Ghorbanian et al. *Journal of Inequalities and Applications* 2014, **2014**:319
http://www.journalofinequalitiesandapplications.com/content/2014/1/319

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH

Open Access

On a fractional differential inclusion via a new integral boundary condition

R Ghorbanian[1,2], Vahid Hedayati[3], Mihai Postolache[4*] and Shahram Rezapour[3]

*Correspondence: emscolar@yahoo.com
[4]Department of Mathematics, University Politehnica of Bucharest, 313 Splaiul Independenţei, Bucharest, 060042, Romania
Full list of author information is available at the end of the article

Abstract

In this paper, we investigate the existence of solution for two systems of fractional differential inclusions via some integral boundary value conditions. For this purpose, we use an endpoint result for multifunctions which has been proved in 2010 by Amini-Harandi (Nonlinear Anal. 72:132-134, 2010). Finally, we give an example for illustrating one of our results.

Keywords: Caputo fractional derivative; endpoint; fractional differential inclusion

1 Introduction

As we know, diverse classes of fractional differential equations have been studied by researchers (see for example, [1–15] and the references therein). Much attention has been devoted to the fractional differential inclusions (see for example, [16–32] and the references therein). Also, there have been provided many applications of this field (see for example, [33, 34] and [35]).

It is the aim of this paper to investigate the existence of solutions for two systems of fractional differential inclusions, subject to some integral boundary value conditions. In this respect, we use an endpoint result for multifunctions due to Amini-Harandi, [36]. We provide an example for illustrating one of our results.

2 Preliminaries

As is well known, the Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f: (0,\infty) \to \mathbb{R}$ is given by $I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds$, provided the right side is pointwise defined on $(0,\infty)$ (see [10, 13] and [14]). The Caputo fractional derivative of order α for a continuous function f is defined by ${}^cD^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha-n+1}}\, ds$, where $n = [\alpha] + 1$ (see [10, 13] and [14]).

Recall that a multifunction $G: J \to P_{cl}(\mathbb{R})$ is said to be measurable whenever the function $t \mapsto d(y, G(t))$ is measurable for all $y \in \mathbb{R}$, where $J = [0,1]$ [37].

Let (X,d) be a metric space. We have the well-known Pompeiu-Hausdorff metric (see [38])

$$H_d: 2^X \times 2^X \to [0,\infty), \quad H_d(A,B) = \max\Big\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b)\Big\},$$

where $d(A,b) = \inf_{a\in A} d(a,b)$. Then $(CB(X),H_d)$ is a metric space and $(C(X),H_d)$ is a generalized metric space, where $CB(X)$ is the set of closed and bounded subsets of X and $C(X)$ is the set of closed subsets of X (see [27]).

Let $T\colon X \to 2^X$ be a multifunction. An element $x \in X$ is called an endpoint of T whenever $Tx = \{x\}$ [36]. Also, we say that T has the approximate endpoint property whenever $\inf_{x\in X}\sup_{y\in Tx} d(x,y) = 0$ [36]. A function $g\colon \mathbb{R} \to \mathbb{R}$ is called upper semi-continuous whenever $\limsup_{n\to\infty} g(\lambda_n) \le g(\lambda)$ for all sequences $\{\lambda_n\}_{n\ge1}$ with $\lambda_n \to \lambda$ [36].

In 2010, Amini-Harandi proved the next result [36].

Lemma 2.1 *Let $\psi\colon [0,\infty) \to [0,\infty)$ be an upper semi-continuous function such that $\psi(t) < t$ and $\liminf_{t\to\infty}(t - \psi(t)) > 0$, for all $t > 0$, (X,d) a complete metric space and $T\colon X \to CB(X)$ a multifunction such that $H_d(Tx,Ty) \le \psi(d(x,y))$ for all $x,y \in X$. Then T has a unique endpoint if and only if T has approximate end point property.*

In 2011, Ahmad *et al.* investigated the fractional inclusion problem ${}^cD^\alpha x(t) \in F(t,x(t))$, via the integral boundary conditions $x^j(0) - \lambda_j x^j(T) = \mu_j \int_0^1 g_j(s,x(s))\,ds$ for $j = 0,1,2$, where F is a multifunction (see for more details [20]).

In this paper, we are going to extend the problem in a sense. In this respect, we first investigate the existence of solution for the fractional differential inclusion problem

$$ {}^cD^\alpha x(t) \in F\bigl(t,x(t),x'(t),x''(t)\bigr), \tag{1} $$

via integral boundary value conditions

$$ \begin{cases} x(0) + x(\eta) + x(1) = \int_0^1 g_0(s,x(s))\,ds, \\ {}^cD^\beta x(0) + {}^cD^\beta x(\eta) + {}^cD^\beta x(1) = \int_0^1 g_1(s,x(s))\,ds, \\ {}^cD^\gamma x(0) + {}^cD^\gamma x(\eta) + {}^cD^\gamma x(1) = \int_0^1 g_2(s,x(s))\,ds, \end{cases} \tag{2} $$

where $t \in J$, $2 < \alpha \le 3$, $0 < \eta,\beta < 1$, $1 < \gamma < 2$, and $F\colon J \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to P_{cp}(\mathbb{R})$ is a multifunction, $g_1,g_2,g_3\colon J \times \mathbb{R} \to \mathbb{R}$ are continuous functions and ${}^cD^q$ is the standard Caputo differentiation. Here, $P_{cp}(\mathbb{R})$ is the set of all compact subsets of $\mathbb{R}$.

Also, we investigate the existence of solution for the fractional differential inclusion problem

$$ {}^cD^\alpha x(t) \in F\bigl(t,x(t),{}^cD^{\gamma_1}x(t),\dots,{}^cD^{\gamma_n}x(t)\bigr), \tag{3} $$

via integral boundary value conditions

$$ \begin{cases} x'(0) + bx'(1) = \sum_{i=1}^n {}^cD^{\gamma_i}x(\eta), \\ x(0) + ax(1) = \sum_{i=1}^n I^{\gamma_i}x(\eta), \end{cases} \tag{4} $$

where $t \in J = [0,1]$, $1 < \alpha \le 2$, $0 < \eta,\gamma_i < 1$, $\alpha - \gamma_i \ge 1$ for all $1 \le i \le n$, $a > \sum_{i=1}^n \frac{\eta^{\gamma_i+1}}{\Gamma(\gamma_i+2)}$, $b > \sum_{i=1}^n \frac{\eta^{1-\gamma_i}}{\Gamma(2-\gamma_i)}$, $n \ge 1$, and $F\colon J \times \mathbb{R}^{n+1} \to P(\mathbb{R})$ is a multifunction.

3 Main results

Now, we are ready to state and prove our main results. First, we give the following one.

Lemma 3.1 *Let $v \in C(J,\mathbb{R})$, $\alpha \in (2,3]$, $\beta \in (0,1)$, $\gamma \in (1,2)$ and $g_0, g_1, g_2 : J \times \mathbb{R} \to \mathbb{R}$ be continuous functions. The unique solution of the fractional differential problem*

$$ {}^cD^\alpha x(t) = v(t) \tag{5}$$

via the boundary value conditions (2) *is given by*

$$\begin{aligned}
x(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{(\alpha-1)} v(s)\,ds + \frac{1}{3}\int_0^1 g_0\bigl(s,x(s)\bigr)\,ds \\
&\quad - \frac{1}{3\Gamma(\alpha)} \biggl[\int_0^1 (1-s)^{\alpha-1} v(s)\,ds + \int_0^\eta (\eta-s)^{\alpha-1} v(s)\,ds \biggr] \\
&\quad + \frac{3\Gamma(2-\beta)t - (\eta+1)\Gamma(2-\beta)}{3(\eta^{1-\beta}+1)} \int_0^1 g_1\bigl(s,x(s)\bigr)\,ds \\
&\quad + \frac{(\eta+1)\Gamma(2-\beta) - 3\Gamma(2-\beta)t}{3(\eta^{1-\beta}+1)\Gamma(\alpha-\beta)} \biggl[\int_0^1 (1-s)^{\alpha-\beta-1} v(s)\,ds + \int_0^\eta (\eta-s)^{\alpha-\beta-1} v(s)\,ds \biggr] \\
&\quad + \biggl(\frac{2(\eta+1)(\eta^{2-\beta}+1)\Gamma(3-\gamma)\Gamma(2-\beta) - (\eta^2+1)\Gamma(3-\gamma)(\eta^{1-\beta}+1)\Gamma(3-\beta)}{6(\eta^{1-\beta}+1)(\eta^{2-\gamma}+1)\Gamma(3-\beta)} \\
&\quad + \frac{-6(\eta^{2-\beta}+1)\Gamma(3-\gamma)\Gamma(2-\beta)t + 3(\eta^{1-\beta}+1)\Gamma(3-\gamma)\Gamma(3-\beta)t^2}{6(\eta^{1-\beta}+1)(\eta^{2-\gamma}+1)\Gamma(3-\beta)} \biggr) \\
&\quad \times \int_0^1 g_2\bigl(s,x(s)\bigr)\,ds \\
&\quad + \biggl(\frac{(\eta^2+1)\Gamma(3-\gamma)(\eta^{1-\beta}+1)\Gamma(3-\beta) - 2(\eta+1)(\eta^{2-\beta}+1)\Gamma(3-\gamma)\Gamma(2-\beta)}{6(\eta^{1-\beta}+1)(\eta^{2-\gamma}+1)\Gamma(3-\beta)\Gamma(\alpha-\gamma)} \\
&\quad + \frac{6(\eta^{2-\beta}+1)\Gamma(3-\gamma)\Gamma(2-\beta)t - 3\Gamma(3-\gamma)\Gamma(3-\beta)(\eta^{1-\beta}+1)t^2}{6(\eta^{1-\beta}+1)(\eta^{2-\gamma}+1)\Gamma(3-\beta)\Gamma(\alpha-\gamma)} \biggr) \\
&\quad \times \biggl[\int_0^1 (1-s)^{\alpha-\gamma-1} v(s)\,ds \int_0^\eta (\eta-s)^{\alpha-\gamma-1} v(s)\,ds \biggr].
\end{aligned}$$

Proof It is known that the general solution of (5) is

$$x(t) = I^\alpha v(t) + c_0 + c_1 t + c_2 t^2,$$

that is

$$x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v(s)\,ds + c_0 + c_1 t + c_2 t^2, \tag{6}$$

where c_0, c_1, c_2 are real arbitrary constants (see [10, 13] and [14]). Thus,

$$ {}^cD^\beta x(t) = \frac{1}{\Gamma(\alpha-\beta)} \int_0^t (t-s)^{\alpha-\beta-1} v(s)\,ds + \frac{c_1 t^{1-\beta}}{\Gamma(2-\beta)} + \frac{2c_2 t^{2-\beta}}{\Gamma(3-\beta)}$$

and ${}^cD^\gamma x(t) = \frac{1}{\Gamma(\alpha-\gamma)} \int_0^t (t-s)^{\alpha-\gamma-1} v(s)\,ds + \frac{2c_2 t^{2-\gamma}}{\Gamma(3-\gamma)}$. Hence,

$$\begin{aligned}
x(0) + x(\eta) + x(1) &= 3c_0 + (1+\eta)c_1 + \bigl(1+\eta^2\bigr)c_2 \\
&\quad + \frac{1}{\Gamma(\alpha)} \biggl[\int_0^1 (1-s)^{\alpha-1} v(s)\,ds + \int_0^\eta (\eta-s)^{\alpha-1} v(s)\,ds \biggr],
\end{aligned}$$

$$\begin{aligned}&{}^cD^\beta x(0)+{}^cD^\beta x(\eta)+{}^cD^\beta x(1)\\&\quad=c_1\frac{\eta^{1-\beta}+1}{\Gamma(2-\beta)}+c_2\frac{2(\eta^{2-\beta}+1)}{\Gamma(3-\beta)}\\&\qquad+\frac{1}{\Gamma(\alpha-\beta)}\biggl[\int_0^1(1-s)^{\alpha-\beta-1}v(s)\,ds+\int_0^\eta(\eta-s)^{\alpha-\beta-1}v(s)\,ds\biggr]\end{aligned}$$

and

$$\begin{aligned}&{}^cD^\gamma x(0)+{}^cD^\gamma x(\eta)+{}^cD^\gamma x(1)\\&\quad=c_2\frac{2(\eta^{2-\gamma}+1)}{\Gamma(3-\gamma)}+\frac{1}{\Gamma(\alpha-\gamma)}\biggl[\int_0^1(1-s)^{\alpha-\gamma-1}v(s)\,ds+\int_0^\eta(\eta-s)^{\alpha-\gamma-1}v(s)\,ds\biggr].\end{aligned}$$

By using the boundary conditions, we obtain

$$\begin{aligned}&3c_0+(1+\eta)c_1+\bigl(1+\eta^2\bigr)c_2\\&\quad=\int_0^1g_0\bigl(s,x(s)\bigr)\,ds-\frac{1}{\Gamma(\alpha)}\biggl[\int_0^1(1-s)^{\alpha-1}v(s)\,ds+\int_0^\eta(\eta-s)^{\alpha-1}v(s)\,ds\biggr],\\&c_1\frac{\eta^{1-\beta}+1}{\Gamma(2-\beta)}+c_2\frac{2(\eta^{2-\beta}+1)}{\Gamma(3-\beta)}\\&\quad=\int_0^1g_1\bigl(s,x(s)\bigr)\,ds-\frac{1}{\Gamma(\alpha-\beta)}\biggl[\int_0^1(1-s)^{\alpha-\beta-1}v(s)\,ds+\int_0^\eta(\eta-s)^{\alpha-\beta-1}v(s)\,ds\biggr]\end{aligned}$$

and

$$\begin{aligned}&c_2\frac{2(\eta^{2-\gamma}+1)}{\Gamma(3-\gamma)}\\&\quad=\int_0^1g_2\bigl(s,x(s)\bigr)\,ds-\frac{1}{\Gamma(\alpha-\gamma)}\biggl[\int_0^1(1-s)^{\alpha-\gamma-1}v(s)\,ds+\int_0^\eta(\eta-s)^{\alpha-\beta-1}v(s)\,ds\biggr].\end{aligned}$$

This is a linear system of equations of triangular form, having c_0, c_1, and c_2 as unknowns. We solve by back substitution and find

$$\begin{aligned}c_0={}&\frac{1}{3}\int_0^1g_0\bigl(s,x(s)\bigr)\,ds-\frac{1}{3\Gamma(\alpha)}\biggl[\int_0^1(1-s)^{\alpha-1}v(s)\,ds+\int_0^\eta(\eta-s)^{\alpha-1}v(s)\,ds\biggr]\\&-\frac{\Gamma(2-\beta)(\eta+1)}{3(\eta^{1-\beta}+1)}\\&\times\int_0^1g_1\bigl(s,x(s)\bigr)\,ds+\frac{(\eta+1)\Gamma(2-\beta)}{3(\eta^{1-\beta}+1)\Gamma(\alpha-\beta)}\biggl[\int_0^1(1-s)^{\alpha-\beta-1}v(s)\,ds\\&+\int_0^\eta(\eta-s)^{\alpha-\beta-1}v(s)\,ds\biggr]\\&+\frac{2(\eta+1)(\eta^{2-\beta}+1)\Gamma(3-\gamma)\Gamma(2-\beta)-(\eta^2+1)\Gamma(3-\gamma)(\eta^{1-\beta}+1)\Gamma(3-\beta)}{6(\eta^{1-\beta}+1)(\eta^{2-\gamma}+1)\Gamma(3-\beta)}\\&\times\int_0^1g_2\bigl(s,x(s)\bigr)\,ds\\&+\biggl(\frac{(\eta^2+1)\Gamma(3-\gamma)(\eta^{1-\beta}+1)\Gamma(3-\beta)-2(\eta+1)(\eta^{2-\beta}+1)\Gamma(3-\gamma)\Gamma(2-\beta)}{6(\eta^{1-\beta}+1)(\eta^{2-\gamma}+1)\Gamma(3-\beta)\Gamma(\alpha-\gamma)}\biggr)\end{aligned}$$

$$\times \left[\int_0^1 (1-s)^{\alpha-\gamma-1} v(s)\,ds + \int_0^\eta (\eta-s)^{\alpha-\gamma-1} v(s)\,ds \right],$$

$$c_1 = \frac{\Gamma(2-\beta)}{(\eta^{1-\beta}+1)} \int_0^1 g_1\bigl(s,x(s)\bigr)\,ds - \frac{\Gamma(2-\beta)}{(\eta^{1-\beta}+1)\Gamma(\alpha-\beta)} \left[\int_0^1 (1-s)^{\alpha-\beta-1} v(s)\,ds \right.$$
$$\left. + \int_0^\eta (\eta-s)^{\alpha-\beta-1} v(s)\,ds \right] - \frac{(\eta^{2-\beta}+1)\Gamma(3-\gamma)\Gamma(2-\beta)}{(\eta^{1-\beta}+1)(\eta^{2-\gamma}+1)\Gamma(3-\beta)} \int_0^1 g_2\bigl(s,x(s)\bigr)\,ds$$
$$+ \frac{(\eta^{2-\beta}+1)\Gamma(3-\gamma)\Gamma(2-\beta)}{(\eta^{1-\beta}+1)(\eta^{2-\gamma}+1)\Gamma(3-\beta)\Gamma(\alpha-\gamma)}$$
$$\times \left[\int_0^1 (1-s)^{\alpha-\gamma-1} v(s)\,ds + \int_0^\eta (\eta-s)^{\alpha-\gamma-1} v(s)\,ds \right],$$

and

$$c_2 = \frac{\Gamma(3-\gamma)}{2(\eta^{2-\gamma}+1)} \int_0^1 g_2\bigl(s,x(s)\bigr)\,ds - \frac{\Gamma(3-\gamma)}{2(\eta^{2-\gamma}+1)\Gamma(\alpha-\gamma)}$$
$$\times \left[\int_0^1 (1-s)^{\alpha-\gamma-1} v(s)\,ds + \int_0^\eta (\eta-s)^{\alpha-\gamma-1} v(s)\,ds \right].$$

Now, we replace c_0, c_1, and c_2 in (6) and find the solution $x(t)$ as we stated. This completes the proof. □

Let $X = C^2([0,1])$ endowed with the norm $\|x\| = \sup_{t\in J}|x(t)| + \sup_{t\in J}|x'(t)| + \sup_{t\in J}|x''(t)|$. Then $(X, \|\cdot\|)$ is a Banach space. For $x \in X$, define

$$S_{F,x} = \bigl\{ v \in L^1[0,1] : v(t) \in F\bigl(t, x(t), x'(t), x''(t)\bigr) \text{ for almost all } t \in [0,1] \bigr\}.$$

For the study of problem (1) and (2), we shall consider the following conditions.

(H1) $F\colon J \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to P_{cp}(\mathbb{R})$ is an integrable bounded multifunction such that $F(\cdot,x,y,z)\colon [0,1] \to P_{cp}(\mathbb{R})$ is measurable for all $x,y,z \in \mathbb{R}$;

(H2) $g_0, g_1, g_2\colon J \times \mathbb{R} \to \mathbb{R}$ be continuous functions, $\psi\colon [0,\infty) \to [0,\infty)$ a nondecreasing upper semi-continuous map such that $\liminf_{t\to\infty}(t-\psi(t)) > 0$ and $\psi(t) < t$ for all $t > 0$;

(H3) There exist $m, m_0, m_1, m_2 \in C(J,[0,\infty))$ such that

$$H_d\bigl(F(t,x_1,x_2,x_3), F(t,y_1,y_2,y_3)\bigr) \le \frac{1}{\Lambda_1+\Lambda_2+\Lambda_3} m(t)\psi\bigl(|x_1-y_1| + |x_2-y_2| + |x_3-y_3|\bigr)$$

and $|g_j(t,x) - g_j(t,y)| \le \frac{1}{\Lambda_1+\Lambda_2+\Lambda_3} m_j(t)\psi(|x-y|)$ for all $t \in J$, $x,y,x_1,x_2,x_3,y_1,y_2,y_3 \in \mathbb{R}$ and $j = 0,1,2$, where

$$\Lambda_1 = \left[\frac{\|m\|_\infty}{\Gamma(\alpha+1)} + \frac{\|m_0\|_\infty}{3} + \frac{2\|m\|_\infty}{3\Gamma(\alpha+1)} + \frac{5\Gamma(2-\beta)\|m_1\|_\infty}{3} + \frac{10\Gamma(2-\beta)\|m\|_\infty}{3\Gamma(\alpha-\beta+1)} \right.$$
$$\left. + \frac{10(2\Gamma(2-\beta)+\Gamma(3-\beta))\Gamma(3-\gamma)(\|m_2\|_\infty\Gamma(\alpha-\gamma+1) + 2\|m\|_\infty)}{3\Gamma(3-\beta)\Gamma(\alpha-\gamma+1)} \right],$$

$$\Lambda_2 = \left[\frac{\|m\|_\infty}{\Gamma(\alpha)} + \frac{2\Gamma(2-\beta)\|m\|_\infty}{\Gamma(\alpha-\beta+1)} \right.$$
$$\left. + \frac{(2\Gamma(2-\beta)+\Gamma(3-\beta))\Gamma(3-\gamma)(\|m_2\|_\infty\Gamma(\alpha-\gamma+1) + 2\|m\|_\infty)}{\Gamma(3-\beta)\Gamma(\alpha-\gamma+1)} \right],$$

and $\Lambda_3 = [\frac{\|m\|_\infty}{\Gamma(\alpha-1)} + \frac{\Gamma(3-\gamma)(\|m_2\|_\infty\Gamma(\alpha-\gamma+1)+2\|m\|_\infty)}{\Gamma(\alpha-\gamma+1)}]$, and finally

(H4) $N : X \to 2^X$ is given by

$$N(x) = \{h \in X : \text{ there exist } v \in S_{F,x} \text{ such that } h(t) = w(t) \text{ for all } t \in J\},$$

where

$$\begin{aligned} w(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{(\alpha-1)} v(s)\,ds + \frac{1}{3}\int_0^1 g_0(s,x(s))\,ds - \frac{1}{3\Gamma(\alpha)}\biggl[\int_0^1 (1-s)^{\alpha-1}v(s)\,ds \\ &\quad + \int_0^\eta (\eta-s)^{\alpha-1}v(s)\,ds\biggr] + \frac{3\Gamma(2-\beta)t-(\eta+1)\Gamma(2-\beta)}{3(\eta^{1-\beta}+1)}\int_0^1 g_1(s,x(s))\,ds \\ &\quad + \frac{(\eta+1)\Gamma(2-\beta)-3\Gamma(2-\beta)t}{3(\eta^{1-\beta}+1)\Gamma(\alpha-\beta)}\biggl[\int_0^1 (1-s)^{\alpha-\beta-1}v(s)\,ds + \int_0^\eta (\eta-s)^{\alpha-\beta-1}v(s)\,ds\biggr] \\ &\quad + \biggl(\frac{2(\eta+1)(\eta^{2-\beta}+1)\Gamma(3-\gamma)\Gamma(2-\beta)-(\eta^2+1)\Gamma(3-\gamma)(\eta^{1-\beta}+1)\Gamma(3-\beta)}{6(\eta^{1-\beta}+1)(\eta^{2-\gamma}+1)\Gamma(3-\beta)} \\ &\quad + \frac{-6(\eta^{2-\beta}+1)\Gamma(3-\gamma)\Gamma(2-\beta)t+3(\eta^{1-\beta}+1)\Gamma(3-\gamma)\Gamma(3-\beta)t^2}{6(\eta^{1-\beta}+1)(\eta^{2-\gamma}+1)\Gamma(3-\beta)}\biggr) \\ &\quad \times \int_0^1 g_2(s,x(s))\,ds \\ &\quad + \biggl(\frac{(\eta^2+1)\Gamma(3-\gamma)(\eta^{1-\beta}+1)\Gamma(3-\beta)-2(\eta+1)(\eta^{2-\beta}+1)\Gamma(3-\gamma)\Gamma(2-\beta)}{6(\eta^{1-\beta}+1)(\eta^{2-\gamma}+1)\Gamma(3-\beta)\Gamma(\alpha-\gamma)} \\ &\quad + \frac{6(\eta^{2-\beta}+1)\Gamma(3-\gamma)\Gamma(2-\beta)t-3\Gamma(3-\gamma)\Gamma(3-\beta)(\eta^{1-\beta}+1)t^2}{6(\eta^{1-\beta}+1)(\eta^{2-\gamma}+1)\Gamma(3-\beta)\Gamma(\alpha-\gamma)}\biggr) \\ &\quad \times \biggl[\int_0^1 (1-s)^{\alpha-\gamma-1}v(s)\,ds + \int_0^\eta (\eta-s)^{\alpha-\gamma-1}v(s)\,ds\biggr]. \end{aligned}$$

Theorem 3.1 *Assume that* (H1)-(H4) *are satisfied. If the multifunction N has the approximate endpoint property, then the boundary value inclusion problem* (1) *and* (2) *has a solution.*

Proof We show that the multifunction $N : X \to P(X)$ has a endpoint which is a solution of the problem (1) and (2).

Note that the multivalued map $t \vdash F(t, x(t), x'(t), x''(t))$ is measurable and has closed values for all $x \in X$. Hence, it has measurable selection and so $S_{F,x}$ is nonempty for all $x \in X$. First, we show that $N(x)$ is closed subset of X for all $x \in X$.

Let $x \in X$ and $\{u_n\}_{n\geq 1}$ be a sequence in $N(x)$ with $u_n \to u$. For each $n \in \mathbb{N}$, choose $v_n \in S_{F,x}$ such that

$$\begin{aligned} u_n(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{(\alpha-1)} v_n(s)\,ds + \frac{1}{3}\int_0^1 g_0(s,x(s))\,ds - \frac{1}{3\Gamma(\alpha)}\biggl[\int_0^1 (1-s)^{\alpha-1}v_n(s)\,ds \\ &\quad + \int_0^\eta (\eta-s)^{\alpha-1}v_n(s)\,ds\biggr] + \frac{3\Gamma(2-\beta)t-(\eta+1)\Gamma(2-\beta)}{3(\eta^{1-\beta}+1)}\int_0^1 g_1(s,x(s))\,ds \\ &\quad + \frac{(\eta+1)\Gamma(2-\beta)-3\Gamma(2-\beta)t}{3(\eta^{1-\beta}+1)\Gamma(\alpha-\beta)}\biggl[\int_0^1 (1-s)^{\alpha-\beta-1}v_n(s)\,ds \\ &\quad + \int_0^\eta (\eta-s)^{\alpha-\beta-1}v_n(s)\,ds\biggr] \end{aligned}$$

$$
\begin{aligned}
&+ \biggl(\frac{2(\eta+1)(\eta^{2-\beta}+1)\Gamma(3-\gamma)\Gamma(2-\beta)-(\eta^2+1)\Gamma(3-\gamma)(\eta^{1-\beta}+1)\Gamma(3-\beta)}{6(\eta^{1-\beta}+1)(\eta^{2-\gamma}+1)\Gamma(3-\beta)} \\
&+ \frac{-6(\eta^{2-\beta}+1)\Gamma(3-\gamma)\Gamma(2-\beta)t+3(\eta^{1-\beta}+1)\Gamma(3-\gamma)\Gamma(3-\beta)t^2}{6(\eta^{1-\beta}+1)(\eta^{2-\gamma}+1)\Gamma(3-\beta)}\biggr) \\
&\times \int_0^1 g_2\bigl(s,x(s)\bigr)\,ds \\
&+ \biggl(\frac{(\eta^2+1)\Gamma(3-\gamma)(\eta^{1-\beta}+1)\Gamma(3-\beta)-2(\eta+1)(\eta^{2-\beta}+1)\Gamma(3-\gamma)\Gamma(2-\beta)}{6(\eta^{1-\beta}+1)(\eta^{2-\gamma}+1)\Gamma(3-\beta)\Gamma(\alpha-\gamma)} \\
&+ \frac{6(\eta^{2-\beta}+1)\Gamma(3-\gamma)\Gamma(2-\beta)t-3\Gamma(3-\gamma)\Gamma(3-\beta)(\eta^{1-\beta}+1)t^2}{6(\eta^{1-\beta}+1)(\eta^{2-\gamma}+1)\Gamma(3-\beta)\Gamma(\alpha-\gamma)}\biggr) \\
&\times \biggl[\int_0^1 (1-s)^{\alpha-\gamma-1}v_n(s)\,ds + \int_0^\eta (\eta-s)^{\alpha-\gamma-1}v_n(s)\,ds\biggr]
\end{aligned}
$$

for all $t \in J$.

Since F has compact values, $\{v_n\}_{n\geq 1}$ has a subsequence which converges to some $v \in L^1[0,1]$. We denote this subsequence again by $\{v_n\}_{n\geq 1}$.

It is easy to check that $v \in S_{F,x}$ and

$$
\begin{aligned}
&u_n(t) \to u(t) \\
&= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{(\alpha-1)}v(s)\,ds + \frac{1}{3}\int_0^1 g_0\bigl(s,x(s)\bigr)\,ds \\
&- \frac{1}{3\Gamma(\alpha)}\biggl[\int_0^1 (1-s)^{\alpha-1}v(s)\,ds + \int_0^\eta (\eta-s)^{\alpha-1}v(s)\,ds\biggr] \\
&+ \frac{3\Gamma(2-\beta)t-(\eta+1)\Gamma(2-\beta)}{3(\eta^{1-\beta}+1)}\int_0^1 g_1\bigl(s,x(s)\bigr)\,ds \\
&+ \frac{(\eta+1)\Gamma(2-\beta)-3\Gamma(2-\beta)t}{3(\eta^{1-\beta}+1)\Gamma(\alpha-\beta)}\biggl[\int_0^1 (1-s)^{\alpha-\beta-1}v(s)\,ds + \int_0^\eta (\eta-s)^{\alpha-\beta-1}v(s)\,ds\biggr] \\
&+ \biggl(\frac{2(\eta+1)(\eta^{2-\beta}+1)\Gamma(3-\gamma)\Gamma(2-\beta)-(\eta^2+1)\Gamma(3-\gamma)(\eta^{1-\beta}+1)\Gamma(3-\beta)}{6(\eta^{1-\beta}+1)(\eta^{2-\gamma}+1)\Gamma(3-\beta)} \\
&+ \frac{-6(\eta^{2-\beta}+1)\Gamma(3-\gamma)\Gamma(2-\beta)t+3(\eta^{1-\beta}+1)\Gamma(3-\gamma)\Gamma(3-\beta)t^2}{6(\eta^{1-\beta}+1)(\eta^{2-\gamma}+1)\Gamma(3-\beta)}\biggr) \\
&\times \int_0^1 g_2\bigl(s,x(s)\bigr)\,ds \\
&+ \biggl(\frac{(\eta^2+1)\Gamma(3-\gamma)(\eta^{1-\beta}+1)\Gamma(3-\beta)-2(\eta+1)(\eta^{2-\beta}+1)\Gamma(3-\gamma)\Gamma(2-\beta)}{6(\eta^{1-\beta}+1)(\eta^{2-\gamma}+1)\Gamma(3-\beta)\Gamma(\alpha-\gamma)} \\
&+ \frac{6(\eta^{2-\beta}+1)\Gamma(3-\gamma)\Gamma(2-\beta)t-3\Gamma(3-\gamma)\Gamma(3-\beta)(\eta^{1-\beta}+1)t^2}{6(\eta^{1-\beta}+1)(\eta^{2-\gamma}+1)\Gamma(3-\beta)\Gamma(\alpha-\gamma)}\biggr) \\
&\times \biggl[\int_0^1 (1-s)^{\alpha-\gamma-1}v(s)\,ds + \int_0^\eta (\eta-s)^{\alpha-\gamma-1}v(s)\,ds\biggr]
\end{aligned}
$$

for all $t \in J$. This implies that $u \in N(x)$ and so N has closed values.

Since F is a compact multivalued map, it is easy to check that $N(x)$ is a bounded set for all $x \in X$.

Now, we show that $H_d(N(x),N(y)) \leq \psi(\|x-y\|)$.

Let $x, y \in X$ and $h_1 \in N(y)$. Choose $v_1 \in S_{F,y}$ such that

$$\begin{aligned}
h_1(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{(\alpha-1)} v_1(s)\,ds + \frac{1}{3}\int_0^1 g_0\bigl(s, y(s)\bigr)\,ds - \frac{1}{3\Gamma(\alpha)}\biggl[\int_0^1 (1-s)^{\alpha-1} v_1(s)\,ds \\
&\quad + \int_0^\eta (\eta-s)^{\alpha-1} v_1(s)\,ds\biggr] + \frac{3\Gamma(2-\beta)t - (\eta+1)\Gamma(2-\beta)}{3(\eta^{1-\beta}+1)} \int_0^1 g_1\bigl(s, y(s)\bigr)\,ds \\
&\quad + \frac{(\eta+1)\Gamma(2-\beta) - 3\Gamma(2-\beta)t}{3(\eta^{1-\beta}+1)\Gamma(\alpha-\beta)}\biggl[\int_0^1 (1-s)^{\alpha-\beta-1} v_1(s)\,ds \\
&\quad + \int_0^\eta (\eta-s)^{\alpha-\beta-1} v_1(s)\,ds\biggr] \\
&\quad + \biggl(\frac{2(\eta+1)(\eta^{2-\beta}+1)\Gamma(3-\gamma)\Gamma(2-\beta) - (\eta^2+1)\Gamma(3-\gamma)(\eta^{1-\beta}+1)\Gamma(3-\beta)}{6(\eta^{1-\beta}+1)(\eta^{2-\gamma}+1)\Gamma(3-\beta)} \\
&\quad + \frac{-6(\eta^{2-\beta}+1)\Gamma(3-\gamma)\Gamma(2-\beta)t + 3(\eta^{1-\beta}+1)\Gamma(3-\gamma)\Gamma(3-\beta)t^2}{6(\eta^{1-\beta}+1)(\eta^{2-\gamma}+1)\Gamma(3-\beta)}\biggr) \\
&\quad \times \int_0^1 g_2\bigl(s, y(s)\bigr)\,ds \\
&\quad + \biggl(\frac{(\eta^2+1)\Gamma(3-\gamma)(\eta^{1-\beta}+1)\Gamma(3-\beta) - 2(\eta+1)(\eta^{2-\beta}+1)\Gamma(3-\gamma)\Gamma(2-\beta)}{6(\eta^{1-\beta}+1)(\eta^{2-\gamma}+1)\Gamma(3-\beta)\Gamma(\alpha-\gamma)} \\
&\quad + \frac{6(\eta^{2-\beta}+1)\Gamma(3-\gamma)\Gamma(2-\beta)t - 3\Gamma(3-\gamma)\Gamma(3-\beta)(\eta^{1-\beta}+1)t^2}{6(\eta^{1-\beta}+1)(\eta^{2-\gamma}+1)\Gamma(3-\beta)\Gamma(\alpha-\gamma)}\biggr) \\
&\quad \times \biggl[\int_0^1 (1-s)^{\alpha-\gamma-1} v_1(s)\,ds + \int_0^\eta (\eta-s)^{\alpha-\gamma-1} v_1(s)\,ds\biggr]
\end{aligned}$$

for almost all $t \in J$.

Since

$$\begin{aligned}
&H_d\bigl(F\bigl(t, x(t), x'(t), x''(t)\bigr), F\bigl(t, y(t), y'(t), y''(t)\bigr)\bigr) \\
&\quad \le \frac{1}{\Lambda_1 + \Lambda_2 + \Lambda_3} m(t)\psi\bigl(\bigl|x(t) - y(t)\bigr| + \bigl|x'(t) - y'(t)\bigr| + \bigl|x''(t) - y''(t)\bigr|\bigr)
\end{aligned}$$

for all $t \in J$, there exists $w \in F(t, x(t), x'(t), x''(t))$ such that

$$\bigl|v_1(t) - w\bigr| \le \frac{1}{\Lambda_1 + \Lambda_2 + \Lambda_3} m(t)\psi\bigl(\bigl|x(t) - y(t)\bigr| + \bigl|x'(t) - y'(t)\bigr| + \bigl|x''(t) - y''(t)\bigr|\bigr)$$

for all $t \in J$.

Consider the multivalued map $U : J \to P(\mathbb{R})$ defined by

$$\begin{aligned}
U(t) = \biggl\{ &w \in \mathbb{R} : \bigl|v_1(t) - w\bigr| \\
&\le \frac{1}{\Lambda_1 + \Lambda_2 + \Lambda_3} m(t)\psi\bigl(\bigl|x(t) - y(t)\bigr| + \bigl|x'(t) - y'(t)\bigr| + \bigl|x''(t) - y''(t)\bigr|\bigr)\biggr\}.
\end{aligned}$$

Since v_1 and $\varphi = m\psi(|x-y| + |x'-y'| + |x''-y''|)(\frac{1}{\Lambda_1+\Lambda_2+\Lambda_3})$ are measurable, the multifunction $U(\cdot) \cap F(\cdot, x(\cdot), x'(\cdot), x''(\cdot))$ is measurable.

Choose $v_2(t) \in F(t, x(t), x'(t), x''(t))$ such that

$$\begin{aligned} &\left|v_1(t) - v_2(t)\right| \\ &\quad \leq \frac{1}{\Lambda_1 + \Lambda_2 + \Lambda_3} m(t)\psi\left(\left|x(t) - y(t)\right| + \left|x'(t) - y'(t)\right| + \left|x''(t) - y''(t)\right|\right) \end{aligned}$$

for all $t \in J$.

Now, consider the element $h_2 \in N(x)$, which is defined by

$$\begin{aligned} h_2(t) = {} & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{(\alpha-1)} v_2(s)\,ds + \frac{1}{3} \int_0^1 g_0\bigl(s, x(s)\bigr)\,ds - \frac{1}{3\Gamma(\alpha)} \biggl[\int_0^1 (1-s)^{\alpha-1} v_2(s)\,ds \\ & + \int_0^\eta (\eta - s)^{\alpha-1} v_2(s)\,ds\biggr] + \frac{3\Gamma(2-\beta)t - (\eta+1)\Gamma(2-\beta)}{3(\eta^{1-\beta}+1)} \int_0^1 g_1\bigl(s, x(s)\bigr)\,ds \\ & + \frac{(\eta+1)\Gamma(2-\beta) - 3\Gamma(2-\beta)t}{3(\eta^{1-\beta}+1)\Gamma(\alpha-\beta)} \biggl[\int_0^1 (1-s)^{\alpha-\beta-1} v_2(s)\,ds \\ & + \int_0^\eta (\eta-s)^{\alpha-\beta-1} v_2(s)\,ds\biggr] \\ & + \biggl(\frac{2(\eta+1)(\eta^{2-\beta}+1)\Gamma(3-\gamma)\Gamma(2-\beta) - (\eta^2+1)\Gamma(3-\gamma)(\eta^{1-\beta}+1)\Gamma(3-\beta)}{6(\eta^{1-\beta}+1)(\eta^{2-\gamma}+1)\Gamma(3-\beta)} \\ & + \frac{-6(\eta^{2-\beta}+1)\Gamma(3-\gamma)\Gamma(2-\beta)t + 3(\eta^{1-\beta}+1)\Gamma(3-\gamma)\Gamma(3-\beta)t^2}{6(\eta^{1-\beta}+1)(\eta^{2-\gamma}+1)\Gamma(3-\beta)}\biggr) \\ & \times \int_0^1 g_2\bigl(s, x(s)\bigr)\,ds \\ & + \biggl(\frac{(\eta^2+1)\Gamma(3-\gamma)(\eta^{1-\beta}+1)\Gamma(3-\beta) - 2(\eta+1)(\eta^{2-\beta}+1)\Gamma(3-\gamma)\Gamma(2-\beta)}{6(\eta^{1-\beta}+1)(\eta^{2-\gamma}+1)\Gamma(3-\beta)\Gamma(\alpha-\gamma)} \\ & + \frac{6(\eta^{2-\beta}+1)\Gamma(3-\gamma)\Gamma(2-\beta)t - 3\Gamma(3-\gamma)\Gamma(3-\beta)(\eta^{1-\beta}+1)t^2}{6(\eta^{1-\beta}+1)(\eta^{2-\gamma}+1)\Gamma(3-\beta)\Gamma(\alpha-\gamma)}\biggr) \\ & \times \biggl[\int_0^1 (1-s)^{\alpha-\gamma-1} v_2(s)\,ds + \int_0^\eta (\eta-s)^{\alpha-\gamma-1} v_2(s)\,ds\biggr] \end{aligned}$$

for all $t \in J$. Thus,

$$\begin{aligned} &\left|h_1(t) - h_2(t)\right| \\ &\quad \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{(\alpha-1)} \left|v_1(s) - v_2(s)\right|\,ds \\ &\qquad + \frac{1}{3} \int_0^1 \left|g_0\bigl(s, y(s)\bigr) - g_0\bigl(s, x(s)\bigr)\right|\,ds + \frac{1}{3\Gamma(\alpha)} \biggl[\int_0^1 (1-s)^{\alpha-1} \left|v_1(s) - v_2(s)\right|\,ds \\ &\qquad + \int_0^\eta (\eta-s)^{\alpha-1} \left|v_1(s) - v_2(s)\right|\,ds\biggr] + \left|\frac{3\Gamma(2-\beta)t - (\eta+1)\Gamma(2-\beta)}{3(\eta^{1-\beta}+1)}\right| \\ &\qquad \times \int_0^1 \left|g_1\bigl(s, y(s)\bigr) - g_1\bigl(s, x(s)\bigr)\right|\,ds + \left|\frac{(\eta+1)\Gamma(2-\beta) - 3\Gamma(2-\beta)t}{3(\eta^{1-\beta}+1)\Gamma(\alpha-\beta)}\right| \\ &\qquad \times \biggl[\int_0^1 (1-s)^{\alpha-\beta-1} \left|v_1(s) - v_2(s)\right|\,ds + \int_0^\eta (\eta-s)^{\alpha-\beta-1} \left|v_1(s) - v_2(s)\right|\,ds\biggr] \end{aligned}$$

$$
\begin{aligned}
&+ \left| \left(\frac{2(\eta+1)(\eta^{2-\beta}+1)\Gamma(3-\gamma)\Gamma(2-\beta) - (\eta^2+1)\Gamma(3-\gamma)(\eta^{1-\beta}+1)\Gamma(3-\beta)}{6(\eta^{1-\beta}+1)(\eta^{2-\gamma}+1)\Gamma(3-\beta)} \right.\right. \\
&+ \left.\left. \frac{-6(\eta^{2-\beta}+1)\Gamma(3-\gamma)\Gamma(2-\beta)t + 3(\eta^{1-\beta}+1)\Gamma(3-\gamma)\Gamma(3-\beta)t^2}{6(\eta^{1-\beta}+1)(\eta^{2-\gamma}+1)\Gamma(3-\beta)} \right) \right| \\
&\times \int_0^1 \left| g_2\big(s, y(s)\big) - g_2\big(s, x(s)\big) \right| ds \\
&+ \left| \left(\frac{(\eta^2+1)\Gamma(3-\gamma)(\eta^{1-\beta}+1)\Gamma(3-\beta) - 2(\eta+1)(\eta^{2-\beta}+1)\Gamma(3-\gamma)\Gamma(2-\beta)}{6(\eta^{1-\beta}+1)(\eta^{2-\gamma}+1)\Gamma(3-\beta)\Gamma(\alpha-\gamma)} \right.\right. \\
&+ \left.\left. \frac{6(\eta^{2-\beta}+1)\Gamma(3-\gamma)\Gamma(2-\beta)t - 3\Gamma(3-\gamma)\Gamma(3-\beta)(\eta^{1-\beta}+1)t^2}{6(\eta^{1-\beta}+1)(\eta^{2-\gamma}+1)\Gamma(3-\beta)\Gamma(\alpha-\gamma)} \right) \right| \\
&\times \left[\int_0^1 (1-s)^{\alpha-\gamma-1} \left| v_1(s) - v_2(s) \right| ds + \int_0^\eta (\eta-s)^{\alpha-\gamma-1} \left| v_1(s) - v_2(s) \right| ds \right] \\
\leq{}& \frac{1}{\Lambda_1+\Lambda_2+\Lambda_3} \psi\big(\|x-y\|\big) \left[\frac{\|m\|_\infty}{\Gamma(\alpha+1)} + \frac{\|m_0\|_\infty}{3} + \frac{2\|m\|_\infty}{3\Gamma(\alpha+1)} + \frac{5\Gamma(2-\beta)\|m_1\|_\infty}{3} \right. \\
&+ \frac{10\Gamma(2-\beta)\|m\|_\infty}{3\Gamma(\alpha-\beta+1)} \\
&+ \left. \frac{10(2\Gamma(2-\beta)+\Gamma(3-\beta))\Gamma(3-\gamma)(\|m_2\|_\infty\Gamma(\alpha-\gamma+1) + 2\|m\|_\infty)}{3\Gamma(3-\beta)\Gamma(\alpha-\gamma+1)} \right] \\
={}& \frac{\Lambda_1}{\Lambda_1+\Lambda_2+\Lambda_3} \psi\big(\|x-y\|\big),
\end{aligned}
$$

$$
\begin{aligned}
&\left| h_1'(t) - h_2'(t) \right| \\
&\quad\leq \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{(\alpha-2)} \left| v_1(s) - v_2(s) \right| ds \\
&\quad+ \frac{\Gamma(2-\beta)}{(\eta^{1-\beta}+1)\Gamma(\alpha-\beta)} \left[\int_0^1 (1-s)^{\alpha-\beta-1} \left| v_1(s) - v_2(s) \right| ds \right. \\
&\quad+ \left. \int_0^\eta (\eta-s)^{\alpha-\beta-1} \left| v_1(s) - v_2(s) \right| ds \right] \\
&\quad+ \left| \left(\frac{(\eta^{2-\beta}+1)\Gamma(3-\gamma)\Gamma(2-\beta) + \Gamma(3-\gamma)(\eta^{1-\beta}+1)\Gamma(3-\beta)t}{(\eta^{1-\beta}+1)(\eta^{2-\gamma}+1)\Gamma(3-\beta)} \right.\right| \\
&\quad\times \int_0^1 \left| g_2\big(s, y(s)\big) - g_2\big(s, x(s)\big) \right| ds \\
&\quad+ \left| \left. \frac{(\eta^{2-\beta}+1)\Gamma(3-\gamma)\Gamma(2-\beta) - \Gamma(3-\gamma)\Gamma(3-\beta)(\eta^{1-\beta}+1)t}{(\eta^{1-\beta}+1)(\eta^{2-\gamma}+1)\Gamma(3-\beta)\Gamma(\alpha-\gamma)} \right) \right| \\
&\quad\times \left[\int_0^1 (1-s)^{\alpha-\gamma-1} \left| v_1(s) - v_2(s) \right| ds + \int_0^\eta (\eta-s)^{\alpha-\gamma-1} \left| v_1(s) - v_2(s) \right| ds \right] \\
&\quad\leq \frac{1}{\Lambda_1+\Lambda_2+\Lambda_3} \psi\big(\|x-y\|\big) \left[\frac{\|m\|_\infty}{\Gamma(\alpha)} + \frac{2\Gamma(2-\beta)\|m\|_\infty}{\Gamma(\alpha-\beta+1)} \right. \\
&\quad+ \left. \frac{(2\Gamma(2-\beta)+\Gamma(3-\beta))\Gamma(3-\gamma)\big(\|m_2\|_\infty\Gamma(\alpha-\gamma+1) + 2\|m\|_\infty\big)}{\Gamma(3-\beta)\Gamma(\alpha-\gamma+1)} \right] \\
&\quad= \frac{\Lambda_2}{\Lambda_1+\Lambda_2+\Lambda_3} \psi\big(\|x-y\|\big),
\end{aligned}
$$

and

$$\begin{aligned}
&\bigl|h_1''(t) - h_2''(t)\bigr| \\
&\quad\le \frac{1}{\Gamma(\alpha-2)}\int_0^t (t-s)^{(\alpha-3)}\bigl|v_1(s)-v_2(s)\bigr|\,ds \\
&\qquad + \frac{\Gamma(3-\gamma)}{(\eta^{2-\gamma}+1)}\int_0^1 \bigl|g_2\bigl(s,y(s)\bigr)-g_2\bigl(s,x(s)\bigr)\bigr|\,ds \\
&\qquad + \frac{\Gamma(3-\gamma)}{(\eta^{2-\gamma}+1)\Gamma(\alpha-\gamma)}\biggl[\int_0^1 (1-s)^{\alpha-\gamma-1}\bigl|v_1(s)-v_2(s)\bigr|\,ds \\
&\qquad + \int_0^\eta (\eta-s)^{\alpha-\gamma-1}\bigl|v_1(s)-v_2(s)\bigr|\,ds\biggr] \\
&\quad\le \frac{1}{\Lambda_1+\Lambda_2+\Lambda_3}\psi\bigl(\|x-y\|\bigr)\biggl[\frac{\|m\|_\infty}{\Gamma(\alpha-1)} + \frac{\Gamma(3-\gamma)(\|m_2\|_\infty\Gamma(\alpha-\gamma+1)+2\|m\|_\infty)}{\Gamma(\alpha-\gamma+1)}\biggr] \\
&\quad= \frac{\Lambda_3}{\Lambda_1+\Lambda_2+\Lambda_3}\psi\bigl(\|x-y\|\bigr).
\end{aligned}$$

Hence,

$$\begin{aligned}
\|h_1-h_2\| &= \sup_{t\in J}\bigl|h_1(t)-h_2(t)\bigr| + \sup_{t\in J}\bigl|h_1'(t)-h_2'(t)\bigr| + \sup_{t\in J}\bigl|h_1''(t)-h_2''(t)\bigr| \\
&\le \frac{1}{\Lambda_1+\Lambda_2+\Lambda_3}\psi\bigl(\|x-y\|\bigr)(\Lambda_1+\Lambda_2+\Lambda_3) = \psi\bigl(\|x-y\|\bigr).
\end{aligned}$$

Thus, it is easy to get $H_d(N(x),N(y)) \le \psi(\|x-y\|)$ for all $x,y\in X$.

Since the multifunction N has approximate endpoint property, by using Lemma 2.1 there exists $x^*\in X$ such that $N(x^*)=\{x^*\}$. Hence by using Lemma 3.1, x^* is a solution of the problem (1) and (2). □

Now, we investigate the existence of solution for the fractional differential inclusion problem

$$^cD^\alpha x(t) \in F\bigl(t,x(t),{}^cD^{\gamma_1}x(t),\dots,{}^cD^{\gamma_n}x(t)\bigr),$$

via integral boundary value conditions

$$x'(0)+bx'(1) = \sum_{i=1}^n {}^cD^{\gamma_i}x(\eta), \qquad x(0)+ax(1) = \sum_{i=1}^n I^{\gamma_i}x(\eta),$$

where $t\in J=[0,1]$, $1<\alpha\le 2$, $n\ge 2$, $0<\eta,\gamma_i<1$, $\alpha-\gamma_i\ge 1$ for all $1\le i\le n$, $a>\sum_{i=1}^n \frac{\eta^{\gamma_i+1}}{\Gamma(\gamma_i+2)}$, $b>\sum_{i=1}^n \frac{\eta^{1-\gamma_i}}{\Gamma(2-\gamma_i)}$ and $F\colon J\times\mathbb{R}^{n+1}\to P(\mathbb{R})$ is a multifunction.

Lemma 3.2 *Let $v\in C(J,\mathbb{R})$, $\alpha\in(1,2]$, $\eta\in(0,1)$, $n\ge 2$ and $\beta_i\in(0,1)$ for $i=1,\dots,n$. The unique solution of the fractional differential problem $^cD^\alpha x(t)=v(t)$ via the boundary value conditions*

$$x(0)+ax(1) = \sum_{i=1}^n I^{\beta_i}x(\eta), \qquad x'(0)+bx'(1) = \sum_{i=1}^n {}^cD^{\beta_i}x(\eta),$$

with $a > \sum_{i=1}^{n} \frac{\eta^{\beta_i+1}}{\Gamma(\beta_i+2)}$, $b > \sum_{i=1}^{n} \frac{\eta^{1-\beta_i}}{\Gamma(2-\beta_i)}$ *is*

$$x(t) = \int_0^1 G(t,s)v(s)\,ds,$$

where $G(t,s)$ *is the Green function given by*

$$\begin{aligned} G(t,s) = {}& \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{1}{A}\sum_{i=1}^{n}\frac{(\eta-s)^{\alpha+\beta_i-1}}{\Gamma(\alpha+\beta_i)} - \frac{a}{A\Gamma(\alpha)}(1-s)^{\alpha-1} - \frac{1}{AB}\left(a - \sum_{i=1}^{n}\frac{\eta^{\beta_i+2}}{\Gamma(\beta_i+3)}\right) \\ & \times \sum_{i=1}^{n}\frac{(\eta-s)^{\alpha-\beta_i-1}}{\Gamma(\alpha-\beta_i)} - \frac{b}{AB}\left(a - \sum_{i=1}^{n}\frac{\eta^{\beta_i+2}}{\Gamma(\beta_i+3)}\right)\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} + \frac{t}{B}\sum_{i=1}^{n}\frac{(\eta-s)^{\alpha-\beta_i-1}}{\Gamma(\alpha-\beta_i)} \\ & - \frac{bt}{B\Gamma(\alpha-1)}(1-s)^{\alpha-2} \end{aligned}$$

whenever $0 \le s \le \eta \le t \le 1$,

$$\begin{aligned} G(t,s) = {}& \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{a}{A\Gamma(\alpha)}(1-s)^{\alpha-1} \\ & - \frac{b}{AB}\left(a - \sum_{i=1}^{n}\frac{\eta^{\beta_i+2}}{\Gamma(\beta_i+3)}\right)\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} - \frac{bt}{B\Gamma(\alpha-1)}(1-s)^{\alpha-2} \end{aligned}$$

whenever $0 \le \eta \le s \le t \le 1$,

$$G(t,s) = -\frac{a}{A\Gamma(\alpha)}(1-s)^{\alpha-1} - \frac{b}{AB}\left(a - \sum_{i=1}^{n}\frac{\eta^{\beta_i+2}}{\Gamma(\beta_i+3)}\right)\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} - \frac{bt}{B\Gamma(\alpha-1)}(1-s)^{\alpha-2}$$

whenever $0 \le \eta \le s \le t \le 1$,

$$\begin{aligned} G(t,s) = {}& \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{1}{A}\sum_{i=1}^{n}\frac{(\eta-s)^{\alpha+\beta_i-1}}{\Gamma(\alpha+\beta_i)} - \frac{a}{A\Gamma(\alpha)}(1-s)^{\alpha-1} - \frac{1}{AB}\left(a - \sum_{i=1}^{n}\frac{\eta^{\beta_i+2}}{\Gamma(\beta_i+3)}\right) \\ & \times \sum_{i=1}^{n}\frac{(\eta-s)^{\alpha-\beta_i-1}}{\Gamma(\alpha-\beta_i)} - \frac{b}{AB}\left(a - \sum_{i=1}^{n}\frac{\eta^{\beta_i+2}}{\Gamma(\beta_i+3)}\right)\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} \\ & + \frac{t}{B}\sum_{i=1}^{n}\frac{(\eta-s)^{\alpha-\beta_i-1}}{\Gamma(\alpha-\beta_i)} - \frac{bt}{B\Gamma(\alpha-1)}(1-s)^{\alpha-2} \end{aligned}$$

whenever $0 \le s \le t \le \eta \le 1$,

$$\begin{aligned} G(t,s) = {}& \frac{1}{A}\sum_{i=1}^{n}\frac{(\eta-s)^{\alpha+\beta_i-1}}{\Gamma(\alpha+\beta_i)} - \frac{a}{A\Gamma(\alpha)}(1-s)^{\alpha-1} - \frac{1}{AB}\left(a - \sum_{i=1}^{n}\frac{\eta^{\beta_i+2}}{\Gamma(\beta_i+3)}\right) \\ & \times \sum_{i=1}^{n}\frac{(\eta-s)^{\alpha-\beta_i-1}}{\Gamma(\alpha-\beta_i)} - \frac{b}{AB}\left(a - \sum_{i=1}^{n}\frac{\eta^{\beta_i+2}}{\Gamma(\beta_i+3)}\right)\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} + \frac{t}{B}\sum_{i=1}^{n}\frac{(\eta-s)^{\alpha-\beta_i-1}}{\Gamma(\alpha-\beta_i)} \\ & - \frac{bt}{B\Gamma(\alpha-1)}(1-s)^{\alpha-2} \end{aligned}$$

whenever $0 \le t \le s \le \eta \le 1$ *and*

$$G(t,s) = -\frac{a}{A\Gamma(\alpha)}(1-s)^{\alpha-1} - \frac{b}{AB}\left(a - \sum_{i=1}^{n} \frac{\eta^{\beta_i+2}}{\Gamma(\beta_i+3)}\right)\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} - \frac{bt}{B\Gamma(\alpha-1)}(1-s)^{\alpha-2}$$

whenever $0 \le t \le \eta \le s \le 1$, *where* $A = 1 + a - \sum_{i=1}^{n} \frac{\eta^{\beta_i+1}}{\Gamma(\beta_i+2)}$ *and* $B = 1 + b - \sum_{i=1}^{n} \frac{\eta^{1-\beta_i}}{\Gamma(2-\beta_i)}$.

Proof It is known that the general solution of the equation ${}^{c}D^{\alpha}x(t) = v(t)$ is

$$x(t) = I^{\alpha}v(t) + c_0 + c_1 t = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{(\alpha-1)}v(s)\,ds + c_0 + c_1 t,$$

where $c_0, c_1 \in \mathbb{R}$ are arbitrary constants (see [10, 13] and [14]). Thus,

$$\begin{aligned}
{}^{c}D^{\beta_i}x(t) &= \frac{1}{\Gamma(\alpha-\beta_i)}\int_0^t (t-s)^{(\alpha-\beta_i-1)}v(s)\,ds + \frac{c_1 t^{1-\beta_i}}{\Gamma(2-\beta_i)}, \\
I^{\beta_i}x(t) &= \frac{1}{\Gamma(\alpha+\beta_i)}\int_0^t (t-s)^{(\alpha+\beta_i-1)}v(s)\,ds + \frac{c_0 t^{1+\beta_i}}{\Gamma(2+\beta_i)} + \frac{c_1 t^{2+\beta_i}}{\Gamma(3+\beta_i)},
\end{aligned}$$

and $x'(t) = \frac{1}{\Gamma(\alpha-1)}\int_0^t (t-s)^{(\alpha-2)}v(s)\,ds + c_1$. Hence,

$$x(0) + ax(1) = (a+1)c_0 + ac_1 + \frac{a}{\Gamma(\alpha)}\int_0^1 (1-s)^{(\alpha-1)}v(s)\,ds$$

and

$$x'(0) + bx'(1) = (1+b)c_1 + \frac{b}{\Gamma(\alpha-1)}\int_0^1 (1-s)^{(\alpha-2)}v(s)\,ds.$$

By using the boundary conditions, we obtain

$$\begin{aligned}
&c_0\left(1 + a - \sum_{i=1}^{n} \frac{\eta^{\beta_i+1}}{\Gamma(\beta_i+2)}\right) + c_1\left(a - \sum_{i=1}^{n} \frac{\eta^{\beta_i+2}}{\Gamma(\beta_i+3)}\right) \\
&\quad = \sum_{i=1}^{n}\int_0^{\eta} \frac{(\eta-s)^{\alpha+\beta_i-1}}{\Gamma(\beta_i+\alpha)}v(s)\,ds - \frac{a}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}v(s)\,ds
\end{aligned}$$

and

$$c_1\left(1 + b - \sum_{i=1}^{n} \frac{\eta^{1-\beta_i}}{\Gamma(2-\beta_i)}\right) = \sum_{i=1}^{n}\int_0^{\eta} \frac{(\eta-s)^{\alpha-\beta_i-1}}{\Gamma(\alpha-\beta_i)}v(s)\,ds - \frac{b}{\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}v(s)\,ds.$$

Thus,

$$\begin{aligned}
c_0 &= \frac{1}{A}\sum_{i=1}^{n}\int_0^{\eta} \frac{(\eta-s)^{\alpha+\beta_i-1}}{\Gamma(\alpha+\beta_i)}v(s)\,ds - \frac{a}{A\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}v(s)\,ds \\
&\quad - \frac{1}{AB}\left(a - \sum_{i=1}^{n} \frac{\eta^{\beta_i+2}}{\Gamma(\beta_i+3)}\right)
\end{aligned}$$

$$\begin{aligned}&\times \sum_{i=1}^{n} \int_0^{\eta} \frac{(\eta - s)^{\alpha-\beta_i-1}}{\Gamma(\alpha - \beta_i)} v(s)\,ds \\ &- \frac{b}{AB\Gamma(\alpha-1)} \left(a - \sum_{i=1}^{n} \frac{\eta^{\beta_i+2}}{\Gamma(\beta_i+3)} \right) \int_0^1 (1-s)^{\alpha-2} v(s)\,ds,\end{aligned}$$

$$c_1 = \frac{1}{B} \sum_{i=1}^{n} \int_0^{\eta} \frac{(\eta - s)^{\alpha-\beta_i-1}}{\Gamma(\alpha - \beta_i)} v(s)\,ds - \frac{b}{B\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} v(s)\,ds.$$

Hence,

$$\begin{aligned} x(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v(s)\,ds + \frac{1}{A} \sum_{i=1}^{n} \int_0^{\eta} \frac{(\eta - s)^{\alpha+\beta_i-1}}{\Gamma(\alpha + \beta_i)} v(s)\,ds \\ &\quad - \frac{a}{A\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} v(s)\,ds \\ &\quad - \frac{1}{AB} \left(a - \sum_{i=1}^{n} \frac{\eta^{\beta_i+2}}{\Gamma(\beta_i+3)} \right) \sum_{i=1}^{n} \int_0^{\eta} \frac{(\eta - s)^{\alpha-\beta_i-1}}{\Gamma(\alpha - \beta_i)} v(s)\,ds \\ &\quad - \frac{b}{AB\Gamma(\alpha-1)} \left(a - \sum_{i=1}^{n} \frac{\eta^{\beta_i+2}}{\Gamma(\beta_i+3)} \right) \\ &\quad \times \int_0^1 (1-s)^{\alpha-2} v(s)\,ds + \frac{t}{B} \sum_{i=1}^{n} \int_0^{\eta} \frac{(\eta - s)^{\alpha-\beta_i-1}}{\Gamma(\alpha - \beta_i)} v(s)\,ds \\ &\quad - \frac{tb}{B\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} v(s)\,ds \\ &= \int_0^1 G(t,s) v(s)\,ds. \end{aligned}$$

This completes the proof. □

Suppose that $X = \{x : x, {}^cD^{\gamma_i}x \in C(J,R) \text{ for all } i = 1,\dots,n\}$ endowed with the norm $\|x\| = \sup_{t\in J} |x(t)| + \sum_{i=1}^{n} \sup_{t\in J} |{}^cD^{\gamma_i}x(t)|$. Then $(X, \|\cdot\|)$ is a Banach space [15]. For $x \in X$, define

$$S_{F,x} = \bigl\{ v \in L^1[0,1] : v(t) \in F\bigl(t, x(t), {}^cD^{\gamma_1}x(t), \dots, {}^cD^{\gamma_n}x(t)\bigr) \text{ for almost all } t \in [0,1] \bigr\}.$$

Now, put

$$\begin{aligned} L_1 &= \frac{1}{\Gamma(\alpha+1)} + \frac{1}{A} \sum_{i=1}^{n} \frac{\eta^{\alpha+\gamma_i}}{\Gamma(\alpha+\gamma_i+1)} + \frac{a}{A\Gamma(\alpha+1)} + \frac{1}{AB} \left(\left| a - \sum_{i=1}^{n} \frac{\eta^{\gamma_i+2}}{\Gamma(\gamma_i+3)} \right| \right) \\ &\quad \times \sum_{i=1}^{n} \frac{\eta^{\alpha-\gamma_i}}{\Gamma(\alpha-\gamma_i+1)} + \frac{b}{AB\Gamma(\alpha)} \left(\left| a - \sum_{i=1}^{n} \frac{\eta^{\gamma_i+2}}{\Gamma(\gamma_i+3)} \right| \right) \\ &\quad + \frac{1}{B} \sum_{i=1}^{n} \frac{\eta^{\alpha-\gamma_i}}{\Gamma(\alpha-\gamma_i+1)} + \frac{b}{B\Gamma(\alpha)} \end{aligned}$$

and $L_2^j = \frac{1}{\Gamma(\alpha-\gamma_j+1)} + \frac{1}{B\Gamma(2-\gamma_j)} \sum_{i=1}^{n} \frac{\eta^{\alpha-\gamma_i}}{\Gamma(\alpha-\gamma_i+1)} + \frac{b}{B\Gamma(2-\gamma_j)\Gamma(\alpha)}$ for all $1 \le j \le n$.

Theorem 3.2 *Let $\psi : [0,\infty) \to [0,\infty)$ a nondecreasing upper semi-continuous map such that $\liminf_{t\to\infty}(t-\psi(t)) > 0$ and $\psi(t) < t$ for all $t > 0$, $F: J \times \mathbb{R}^{n+1} \to P_{cp}(\mathbb{R})$ a multifunction such that $F(\cdot, x_1, x_2, \dots, x_{n+1}): [0,1] \to P_{cp}(\mathbb{R})$ is measurable and integrable bounded for all $x_1, x_2, \dots, x_{n+1} \in \mathbb{R}$. Assume that there exists $m \in C(J,[0,\infty))$ such that*

$$\begin{aligned}&H_d\bigl(F(t,x_1,x_2,\dots,x_{n+1}) - F(t,y_1,y_2,\dots,y_{n+1})\bigr)\\&\quad\le m(t)\psi\bigl(|x_1-y_1| + |x_2-y_2| + \cdots + |x_{n+1}-y_{n+1}|\bigr)\left(\frac{1}{\|m\|_\infty(L_1+\sum_{j=1}^n L_2^j)}\right).\end{aligned}$$

Define $\Omega: X \to 2^X$ by

$$\Omega(x) = \left\{h \in X:\ \text{there exist } v \in S_{F,x} \text{ such that } h(t) = \int_0^1 G(t,s)v(s)\,ds \text{ for all } t \in J\right\}.$$

If the multifunction Ω has the approximate endpoint property, then the boundary value inclusion problem (3) *and* (4) *has a solution.*

Proof We show that the multifunction $\Omega: X \to P(X)$ has a endpoint which is a solution of the problem (3) and (4).

First, we show that $\Omega(x)$ is closed subset of X for all $x \in X$.

Let $x \in X$ and $\{u_n\}_{n\ge1}$ be a sequence in $\Omega(x)$ with $u_n \to u$. For each $n \in \mathbb{N}$, choose $v_n \in S_{F,x}$ such that $u_n(t) = \int_0^1 G(t,s)v_n(s)\,ds$ for all $t \in J$. Since F has compact values, $\{v_n\}_{n\ge1}$ has a subsequence which converges to some $v \in L^1[0,1]$. We denote this subsequence again by $\{v_n\}_{n\ge1}$.

It is easy to check that $v \in S_{F,x}$ and $u_n(t) \to u(t) = \int_0^1 G(t,s)v(s)\,ds$ for all $t \in J$. This implies that $u \in \Omega(x)$ and so Ω has closed values. Since F is a compact multivalued map, it is easy to check that $\Omega(x)$ is a bounded set for all $x \in X$.

Now, we show that for all $x, y \in X$, $H_d(\Omega(x),\Omega(y)) \le \psi(\|x-y\|)$.

Let $x, y \in X$ and $h_1 \in \Omega(y)$. Choose $v_1 \in S_{F,y}$ such that $h_1(t) = \int_0^1 G(t,s)v_1(s)\,ds$ for almost all $t \in J$. Since

$$\begin{aligned}&H_d\bigl(F\bigl(t,x(t),{}^cD^{\gamma_1}x(t),\dots,{}^cD^{\gamma_n}x(t)\bigr),F\bigl(t,y(t),{}^cD^{\gamma_1}y(t),\dots,{}^cD^{\gamma_n}y(t)\bigr)\bigr)\\&\quad\le m(t)\psi\bigl(\bigl|x(t)-y(t)\bigr| + \bigl|{}^cD^{\gamma_1}x(t) - {}^cD^{\gamma_1}y(t)\bigr| + \cdots + \bigl|{}^cD^{\gamma_n}x(t) - {}^cD^{\gamma_n}y(t)\bigr|\bigr)\\&\qquad\times\left(\frac{1}{\|m\|_\infty(L_1+\sum_{j=1}^n L_2^j)}\right)\end{aligned}$$

for all $t \in J$, there exists $w \in F(t,x(t),{}^cD^{\gamma_1}x(t),\dots,{}^cD^{\gamma_n}x(t))$ such that

$$\begin{aligned}\bigl|v_1(t)-w\bigr| &\le m(t)\psi\bigl(\bigl|x(t)-y(t)\bigr| + \bigl|{}^cD^{\gamma_1}x(t) - {}^cD^{\gamma_1}y(t)\bigr| + \cdots\\&\quad+ \bigl|{}^cD^{\gamma_n}x(t) - {}^cD^{\gamma_n}y(t)\bigr|\bigr)\left(\frac{1}{\|m\|_\infty(L_1+\sum_{j=1}^n L_2^j)}\right)\end{aligned}$$

for all $t \in J$. Consider the multivalued map $U: J \to P(\mathbb{R})$ defined by the rule

$$U(t) = \left\{w \in \mathbb{R} : \left|v_1(t) - w\right| \le m(t)\psi\left(\left|x(t) - y(t)\right| + \left|{}^cD^{\gamma_1}x(t) - {}^cD^{\gamma_1}y(t)\right| + \cdots \right.\right.$$
$$\left.\left. + \left|{}^cD^{\gamma_n}x(t) - {}^cD^{\gamma_n}y(t)\right|\right)\left(\frac{1}{\|m\|_\infty(L_1 + \sum_{j=1}^n L_2^j)}\right)\right\}.$$

Since v_1 and

$$\varphi = m\psi\left(|x - y| + \left|{}^cD^{\gamma_1}x - {}^cD^{\gamma_1}y\right| + \cdots + \left|{}^cD^{\gamma_n}x - {}^cD^{\gamma_n}y\right|\right)\left(\frac{1}{\|m\|_\infty(L_1 + \sum_{j=1}^n L_2^j)}\right)$$

are measurable, the multifunction

$$U(\cdot) \cap F\big(t, x(\cdot), {}^cD^{\gamma_1}x(\cdot), \ldots, {}^cD^{\gamma_n}x(\cdot)\big)$$

is measurable.

Choose $v_2(t) \in F(t, x(t), {}^cD^{\gamma_1}x(t), \ldots, {}^cD^{\gamma_n}x(t))$ such that

$$\left|v_1(t) - v_2(t)\right| \le m(t)\psi\left(\left|x(t) - y(t)\right| + \left|{}^cD^{\gamma_1}x(t) - {}^cD^{\gamma_1}y(t)\right| + \cdots \right.$$
$$\left. + \left|{}^cD^{\gamma_n}x(t) - {}^cD^{\gamma_n}y(t)\right|\right)\left(\frac{1}{\|m\|_\infty(L_1 + \sum_{j=1}^n L_2^j)}\right)$$

for all $t \in J$. Now, consider the element $h_2 \in \Omega(x)$ which is defined by $h_2(t) = \int_0^1 G(t,s) \times v_2(s)\,ds$ for all $t \in J$.

Thus,

$$\begin{aligned}
&\left|h_1(t) - h_2(t)\right| \\
&\quad = \left|\int_0^1 G(t,s)v_1(s)\,ds - \int_0^1 G(t,s)v_2(s)\,ds\right| \\
&\quad = \left|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}v_1(s)\,ds\right. \\
&\qquad + \frac{1}{A}\sum_{i=1}^n \int_0^\eta \frac{(\eta - s)^{\alpha+\gamma_i-1}}{\Gamma(\alpha+\gamma_i)}v_1(s)\,ds - \frac{a}{A\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}v_1(s)\,ds \\
&\qquad - \frac{1}{AB}\left(a - \sum_{i=1}^n \frac{\eta^{\gamma_i+2}}{\Gamma(\gamma_i+3)}\right) \\
&\qquad \times \sum_{i=1}^n \int_0^\eta \frac{(\eta - s)^{\alpha-\gamma_i-1}}{\Gamma(\alpha-\gamma_i)}v_1(s)\,ds - \frac{b}{AB\Gamma(\alpha-1)}\left(a - \sum_{i=1}^n \frac{\eta^{\gamma_i+2}}{\Gamma(\gamma_i+3)}\right) \\
&\qquad \times \int_0^1 (1-s)^{\alpha-2}v_1(s)\,ds \\
&\qquad + \frac{t}{B}\sum_{i=1}^n \int_0^\eta \frac{(\eta - s)^{\alpha-\gamma_i-1}}{\Gamma(\alpha-\gamma_i)}v_1(s)\,ds - \frac{tb}{B\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}v_1(s)\,ds \\
&\qquad - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}v_2(s)\,ds
\end{aligned}$$

$$\begin{aligned}
&- \frac{1}{A}\sum_{i=1}^{n}\int_0^{\eta}\frac{(\eta-s)^{\alpha+\gamma_i-1}}{\Gamma(\alpha+\gamma_i)}\nu_2(s)\,ds + \frac{a}{A\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}\nu_2(s)\,ds \\
&+ \frac{1}{AB}\left(a-\sum_{i=1}^{n}\frac{\eta^{\gamma_i+2}}{\Gamma(\gamma_i+3)}\right) \\
&\times \sum_{i=1}^{n}\int_0^{\eta}\frac{(\eta-s)^{\alpha-\gamma_i-1}}{\Gamma(\alpha-\gamma_i)}\nu_2(s)\,ds + \frac{b}{AB\Gamma(\alpha-1)}\left(a-\sum_{i=1}^{n}\frac{\eta^{\gamma_i+2}}{\Gamma(\gamma_i+3)}\right) \\
&\times \int_0^1(1-s)^{\alpha-2}\nu_2(s)\,ds \\
&- \frac{t}{B}\sum_{i=1}^{n}\int_0^{\eta}\frac{(\eta-s)^{\alpha-\gamma_i-1}}{\Gamma(\alpha-\gamma_i)}\nu_2(s)\,ds + \frac{tb}{B\Gamma(\alpha-1)}\int_0^1(1-s)^{\alpha-2}\nu_2(s)\,ds\bigg| \\
\le{}& \frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\big|\nu_1(s)-\nu_2(s)\big|\,ds \\
&+ \frac{1}{A}\sum_{i=1}^{n}\int_0^{\eta}\frac{(\eta-s)^{\alpha+\gamma_i-1}}{\Gamma(\alpha+\gamma_i)}\big|\nu_1(s)-\nu_2(s)\big|\,ds \\
&+ \frac{a}{A\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}\big|\nu_1(s)-\nu_2(s)\big|\,ds \\
&+ \frac{1}{AB}\left(\left|a-\sum_{i=1}^{n}\frac{\eta^{\gamma_i+2}}{\Gamma(\gamma_i+3)}\right|\right)\sum_{i=1}^{n}\int_0^{\eta}\frac{(\eta-s)^{\alpha-\gamma_i-1}}{\Gamma(\alpha-\gamma_i)}\big|\nu_1(s)-\nu_2(s)\big|\,ds \\
&+ \frac{b}{AB\Gamma(\alpha-1)}\left(\left|a-\sum_{i=1}^{n}\frac{\eta^{\gamma_i+2}}{\Gamma(\gamma_i+3)}\right|\right) \\
&\times \int_0^1(1-s)^{\alpha-2}\big|\nu_1(s)-\nu_2(s)\big|\,ds + \frac{t}{B}\sum_{i=1}^{n}\int_0^{\eta}\frac{(\eta-s)^{\alpha-\gamma_i-1}}{\Gamma(\alpha-\gamma_i)}\big|\nu_1(s)-\nu_2(s)\big|\,ds \\
&+ \frac{tb}{B\Gamma(\alpha-1)}\int_0^1(1-s)^{\alpha-2}\big|\nu_1(s)-\nu_2(s)\big|\,ds \\
\le{}& \left(\frac{L_1}{L_1+\sum_{j=1}^{n}L_2^j}\right)\psi\big(\|x-y\|\big),
\end{aligned}$$

and

$$\begin{aligned}
&\big|{}^cD^{\gamma_j}h_1(t)-{}^cD^{\gamma_j}h_2(t)\big| \\
&\quad\le \frac{1}{\Gamma(\alpha-\gamma_j)}\int_0^t(t-s)^{\alpha-\gamma_j-1}\big|\nu_1(s)-\nu_2(s)\big|\,ds + \frac{t^{1-\gamma_j}}{B\Gamma(2-\gamma_j)} \\
&\qquad\times \sum_{i=1}^{n}\int_0^{\eta}\frac{(\eta-s)^{\alpha-\gamma_i-1}}{\Gamma(\alpha-\gamma_i)}\big|\nu_1(s)-\nu_2(s)\big|\,ds \\
&\qquad+ \frac{bt^{1-\gamma_j}}{B\Gamma(2-\gamma_j)\Gamma(\alpha-1)}\int_0^1(1-s)^{\alpha-2}\big|\nu_1(s)-\nu_2(s)\big|\,ds \\
&\quad\le \left(\frac{L_2^j}{L_1+\sum_{j=1}^{n}\Lambda_2^j}\right)\psi\big(\|x-y\|\big)
\end{aligned}$$

for all $1 \le j \le n$. Hence,

$$\begin{aligned}\|h_1 - h_2\| &= \sup_{t\in J}\bigl|h_1(t) - h_2(t)\bigr| + \sup_{t\in J}\sum_{i=1}^{n}\bigl|{}^cD^{\gamma_i}h_1(t) - {}^cD^{\gamma_i}h_2(t)\bigr| \\ &\le \psi\bigl(\|x-y\|\bigr)\left(\frac{L_1}{L_1 + \sum_{j=1}^n L_2^j} + \sum_{i=1}^{n}\frac{L_2^i}{L_1 + \sum_{j=1}^n L_2^j}\right) = \psi\bigl(\|x-y\|\bigr).\end{aligned}$$

Analogously, interchanging the roles of x, y, we obtain $H_d(\Omega(x),\Omega(y)) \le \psi(\|x-y\|)$. Since the multifunction Ω has the approximate endpoint property, by using Lemma 3.2 there exists $x^* \in X$ such that $\Omega(x^*) = \{x^*\}$. □

4 Example

Here, we give an example to illustrate our first main result.

Example 4.1 Consider the fractional differential inclusion problem via the integral boundary conditions

$$\begin{cases} {}^cD^{\frac{5}{2}}x(t) \in [0, \frac{t}{1{,}000}\sin x(t) + \frac{1}{1{,}000}\cos x'(t) + \frac{1}{1{,}000}\frac{|x''(t)|}{1+|x''(t)|}], \\ x(0) + x(\frac{1}{2}) + x(1) = \int_0^1 \frac{s}{300}\sin x(s)\,ds, \\ {}^cD^{\frac{1}{2}}x(0) + {}^cD^{\frac{1}{2}}x(\frac{1}{2}) + {}^cD^{\frac{1}{2}}x(1) = \int_0^1 \frac{e^{s-1}}{300}\sin x(s)\,ds, \\ {}^cD^{\frac{3}{2}}x(0) + {}^cD^{\frac{3}{2}}x(\frac{3}{2}) + {}^cD^{\frac{3}{2}}x(1) = \int_0^1 \frac{2s+1}{286\pi}\sin x(s)\,ds, \end{cases}$$

where $t \in [0,1]$. Define the maps

$$\begin{aligned} &F\colon [0,1]\times\mathbb{R}^3 \to P(\mathbb{R}), \quad F(t,x,y,z) = \left[0, \frac{t}{1{,}000}\sin x + \frac{1}{1{,}000}\cos y + \frac{1}{1{,}000}\frac{|z|}{1+|z|}\right], \\ &g_0\colon [0,1]\times\mathbb{R} \to \mathbb{R}, \quad g_0(t,x) = \frac{t}{300}\sin x, \\ &g_1\colon [0,1]\times\mathbb{R} \to \mathbb{R}, \quad g_1(t,x) = \frac{e^{t-1}}{300}\sin x, \\ &g_2\colon [0,1]\times\mathbb{R} \to \mathbb{R}, \quad g_2(t,x) = \frac{2t+1}{286\pi}\sin x, \end{aligned}$$

and $N\colon C^2([0,1]) \to 2^{C^2([0,1])}$ by the rule

$$N(x) = \bigl\{h \in C^2\bigl([0,1]\bigr) : \text{ there exist } v \in S_{F,x} \text{ such that } h(t) = w(t) \text{ for all } t \in [0,1]\bigr\},$$

where

$$\begin{aligned} w(t) = {}&\frac{1}{\Gamma(\frac{5}{2})}\int_0^t (t-s)^{\frac{3}{2}}v(s)\,ds + \frac{1}{3}\int_0^1 \frac{s}{300}\sin x(s)\,ds \\ &- \frac{1}{3\Gamma(\frac{5}{2})}\left[\int_0^1 (1-s)^{\frac{3}{2}}v(s)\,ds + \int_0^{\frac{1}{2}}\left(\frac{1}{2}-s\right)^{\frac{3}{2}}v(s)\,ds\right] \\ &+ \frac{3\Gamma(\frac{3}{2})t - \frac{3}{2}\Gamma(\frac{3}{2})}{3(\frac{1}{2}^{\frac{1}{2}}+1)}\int_0^1 \frac{e^{s-1}}{300}\sin x(s)\,ds + \frac{\frac{3}{2}\Gamma(\frac{3}{2}) - 3\Gamma(\frac{3}{2})t}{3(\frac{1}{2}^{\frac{1}{2}}+1)} \end{aligned}$$

$$\times \left[\int_0^1 (1-s)\nu(s)\,ds + \int_0^{\frac{1}{2}} \left(\frac{1}{2}-s\right)\nu(s)\,ds \right]$$

$$+ \left(\frac{3(\frac{1}{2}^{\frac{5}{2}}+1)\Gamma(\frac{3}{2})^2 - \frac{5}{4}\Gamma(\frac{3}{2})(\frac{1}{2}^{\frac{1}{2}}+1)\Gamma(\frac{5}{2})}{6(\frac{1}{2}^{\frac{1}{2}}+1)^2\Gamma(\frac{5}{2})} \right.$$

$$\left. + \frac{-6(\frac{1}{2}^{\frac{3}{2}}+1)\Gamma(\frac{3}{2})^2 t + 3(\frac{1}{2}^{\frac{1}{2}}+1)\Gamma(\frac{3}{2})\Gamma(\frac{5}{2})t^2}{6(\frac{1}{2}^{\frac{1}{2}}+1)^2\Gamma(\frac{5}{2})} \right) \int_0^1 \frac{2s+1}{286\pi} \sin x(s)\,ds$$

$$+ \left(\frac{(\frac{5}{4})\Gamma(\frac{3}{2})(\frac{1}{2}^{\frac{1}{2}}+1)\Gamma(\frac{3}{2}) - 3(\frac{1}{2}^{\frac{3}{2}}+1)\Gamma(\frac{3}{2})\Gamma(\frac{3}{2})}{6(\frac{1}{2}^{\frac{1}{2}}+1)^2\Gamma(\frac{5}{2})} \right.$$

$$\left. + \frac{6(\frac{1}{2}^{\frac{3}{2}}+1)\Gamma(\frac{3}{2})\Gamma(\frac{3}{2})t - 3\Gamma(\frac{3}{2})\Gamma(\frac{5}{2})(\frac{1}{2}^{\frac{1}{2}}+1)t^2}{6(\frac{1}{2}^{\frac{1}{2}}+1)^2\Gamma(\frac{5}{2})} \right)$$

$$\times \left[\int_0^1 \nu(s)\,ds + \int_0^{\frac{1}{2}} \nu(s)\,ds \right].$$

Put $m(t) = \frac{3t}{1{,}000}$, $m_0(t) = \frac{t}{100}$, $m_1(t) = \frac{e^{t-1}}{100}$, $m_2(t) = \frac{2t+1}{100\pi}$, $\psi(t) = \frac{t}{3}$, $\alpha = \frac{5}{2}$, $\beta = \frac{1}{2}$, $\gamma = \frac{3}{2}$, $\eta = \frac{1}{2}$. Then we have $\Lambda_1 \cong 0.135$, $\Lambda_2 \cong 0.037$, and $\Lambda_3 \cong 0.017$.

It is easy to check that

$$H_d\big(F(t,x_1,x_2,x_3), F(t,y_1,y_2,y_3)\big) \le \frac{1}{\Lambda_1+\Lambda_2+\Lambda_3} m(t)\psi\big(|x_1-y_1|+|x_2-y_2|+|x_3-y_3|\big)$$

and $|g_j(t,x) - g_j(t,y)| \le \frac{1}{\Lambda_1+\Lambda_2+\Lambda_3} m_j(t)\psi(|x-y|)$ for all $t \in [0,1]$, $x,y,x_1,x_2,x_3,y_1,y_2,y_3 \in \mathbb{R}$ and $j = 0,1,2$. Note that $\sup_{x\in N(0)} \|x\| = 0$ and so $\inf_{x\in C^2([0,1])} \sup_{y\in N(x)} \|x-y\| = 0$. Hence, N has the approximate endpoint property. Now by using Theorem 3.1, the system of fractional differential inclusions has at least one solution.

5 Concluding remarks

This work contains our dedicated study to develop and improve methods for studying two fractional differential inclusions via integral boundary value conditions. We introduced our result by using an endpoint result for multifunctions, due to Amini-Harandi [36]. This study is motivated by relevant applications for solving many real-world problems which give rise to mathematical models in the sphere of boundary value problems.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to this work.

Author details

[1]Department of Mathematics, Tabriz Branch, Islamic Azad University, Tabriz, Iran. [2]Education District 5 Tabriz, Tabriz, Iran. [3]Department of Mathematics, Azarbaijan Shahid Madani University, Azarshahr, Tabriz, Iran. [4]Department of Mathematics, University Politehnica of Bucharest, 313 Splaiul Independenţei, Bucharest, 060042, Romania.

Acknowledgements

Research of the second and fourth authors was supported by Azarbaijan Shahid Madani University.

Received: 7 April 2014 Accepted: 31 July 2014 Published: 21 August 2014

References

1. Ahmad, B, Nieto, JJ: Existence of solutions for anti-periodic boundary value problems involving fractional differential equations via Leray-Schauder degree theory. Topol. Methods Nonlinear Anal. **35**, 295-304 (2010)
2. Bai, Z, Sun, W: Existence and multiplicity of positive solutions for singular fractional boundary value problems. Comput. Math. Appl. **63**, 1369-1381 (2012)
3. Baleanu, D, Agarwal, RP, Mohammadi, H, Rezapour, S: Some existence results for a nonlinear fractional differential equation on partially ordered Banach spaces. Bound. Value Probl. **2013**, Article ID 112 (2013)
4. Baleanu, D, Mohammadi, H, Rezapour, S: The existence of solutions for a nonlinear mixed problem of singular fractional differential equations. Adv. Differ. Equ. **2013**, Article ID 359 (2013)
5. Baleanu, D, Mohammadi, H, Rezapour, S: Positive solutions of a boundary value problem for nonlinear fractional differential equations. Abstr. Appl. Anal. **2012**, Article ID 837437 (2012)
6. Baleanu, D, Mohammadi, H, Rezapour, S: Some existence results on nonlinear fractional differential equations. Philos. Trans. R. Soc. A, Math. Phys. Eng. Sci. **371**, 20120144 (2013)
7. Baleanu, D, Mohammadi, H, Rezapour, S: On a nonlinear fractional differential equation on partially ordered metric spaces. Adv. Differ. Equ. **2013**, Article ID 83 (2013)
8. Baleanu, D, Nazemi, Z, Rezapour, S: The existence of positive solutions for a new coupled system of multi-term singular fractional integro-differential boundary value problems. Abstr. Appl. Anal. **2013**, Article ID 368659 (2013)
9. Baleanu, D, Nazemi, Z, Rezapour, S: Existence and uniqueness of solutions for multi-term nonlinear fractional integro-differential equations. Adv. Differ. Equ. **2013**, Article ID 368 (2013)
10. Kilbas, AA, Srivastava, HM, Trujillo, JJ: Theory and Applications of Fractional Differential Equations. North-Holland Mathematics Studies. Elsevier, Amsterdam (2006)
11. Miller, KS, Ross, B: An Introduction to the Fractional Calculus and Differential Equations. Wiley, New York (1993)
12. Mohammadi, H, Rezapour, S: Two existence results for nonlinear fractional differential equations by using fixed point theory on ordered Gauge spaces. J. Adv. Math. Stud. **6**(2), 154-158 (2013)
13. Podlubny, I: Fractional Differential Equations. Academic Press, New York (1999)
14. Samko, SG, Kilbas, AA, Marichev, OI: Fractional Integrals and Derivatives: Theory and Applications. Gordon & Breach, Yverdon (1993)
15. Su, X: Boundary value problem for a coupled system of nonlinear fractional differential equations. Appl. Math. Lett. **22**, 64-69 (2009)
16. Agarwal, RP, Ahmad, B: Existence theory for anti-periodic boundary value problems of fractional differential equations and inclusions. J. Appl. Math. Comput. **62**, 1200-1214 (2011)
17. Agarwal, RP, Belmekki, M, Benchohra, M: A survey on semilinear differential equations and inclusions involving Riemann-Liouville fractional derivative. Adv. Differ. Equ. **2009**, Article ID 981728 (2009)
18. Agarwal, RP, Benchohra, M, Hamani, S: A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions. Acta Appl. Math. **109**, 973-1033 (2010)
19. Ahmad, B, Ntouyas, SK: Boundary value problem for fractional differential inclusions with four-point integral boundary conditions. Surv. Math. Appl. **6**, 175-193 (2011)
20. Ahmad, B, Ntouyas, SK, Alsedi, A: On fractional differential inclusions with anti-periodic type integral boundary conditions. Bound. Value Probl. **2013**, Article ID 82 (2013)
21. Alsaedi, A, Ntouyas, SK, Ahmad, B: Existence of solutions for fractional differential inclusions with separated boundary conditions in Banach space. Abstr. Appl. Anal. **2013**, Article ID 869837 (2013)
22. Aubin, J, Ceuina, A: Differential Inclusions: Set-Valued Maps and Viability Theory. Springer, Berlin (1984)
23. Bragdi, M, Debbouche, A, Baleanu, D: Existence of solutions for fractional differential inclusions with separated boundary conditions in Banach space. Adv. Math. Phys. **2013**, Article ID 426061 (2013)
24. Benchohra, M, Hamidi, N: Fractional order differential inclusions on the Half-Lin. Surv. Math. Appl. **5**, 99-111 (2010)
25. Benchohra, M, Ntouyas, SK: On second order differential inclusions with periodic boundary conditions. Acta Math. Univ. Comen. **LXIX**(2), 173-181 (2000)
26. El-Sayed, AMA, Ibrahim, AG: Multivalued fractional differential equations. Appl. Math. Comput. **68**, 15-25 (1995)
27. Kisielewicz, M: Differential Inclusions and Optimal Control. Kluwer Academic, Dordrecht (1991)
28. Liu, X, Liu, Z: Existence result for fractional differential inclusions with multivalued term depending on lower-order derivative. Abstr. Appl. Anal. **2012**, Article ID 423796 (2012)
29. Nieto, JJ, Ouahab, A, Prakash, P: Extremal solutions and relaxation problems for fractional differential inclusions. Abstr. Appl. Anal. **2013**, Article ID 292643 (2013)
30. Phung, PD, Truong, LX: On a fractional differential inclusion with integral boundary conditions in Banach space. Fract. Calc. Appl. Anal. **16**(3), 538-558 (2013)
31. Ouahab, A: Some results for fractional boundary value problem of differential inclusions. Nonlinear Anal. **69**, 3877-3896 (2008)
32. Wang, J, Ibrahim, AG: Existence and controllability results for nonlocal fractional impulsive differential inclusions in Banach spaces. J. Funct. Spaces Appl. **2013**, Article ID 518306 (2013)
33. Heikkila, S, Lakshmikantam, V: Monotone Iterative Technique for Nonlinear Discontinuous Differential Equations. Dekker, New York (1994)
34. Heymans, N, Podlubny, I: Physical interpretation of initial conditions for fractional differential equations with Riemann-Liouville fractional derivatives. Rheol. Acta **45**(5), 765-772 (2006)
35. Lakshmikantham, V, Leela, S, Vasundhara Devi, J: Theory of Fractional Dynamic Systems. Cambridge Academic Publishers, Cambridge (2009)
36. Amini-Harandi, A: Endpoints of set-valued contractions in metric spaces. Nonlinear Anal. **72**, 132-134 (2010)
37. Deimling, K: Multi-Valued Differential Equations. de Gruyter, Berlin (1992)
38. Berinde, V, Pacurar, M: The role of the Pompeiu-Hausdorff metric in fixed point theory. Creative Math. Inform. **22**(2), 35-42 (2013)

doi:10.1186/1029-242X-2014-319
Cite this article as: Ghorbanian et al.: **On a fractional differential inclusion via a new integral boundary condition.** *Journal of Inequalities and Applications* 2014 **2014**:319.