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Strong convergence theorems for quasi-nonexpansive mappings and maximal monotone operators in Hilbert spaces

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Celebration of Professor SS Chang's 80th birthday

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Abstract
We present the strong convergence theorem for the iterative scheme for finding a common element of the fixed-point set of a quasi-nonexpansive mapping and the zero set of the sums of maximal monotone operators in Hilbert spaces. Our results extend and improve the recent results of Takahashi *et al.* (J. Optim. Theory Appl. 147:27-41, 2010) and Takahashi and Takahashi (Nonlinear Anal. 69:1025-1033, 2008).

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1 Introduction

Let H be a real Hilbert space and let K be a nonempty closed convex subset of H. Let $T : K \to K$ be a mapping. We denote by $\mathrm{Fix}(T)$ the fixed-point set of T, that is, $\mathrm{Fix}(T) = \{x \in K : Tx = x\}$. A mapping $T : K \to K$ is nonexpansive if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in K$. Approximation methods for fixed points of nonexpansive mappings have attracted considerable attention (see [1–5]). A mapping $T : K \to K$ is quasi-nonexpansive if $\mathrm{Fix}(T) \ne \emptyset$ and $\|Tx - y\| \le \|x - y\|$ for all $x \in K$ and $y \in \mathrm{Fix}(T)$. It is well known that the fixed-point set of a quasi-nonexpansive mapping is closed and convex (see [6, 7]). There are some quasi-nonexpansive mappings which are not nonexpansive (see [8–10]). For example, the level set of a continuous convex function is characterized as the fixed-point set of a nonlinear mapping called the subgradient projection, which is not nonexpansive but quasi-nonexpansive. Quasi-nonexpansive mappings have been discussed in the recent literature (see [9–11]).

We say that a mapping $T : K \to K$ is demiclosed at zero if for any sequence $\{x_n\} \subset K$ which converges weakly to x, the strong convergence of the sequence $\{Tx_n\}$ to zero implies $Tx = 0$. It is well known that $I - T$ is demiclosed whenever T is nonexpansive. In fact, this property is satisfied for more general mappings (see [12, 13]).

Let B be a mapping from H into 2^H. The effective domain of B is denoted by $\mathrm{dom}(B)$, namely, $\mathrm{dom}(B) = \{x \in H : Bx \ne \emptyset\}$. The graph of B is

$$\mathrm{Gra}(B) = \big\{(v, r) \in H \times H : r \in Bv\big\}.$$

A multi-valued mapping B is said to be monotone if

$$\langle x - y, u - v\rangle \geq 0 \quad \text{for all } x, y \in \operatorname{dom}(B), u \in Bx, \text{ and } v \in By.$$

A monotone operator B is said to be maximal if its graph is not properly contained in the graph of any other monotone operator. For a maximal monotone operator B on H and $r > 0$, we define a single-valued operator $J_{rB} = (I + rB)^{-1} : H \to \operatorname{dom}(B)$, which is called the resolvent of B for r. It is well known that J_{rB} is firmly nonexpansive, that is,

$$\langle x - y, J_{rB}x - J_{rB}y\rangle \geq \|J_{rB}x - J_{rB}y\|^2 \quad \text{for any } x, y \in H.$$

A basic problem for maximal monotone operator B is to

$$\text{find } x \in H \quad \text{such that} \quad 0 \in Bx. \tag{1.1}$$

The classical method for solving problem (1.1) is the proximal point algorithm which was first introduced by Martinet [14]. Rockafellar [15] obtained the weak convergence of the proximal point algorithm for maximal monotone operators. Güler [16] constructed a proximal point iteration that converges weakly but not strongly. Some researchers have devoted their work to modifications of the proximal point algorithm in order to obtain the strong convergence theorem (see [17, 18]). For a positive constant α, a mapping $A : K \to H$ is said to be α-inverse strongly monotone if

$$\langle x - y, Ax - Ay\rangle \geq \alpha\|Ax - Ay\|^2 \quad \text{for all } x, y \in K.$$

We write $(A + B)^{-1}0$ for the zero set of $A + B$, that is, $(A + B)^{-1}0 = \{x \in K : 0 \in (A + B)x\}$, where the mapping $A : C \to H$ is inverse strongly monotone and B is maximal monotone. It is well known that $(A + B)^{-1}0 = \operatorname{Fix}(J_{\lambda B}(I - \lambda A))$ for all $\lambda > 0$ (see [19]). Takahashi *et al.* [20] presented the following iterative sequence. Let $u \in K$, $x_1 = x \in K$ and let $\{x_n\}$ be a sequence generated by

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)J_{\lambda_n B}(x_n - \lambda_n Ax_n).$$

Under appropriate conditions they proved that the sequence $\{x_n\}$ converges strongly to a point $z_0 \in (A + B)^{-1}0$. Lin and Takahashi [21] introduced an iterative sequence that converges strongly to an element of $(A + B)^{-1}0 \cap F^{-1}0$, where F is another maximal monotone operator. Takahashi *et al.* [22] established an iterative scheme for finding a point of $(A + B)^{-1}0 \cap \operatorname{Fix}(T)$ as follows. Let $x_1 = x \in K$ and let $\{x_n\}$ be a sequence generated by

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)T\big[\alpha_n x + (1 - \alpha_n)J_{\lambda_n B}(x_n - \lambda_n Ax_n)\big],$$

where $T : K \to K$ is a nonexpansive mapping.

Motivated by the above results, especially by Chuang *et al.* [11] and Takahashi *et al.* [22], we obtain the strong convergence theorem for the iterative scheme for finding a common element of the fixed-point set of a quasi-nonexpansive mapping and the zero set of the sums of maximal monotone operators in Hilbert spaces. Our results extend and improve the recent results of [22] and [23].

The rest of this paper is organized as follows. Section 2 contains some important facts and tools. In Section 3, we introduce a new iterative scheme for finding a common element of the fixed-point set of a quasi-nonexpansive mapping and the zero set of the sums of maximal monotone operators, and we prove strong convergence theorem in Hilbert spaces.

2 Preliminaries

Throughout this paper, let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, and let K be a nonempty closed convex subset of H. Let $\mathbb{N}$ be the set of positive integers. We denote the strong convergence and the weak convergence of $\{x_n\}$ to x by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. For any $x \in H$, there exists a unique point $P_K x \in K$ such that

$$\|x - P_K x\| \leq \|x - y\|, \quad \forall y \in K.$$

P_K is called the metric projection of H onto K. Note that P_K is a nonexpansive mapping. For $x \in H$ and $z \in K$, we have

$$z = P_K x \quad \Longleftrightarrow \quad \langle x - z, y - z\rangle \leq 0 \quad \text{for every } y \in K. \tag{2.1}$$

Let f be a proper lower semicontinuous convex function of H into $(-\infty, +\infty]$. The subdifferential ∂f of f is defined as

$$\partial f(x) = \bigl\{z \in H : f(y) - f(x) \geq \langle z, y - x\rangle, \forall y \in H\bigr\} \tag{2.2}$$

for all $x \in H$. Rockafellar [24] claimed that ∂f is a maximal monotone operator. Let δ_K be the indicator function of K, *i.e.*,

$$\delta_K(x) = \begin{cases} 0, & x \in K, \\ +\infty, & x \notin K. \end{cases}$$

The subdifferential $\partial\delta_K$ of δ_K is a maximal monotone operator since δ_K is a proper lower semicontinuous convex function on H. The resolvent $J_{r\partial\delta_K}$ of $\partial\delta_K$ for r is P_K (see [21]).

Let $A : K \to H$ be a nonlinear mapping. The variational inequality problem is to find $\bar{x} \in K$ such that

$$\langle A\bar{x}, y - \bar{x}\rangle \geq 0 \quad \text{for every } y \in K. \tag{2.3}$$

The solution set of (2.3) is denoted by $VI(K,A)$. Some methods have been proposed to study the variational inequality problem (see [25–28] and the references therein). It is easy to see that $VI(K,A) = (A + \partial\delta_K)^{-1}0$, where A is an inverse strongly monotone mapping of K into H (for more details, see [21]).

We collect some useful lemmas.

Lemma 2.1 [29] *Let $A : K \to H$ be an α-inverse strongly monotone mapping. For all $x, y \in K$ and $\lambda > 0$, we have*

$$\bigl\|(I - \lambda A)x - (I - \lambda A)y\bigr\|^2 \leq \|x - y\|^2 + \lambda(\lambda - 2\alpha)\|Ax - Ay\|^2.$$

In particular, if $0 < \lambda \leq 2\alpha$, then $I - \lambda A$ is a nonexpansive mapping.

Lemma 2.2 [30] *Let $\{\Gamma_n\}$ be a sequence of real numbers that does not decrease at infinity in the sense that there exists a subsequence $\{\Gamma_{n_j}\}$ of $\{\Gamma_n\}$ such that*

$$\Gamma_{n_j} < \Gamma_{n_j+1} \quad \text{for all } j \in \mathbb{N}.$$

Define the sequence $\{\tau(n)\}_{n\geq n_0}$ of integers as follows:

$$\tau(n) = \max_k\{k \leq n : \Gamma_k < \Gamma_{k+1}\},$$

where $n_0 \in \mathbb{N}$ such that $\{k \leq n_0 : \Gamma_k < \Gamma_{k+1}\} \neq \emptyset$. Then, for all $n \geq n_0$, the following hold:

(1) $\tau(n) \leq \tau(n+1) \leq \cdots$ *and* $\tau(n) \to \infty$;

(2) $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ *and* $\Gamma_n \leq \Gamma_{\tau(n)+1}$.

Lemma 2.3 [22] *Let B be a maximal monotone operator on H. Then the following holds:*

$$\frac{s-t}{s}\langle J_{sB}x - J_{tB}x, J_{sB}x - x\rangle \geq \|J_{sB}x - J_{tB}x\|^2$$

for all $s, t > 0$ and $x \in H$.

The following lemma is an immediate consequence of the inner product on H.

Lemma 2.4 *For all $x, y \in H$, the inequality $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle$ holds.*

Lemma 2.5 [31] *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying $a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n\beta_n$, where*

(i) $\{\alpha_n\} \subset (0,1)$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $\limsup_{n\to\infty} \beta_n \leq 0$.

Then $\lim_{n\to\infty} a_n = 0$.

3 Strong convergence theorems

In this section, a new iterative scheme for finding a common element of the fixed-point set of a quasi-nonexpansive mapping and the zero set of the sums of maximal monotone operators is presented.

Theorem 3.1 *Let K be a nonempty closed convex subset of a real Hilbert space H. Let $A : K \to H$ and $C : K \to H$ be α-inverse strongly monotone and γ-inverse strongly monotone, respectively. Suppose that B and D are maximal monotone operators on H such that the domains of B and D are contained in K and that $T : K \to K$ is a quasi-nonexpansive mapping such that $I - T$ is demiclosed at zero. Assume that $\Omega := \mathrm{Fix}(T) \cap (A+B)^{-1}0 \cap (C+D)^{-1}0 \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $(0,1)$ and let $\{u_n\}$ be a sequence in K. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_1 \in K \quad \text{chosen arbitrarily,} \\ y_n = \alpha_n u_n + (1-\alpha_n)J_{\lambda_n B}(x_n - \lambda_n A x_n), \\ z_n = J_{\psi_n D}(y_n - \psi_n C y_n), \\ x_{n+1} = \beta_n x_n + (1-\beta_n)T z_n. \end{cases} \tag{3.1}$$

Suppose the following conditions are satisfied:

(c1) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$;

(c2) $\lim_{n\to\infty} u_n = u$;

(c3) $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$;

(c4) $0 < a \le \lambda_n \le b < 2\alpha$;

(c5) $0 < c \le \psi_n \le d < 2\gamma$.

Then the sequence $\{x_n\}$ *converges strongly to* $P_\Omega u$.

Proof Observe that the set Ω is closed and convex since $\mathrm{Fix}(T)$, $(A+B)^{-1}0$ and $(C+D)^{-1}0$ are closed and convex.

By Lemma 2.1, for any $p\in\Omega$, we have

$$\|z_n - p\| = \left\|J_{\psi_n D}(y_n - \psi_n Cy_n) - J_{\psi_n D}(p - \psi_n Cp)\right\| \le \|y_n - p\|$$

and

$$\begin{aligned}\|y_n - p\| &\le \alpha_n\|u_n - p\| + (1-\alpha_n)\left\|J_{\lambda_n B}(I - \lambda_n A)x_n - p\right\| \\ &\le \alpha_n\|u_n - p\| + (1-\alpha_n)\|x_n - p\|.\end{aligned}$$

It follows that

$$\begin{aligned}\|x_{n+1} - p\| &\le \beta_n\|x_n - p\| + (1-\beta_n)\|z_n - p\| \\ &\le \left[1 - \alpha_n(1-\beta_n)\right]\|x_n - p\| + \alpha_n(1-\beta_n)\|u_n - p\| \\ &\le \max\left\{\|x_n - p\|, \|u_n - p\|\right\}.\end{aligned}$$

The sequence $\{u_n\}$ is bounded due to condition (c2). Hence there exists a positive number L such that $\sup_n\{\|u_n - p\|\} \le L$. By a simple inductive process, we have

$$\|x_{n+1} - p\| \le \max\left\{\|x_1 - p\|, L\right\},$$

which shows that $\{x_n\}$ is bounded. So are $\{y_n\}$ and $\{z_n\}$.

Note that

$$2\langle x_{n+1} - x_n, x_n - p\rangle = \|x_{n+1} - p\|^2 - \|x_n - p\|^2 - \|x_{n+1} - x_n\|^2$$

and

$$x_{n+1} - x_n = (1-\beta_n)(Tz_n - x_n).$$

Thus we get

$$\begin{aligned}&\|x_{n+1} - p\|^2 - \|x_n - p\|^2 - \|x_{n+1} - x_n\|^2 \\ &\quad = 2\langle x_{n+1} - x_n, x_n - p\rangle \\ &\quad = (1-\beta_n)\left[\|Tz_n - p\|^2 - \|Tz_n - x_n\|^2 - \|x_n - p\|^2\right]\end{aligned}$$

$$\begin{aligned}
&\leq (1-\beta_n)\big[\|z_n-p\|^2-\|Tz_n-x_n\|^2-\|x_n-p\|^2\big]\\
&\leq \alpha_n(1-\beta_n)\|u_n-p\|^2-(1-\beta_n)\|Tz_n-x_n\|^2
\end{aligned}$$

and

$$\begin{aligned}
&\|x_{n+1}-p\|^2-\|x_n-p\|^2-(1-\beta_n)^2\|Tz_n-x_n\|^2\\
&\quad\leq \alpha_n(1-\beta_n)\|u_n-p\|^2-(1-\beta_n)\|Tz_n-x_n\|^2.
\end{aligned}$$

This implies that

$$(1-\beta_n)\beta_n\|Tz_n-x_n\|^2\leq \|x_n-p\|^2-\|x_{n+1}-p\|^2+\alpha_n(1-\beta_n)\|u_n-p\|^2. \tag{3.2}$$

Set $\Gamma_n=\|x_n-p_0\|^2$, where $p_0=P_\Omega u$. We divide the rest proof into two cases.

Case 1. Suppose that $\Gamma_{n+1}\leq\Gamma_n$ for all $n\in\mathbb{N}$. In this case, the limit $\lim_{n\to\infty}\Gamma_n$ exists and then $\lim_{n\to\infty}(\Gamma_{n+1}-\Gamma_n)=0$. We obtain

$$\lim_{n\to\infty}\|Tz_n-x_n\|=0, \tag{3.3}$$

which implies

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=\lim_{n\to\infty}(1-\beta_n)\|Tz_n-x_n\|=0. \tag{3.4}$$

Note that

$$\begin{aligned}
&\big\|J_{\lambda_n B}(x_n-\lambda_n Ax_n)-p_0\big\|^2\\
&\quad\leq \big\|(I-\lambda_n A)x_n-(I-\lambda_n A)p_0\big\|^2\\
&\quad\leq \|x_n-p_0\|^2+\lambda_n(\lambda_n-2\alpha)\|Ax_n-Ap_0\|^2.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\|x_{n+1}-p_0\|^2\\
&\quad\leq \beta_n\|x_n-p_0\|^2+(1-\beta_n)\big[\alpha_n\|u_n-p_0\|^2+(1-\alpha_n)\big\|J_{\lambda_n B}(x_n-\lambda_n Ax_n)-p_0\big\|^2\big]\\
&\quad\leq \beta_n\|x_n-p_0\|^2+\alpha_n(1-\beta_n)\|u_n-p_0\|^2\\
&\qquad+(1-\alpha_n)(1-\beta_n)\big[\|x_n-p_0\|^2+\lambda_n(\lambda_n-2\alpha)\|Ax_n-Ap_0\|^2\big]\\
&\quad\leq \|x_n-p_0\|^2+\alpha_n(1-\beta_n)\|u_n-p_0\|^2\\
&\qquad+(1-\alpha_n)(1-\beta_n)\lambda_n(\lambda_n-2\alpha)\|Ax_n-Ap_0\|^2,
\end{aligned}$$

which yields

$$\begin{aligned}
&(1-\alpha_n)(1-\beta_n)\lambda_n(2\alpha-\lambda_n)\|Ax_n-Ap_0\|^2\\
&\quad\leq \|x_n-p_0\|^2-\|x_{n+1}-p_0\|^2+\alpha_n(1-\beta_n)\|u_n-p_0\|^2.
\end{aligned}$$

Therefore we get

$$\lim_{n\to\infty} \|Ax_n - Ap_0\| = 0. \tag{3.5}$$

In a similar way, we have

$$\lim_{n\to\infty} \|Cy_n - Cp_0\| = 0. \tag{3.6}$$

Letting $h_n = J_{\lambda_n B}(I - \lambda_n A)x_n$, we have

$$\begin{aligned}
\|h_n - p_0\|^2 &= \big\|J_{\lambda_n B}(I - \lambda_n A)x_n - J_{\lambda_n B}(I - \lambda_n A)p_0\big\|^2 \\
&\le \big\langle (I - \lambda_n A)x_n - (I - \lambda_n A)p_0, h_n - p_0\big\rangle \\
&\le \frac{1}{2}\big[\|x_n - p_0\|^2 + \|h_n - p_0\|^2 - \big\|(x_n - h_n) - \lambda_n(Ax_n - Ap_0)\big\|^2\big],
\end{aligned}$$

from which one deduces that

$$\|h_n - p_0\|^2 \le \|x_n - p_0\|^2 - \big\|(x_n - h_n) - \lambda_n(Ax_n - Ap_0)\big\|^2.$$

Using (3.1), we see that

$$\begin{aligned}
&\|x_{n+1} - p_0\|^2 \\
&\quad\le \beta_n\|x_n - p_0\|^2 + (1 - \beta_n)\big[\alpha_n\|u_n - p_0\|^2 + (1 - \alpha_n)\|h_n - p_0\|^2\big] \\
&\quad\le \beta_n\|x_n - p_0\|^2 + \alpha_n(1 - \beta_n)\|u_n - p_0\|^2 \\
&\qquad + (1 - \alpha_n)(1 - \beta_n)\big(\|x_n - p_0\|^2 - \|x_n - h_n\|^2 + 2\lambda_n\langle Ax_n - Ap_0, x_n - h_n\rangle\big) \\
&\quad\le \|x_n - p_0\|^2 + \alpha_n(1 - \beta_n)\|u_n - p_0\|^2 \\
&\qquad - (1 - \alpha_n)(1 - \beta_n)\|x_n - h_n\|^2 + 2(1 - \alpha_n)(1 - \beta_n)\lambda_n\langle Ax_n - Ap_0, x_n - h_n\rangle.
\end{aligned}$$

Thus,

$$\begin{aligned}
&(1 - \alpha_n)(1 - \beta_n)\|x_n - h_n\|^2 \\
&\quad\le \|x_n - p_0\|^2 - \|x_{n+1} - p_0\|^2 + \alpha_n(1 - \beta_n)\|u_n - p_0\|^2 \\
&\qquad + 2(1 - \alpha_n)(1 - \beta_n)\lambda_n\langle Ax_n - Ap_0, x_n - h_n\rangle.
\end{aligned}$$

It follows from (3.5) that

$$\lim_{n\to\infty} \|x_n - h_n\| = 0. \tag{3.7}$$

This implies that

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{3.8}$$

By (3.6), a similar argument shows that

$$\lim_{n\to\infty} \|y_n - z_n\| = 0. \tag{3.9}$$

As

$$\|Tz_n - z_n\| \le \|Tz_n - x_n\| + \|x_n - y_n\| + \|y_n - z_n\|,$$

combining (3.3), (3.8) and (3.9) gives

$$\lim_{n\to\infty} \|Tz_n - z_n\| = 0. \tag{3.10}$$

Next we prove that

$$\limsup_{n\to\infty} \langle u - p_0, y_n - p_0\rangle \le 0. \tag{3.11}$$

To show this inequality, we choose a subsequence $\{y_{n_j}\}$ of $\{y_n\}$ such that

$$\limsup_{n\to\infty} \langle u - p_0, y_n - p_0\rangle = \lim_{j\to\infty} \langle u - p_0, y_{n_j} - p_0\rangle. \tag{3.12}$$

In view of the boundedness of $\{y_{n_j}\}$, without loss of generality, we assume that $y_{n_j} \rightharpoonup \omega$. Now we show that $\omega \in \Omega$. According to the fact that $\{y_n\}$ is contained in K and K is a closed convex set, one has $\omega \in \Omega$.

Note that the expressions (3.8) and (3.9) yield $x_{n_j} \rightharpoonup \omega$ and $z_{n_j} \rightharpoonup \omega$. By the fact that $I - T$ is demiclosed at zero, the expression (3.10) implies $\omega \in \operatorname{Fix}(T)$.

We prove that $\omega \in (A+B)^{-1}0$. Due to (c4), there is a subsequence $\{\lambda_{n_{j_i}}\}$ of $\{\lambda_{n_j}\}$ such that $\lambda_{n_{j_i}} \to \lambda_0 \in [a,b]$. Without loss of generality, we assume that $\lambda_{n_j} \to \lambda_0$. Observe that

$$\begin{aligned}
&\big\|x_{n_j} - J_{\lambda_0 B}(I - \lambda_0 A)x_{n_j}\big\| \\
&\quad\le \|x_{n_j} - y_{n_j}\| + \big\|y_{n_j} - \big[\alpha_{n_j}u_{n_j} + (1-\alpha_{n_j})J_{\lambda_0 B}(I-\lambda_0 A)x_{n_j}\big]\big\| \\
&\qquad + \big\|\alpha_{n_j}u_{n_j} + (1-\alpha_{n_j})J_{\lambda_0 B}(I-\lambda_0 A)x_{n_j} - J_{\lambda_0 B}(I-\lambda_0 A)x_{n_j}\big\| \\
&\quad\le \|x_{n_j} - y_{n_j}\| + (1-\alpha_{n_j})\big\|J_{\lambda_{n_j} B}(I-\lambda_{n_j} A)x_{n_j} - J_{\lambda_0 B}(I-\lambda_0 A)x_{n_j}\big\| \\
&\qquad + \alpha_{n_j}\big\|u_{n_j} - J_{\lambda_0 B}(I-\lambda_0 A)x_{n_j}\big\| \\
&\quad\le \|x_{n_j} - y_{n_j}\| + (1-\alpha_{n_j})\big[\big\|J_{\lambda_{n_j} B}(I-\lambda_{n_j} A)x_{n_j} - J_{\lambda_0 B}(I-\lambda_{n_j} A)x_{n_j}\big\| \\
&\qquad + \big\|J_{\lambda_0 B}(I-\lambda_{n_j} A)x_{n_j} - J_{\lambda_0 B}(I-\lambda_0 A)x_{n_j}\big\|\big] + \alpha_{n_j}\big\|u_{n_j} - J_{\lambda_0 B}(I-\lambda_0 A)x_{n_j}\big\| \\
&\quad\le \|x_{n_j} - y_{n_j}\| + (1-\alpha_{n_j})\bigg[\frac{|\lambda_0 - \lambda_{n_j}|}{\lambda_0}\big\|J_{\lambda_0 B}(I-\lambda_{n_j} A)x_{n_j} - (I-\lambda_{n_j} A)x_{n_j}\big\| \\
&\qquad + |\lambda_{n_j} - \lambda_0|\|Ax_{n_j}\|\bigg] + \alpha_{n_j}\big\|u_{n_j} - J_{\lambda_0 B}(I-\lambda_0 A)x_{n_j}\big\|.
\end{aligned}$$

Hence,

$$\lim_{j\to\infty} \big\|x_{n_j} - J_{\lambda_0 B}(I-\lambda_0 A)x_{n_j}\big\| = 0. \tag{3.13}$$

Since $J_{\lambda_0 B}(I-\lambda_0 A)$ is nonexpansive, the demiclosedness for a nonexpansive mapping implies that $\omega \in \operatorname{Fix}(J_{\lambda_0 B}(I-\lambda_0 A))$, that is, $\omega \in (A+B)^{-1}0$.

Note that

$$\begin{aligned}
&\big\|y_{n_i} - J_{\psi_0 D}(I - \psi_0 C)y_{n_i}\big\| \\
&\quad \le \|y_{n_i} - z_{n_i}\| + \big\|J_{\psi_{n_i} D}(I - \psi_{n_i} C)y_{n_i} - J_{\psi_0 D}(I - \psi_0 C)y_{n_i}\big\| \\
&\quad \le \|y_{n_i} - z_{n_i}\| + \big\|J_{\psi_{n_i} D}(I - \psi_{n_i} C)y_{n_i} - J_{\psi_{n_i} D}(I - \psi_0 C)y_{n_i}\big\| \\
&\qquad + \big\|J_{\psi_{n_i} D}(I - \psi_0 C)y_{n_i} - J_{\psi_0 D}(I - \psi_0 C)y_{n_i}\big\|.
\end{aligned}$$

Using a similar argument, we get $\omega \in (C + D)^{-1}0$. In fact, we have obtained $\omega \in \Omega$.

By (3.12) and (2.1), we have

$$\begin{aligned}
\limsup_{n\to\infty}\langle u - p_0, y_n - p_0\rangle &= \lim_{j\to\infty}\langle u - p_0, y_{n_j} - p_0\rangle \\
&= \langle u - p_0, \omega - p_0\rangle \\
&\le 0.
\end{aligned}$$

The inequality (3.11) is obtained.

Finally, we prove that $x_n \to p_0$. With the help of Lemma 2.4, we obtain

$$\begin{aligned}
&\|x_{n+1} - p_0\|^2 \\
&\quad \le \beta_n\|x_n - p_0\|^2 + (1 - \beta_n)\|y_n - p_0\|^2 \\
&\quad \le \beta_n\|x_n - p_0\|^2 + (1 - \beta_n)\big[(1 - \alpha_n)\|x_n - p_0\|^2 + 2\alpha_n\langle u_n - p_0, y_n - p_0\rangle\big] \\
&\quad \le \big[1 - \alpha_n(1 - \beta_n)\big]\|x_n - p_0\|^2 + 2\alpha_n(1 - \beta_n)\big(\langle u_n - u, y_n - p_0\rangle + \langle u - p_0, y_n - p_0\rangle\big).
\end{aligned}$$

It follows from (3.11) and Lemma 2.5 that $\{x_n\}$ converges strongly to p_0.

Case 2. Suppose that there exists a subsequence $\{\Gamma_{n_i}\}$ of $\{\Gamma_n\}$ such that

$$\Gamma_{n_i} < \Gamma_{n_i+1} \quad \text{for all } i \in \mathbb{N}.$$

We define $\tau : \mathbb{N} \to \mathbb{N}$ by

$$\tau(n) = \max\{k \le n : \Gamma_k < \Gamma_{k+1}\}.$$

Lemma 2.2 shows that $\Gamma_{\tau(n)} \le \Gamma_{\tau(n)+1}$. Therefore we have from (3.2)

$$\lim_{n\to\infty}\|Tz_{\tau(n)} - x_{\tau(n)}\| = 0 \tag{3.14}$$

and

$$\lim_{n\to\infty}\|x_{\tau(n)+1} - x_{\tau(n)}\| = 0. \tag{3.15}$$

As in the proof of Case 1, we obtain

$$\limsup_{n\to\infty}\langle u - p_0, y_{\tau(n)} - p_0\rangle \le 0. \tag{3.16}$$

Observe that

$$\begin{aligned}
&\|x_{\tau(n)+1} - p_0\|^2 \\
&\quad \le \big[1 - \alpha_{\tau(n)}(1 - \beta_{\tau(n)})\big]\|x_{\tau(n)} - p_0\|^2 \\
&\qquad + 2\alpha_{\tau(n)}(1 - \beta_{\tau(n)})\big(\langle u_{\tau(n)} - u, y_{\tau(n)} - p_0\rangle + \langle u - p_0, y_{\tau(n)} - p_0\rangle\big).
\end{aligned}$$

It follows that

$$\|x_{\tau(n)} - p_0\|^2 \le 2\big(\langle u_{\tau(n)} - u, y_{\tau(n)} - p_0\rangle + \langle u - p_0, y_{\tau(n)} - p_0\rangle\big), \tag{3.17}$$

which implies that

$$\limsup_{n\to\infty} \|x_{\tau(n)} - p_0\|^2 \le 0.$$

Thus we get

$$\lim_{n\to\infty} \|x_{\tau(n)} - p_0\| = 0. \tag{3.18}$$

It follows from (3.15) and (3.18) that

$$\lim_{n\to\infty} \|x_{\tau(n)+1} - p_0\| = 0. \tag{3.19}$$

Lemma 2.2 implies that

$$\lim_{n\to\infty} \|x_n - p_0\| = 0.$$

The proof is completed. □

The following result is a direct consequence of Theorem 3.1.

Corollary 3.2 *Let K be a nonempty closed convex subset of a real Hilbert space H. Let A be an α-inverse strongly monotone operator of K into H and let B be a maximal monotone operator on H such that the domain of B is contained in K. Let $T : K \to K$ be a quasi-nonexpansive mapping such that $I - T$ is demiclosed at zero. Assume that* $\mathrm{Fix}(T) \cap (A + B)^{-1}0 \neq \emptyset$. *Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in* $(0,1)$ *and let $\{u_n\}$ be a sequence in K. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_1 \in K \quad \text{chosen arbitrarily}, \\ y_n = \alpha_n u_n + (1 - \alpha_n) J_{\lambda_n B}(x_n - \lambda_n A x_n), \\ x_{n+1} = \beta_n x_n + (1 - \beta_n) T y_n. \end{cases} \tag{3.20}$$

If conditions (c1)-(c4) *are satisfied, then the sequence $\{x_n\}$ converges strongly to the element* $P_{\mathrm{Fix}(T)\cap(A+B)^{-1}0}u$.

Proof Letting $C = 0$ and $D = \partial\delta_K$ in Theorem 3.1, the desired result follows. □

Let us consider the variational inequality problem. Recall that the subdifferential $\partial\delta_K$ of δ_K is a maximal monotone operator and $VI(K,A) = (A + \partial\delta_K)^{-1}0$, where A is an inverse strongly monotone mapping. We obtain the following result.

Corollary 3.3 *Let K be a nonempty closed convex subset of a real Hilbert space H. Let A be an α-inverse strongly monotone operator of K into H and let $T : K \to K$ be a quasi-nonexpansive mapping such that $I - T$ is demiclosed at zero. Assume that* $\mathrm{Fix}(T) \cap VI(K,A) \neq \emptyset$. *Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in* $(0,1)$ *and let $\{u_n\}$ be a sequence in K. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_1 \in K \quad \text{chosen arbitrarily}, \\ y_n = \alpha_n u_n + (1-\alpha_n)P_K(x_n - \lambda_n A x_n), \\ x_{n+1} = \beta_n x_n + (1-\beta_n)Ty_n. \end{cases} \tag{3.21}$$

If conditions (c1)-(c4) *are satisfied, then the sequence $\{x_n\}$ converges strongly to the element $P_{\mathrm{Fix}(T)\cap VI(K,A)}u$.*

Proof Corollary 3.2 easily yields the desired result. □

Remark Corollaries 3.2 and 3.3 improve and extend Theorem 3.1 of Takahashi *et al.* [22] and Theorem 4.2 of Takahashi and Takahashi [23] in the following aspects, respectively.

(1) The nonexpansive mapping is extended to the quasi-nonexpansive mapping.
(2) The constant vector u is replaced by the variables u_n with $\lim_{n\to\infty} u_n = u$.
(3) The condition $\lim_{n\to\infty}(\lambda_{n+1} - \lambda_n) = 0$ is removed.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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