j, e_j, e_i) = c - \alpha(e_i, e_i) - \alpha(e_j, e_j). \tag{3.2}$$

From (3.1) and (3.2), we get

$$c - \alpha(e_i, e_i) - \alpha(e_j, e_j)$$
$$= R(e_i, e_j, e_j, e_i) + g\big(h(e_i, e_j), h(e_i, e_j)\big) - g\big(h(e_i, e_i), h(e_j, e_j)\big).$$

By summation over $1 \le i, j \le n$, it follows from the previous relation that

$$2\tau = n^2 \|H\|^2 - n\mathcal{C} + n(n-1)c - 2(n-1)\operatorname{trace}(\alpha). \tag{3.3}$$

We define now the following function, denoted by $\mathcal{P}$, which is a quadratic polynomial in the components of the second fundamental form:

$$\mathcal{P} = \frac{1}{2}n(n-1)\mathcal{C} + \frac{1}{2}(n+1)(n-1)\mathcal{C}(L) - 2\tau + n(n-1)c - 2(n-1)\operatorname{trace}(\alpha).$$

Without loss of generality, by assuming that L is spanned by $e_1, \ldots, e_{n-1}$, one derives that

$$\mathcal{P} = \frac{n+1}{2} \sum_{\alpha=n+1}^{m} \left(\sum_{i,j=1}^{n} \big(h_{ij}^{\alpha}\big)^2 \right) + \frac{n+1}{2} \sum_{\alpha=n+1}^{m} \left(\sum_{i,j=1}^{n-1} \big(h_{ij}^{\alpha}\big)^2 \right)$$
$$- \sum_{\alpha=n+1}^{m} \left(\sum_{i=1}^{n} h_{ii}^{\alpha} \right)^2$$

and now we easily obtain

$$\mathcal{P} = \sum_{\alpha=n+1}^{m} \sum_{i=1}^{n-1} \Big[n\big(h_{ii}^{\alpha}\big)^2 + (n+1)\big(h_{in}^{\alpha}\big)^2 \Big]$$
$$+ \sum_{\alpha=n+1}^{m} \left[2(n+1) \sum_{i<j=1}^{n-1} \big(h_{ij}^{\alpha}\big)^2 - 2 \sum_{i<j=1}^{n} h_{ii}^{\alpha} h_{jj}^{\alpha} + \frac{n-1}{2} \big(h_{nn}^{\alpha}\big)^2 \right]. \tag{3.4}$$

From (3.4), it follows that the critical points

$$h^c = \big(h_{11}^{n+1}, h_{12}^{n+1}, \ldots, h_{nn}^{n+1}, \ldots, h_{11}^{m}, \ldots, h_{nn}^{m} \big)$$

of $\mathcal{P}$ are the solutions of the following system of linear homogeneous equations:

$$\begin{cases} \frac{\partial \mathcal{P}}{\partial h_{ii}^{\alpha}} = 2(n+1)h_{ii}^{\alpha} - 2\sum_{k=1}^{n} h_{kk}^{\alpha} = 0, \\ \frac{\partial \mathcal{P}}{\partial h_{nn}^{\alpha}} = (n-1)h_{nn}^{\alpha} - 2\sum_{k=1}^{n-1} h_{kk}^{\alpha} = 0, \\ \frac{\partial \mathcal{P}}{\partial h_{ij}^{\alpha}} = 4(n+1)h_{ij}^{\alpha} = 0, \\ \frac{\partial \mathcal{P}}{\partial h_{in}^{\alpha}} = 2(n+1)h_{in}^{\alpha} = 0, \end{cases} \tag{3.5}$$

with $i, j \in \{1, \ldots, n-1\}$, $i \ne j$ and $\alpha \in \{n+1, \ldots, m\}$. Thus, every solution h^c has $h_{ij}^{\alpha} = 0$ for $i \ne j$, and the determinant which corresponds to the first two sets of equations of the above system is zero (there exist solutions for non-totally geodesic submanifolds). Moreover, it

is easy to see that the Hessian matrix of $\mathcal{P}$ has the form

$$\mathcal{H}(\mathcal{P}) = \begin{pmatrix} H_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & H_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & H_3 \end{pmatrix},$$

where

$$H_1 = \begin{pmatrix} 2n & -2 & \cdots & -2 & -2 \\ -2 & 2n & \cdots & -2 & -2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -2 & -2 & \cdots & 2n & -2 \\ -2 & -2 & \cdots & -2 & n-1 \end{pmatrix},$$

$\mathbf{0}$ denotes the null matrix of corresponding dimensions and H_2, H_3 are the next diagonal matrices

$$H_2 = \text{diag}\big(4(n+1), 4(n+1), \ldots, 4(n+1)\big),$$

$$H_3 = \text{diag}\big(2(n+1), 2(n+1), \ldots, 2(n+1)\big).$$

Therefore, we find that $\mathcal{H}(\mathcal{P})$ has the following eigenvalues:

$$\lambda_{11} = 0, \qquad \lambda_{22} = n + 3, \qquad \lambda_{33} = \cdots = \lambda_{nn} = 2(n+1),$$

$$\lambda_{ij} = 4(n+1), \qquad \lambda_{in} = 2(n+1), \quad \forall i, j \in \{1, \ldots, n-1\}, i \neq j.$$

Therefore, $\mathcal{P}$ is parabolic and reaches a minimum $\mathcal{P}(h^c) = 0$ for the solution h^c of the system (3.5). It follows that $\mathcal{P} \geq 0$, and, hence,

$$2\tau \leq \frac{1}{2} n(n-1)\mathcal{C} + \frac{1}{2}(n+1)(n-1)\mathcal{C}(L) + n(n-1)c - 2(n-1)\,\text{trace}(\alpha).$$

Hence, we deduce that

$$\rho \leq \frac{1}{2}\mathcal{C} + \frac{n+1}{2n}\mathcal{C}(L) + c - \frac{2}{n}\,\text{trace}(\alpha)$$

for every tangent hyperplane L of M. Taking the infimum over all tangent hyperplane L, the theorem trivially follows.

Moreover, we can easily check that the equality sign holds in the theorem if and only if

$$h_{ij}^\alpha = 0, \quad \forall i, j \in \{1, \ldots, n\}, i \neq j \text{ and } \alpha \in \{n+1, \ldots, m\} \tag{3.6}$$

and

$$h_{nn}^\alpha = 2h_{11}^\alpha = \cdots = 2h_{n-1n-1}^\alpha, \quad \forall \alpha \in \{n+1, \ldots, m\}. \tag{3.7}$$

From (3.6) and (3.7), we conclude that the equality holds if and only if the submanifold M is invariantly quasi-umbilical with trivial normal connection in N, such that with respect to

suitable orthonormal tangent and normal orthonormal frames, the shape operators take the forms below:

$$A_{n+1} = \begin{pmatrix} 2a & 0 & 0 & \cdots & 0 & 0 \\ 0 & 2a & 0 & \cdots & 0 & 0 \\ 0 & 0 & 2a & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & a & \cdots & 2a & 0 \\ 0 & 0 & a & \cdots & 0 & a \end{pmatrix}, \qquad A_{n+2} = \cdots = A_m = 0. \tag{3.8}$$

Remark We have a slightly modified coefficient in the definition of $\delta_C(n-1)$; in fact, it was used the coefficient $\frac{n+1}{2n(n-1)}$, as in [22, 23, 25], instead of $\frac{n+1}{2n}$, like in the present paper because we are working on the generalized normalized δ-Casorati curvature $\delta_C(r; n-1)$ for a positive real number $r \neq n(n-1)$, as in [24].

Authors' contributions
All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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