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On the existence and essential components of solution sets for systems of generalized quasi-variational relation problems

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Abstract
In this paper, we study the existence of a solution for a system of quasi-variational relation problems (in short, (SQVR)). Moreover, we discuss the existence of essentially connected components of the solution set for (SQVR). Then the obtained results are applied to systems of quasi-variational inclusions and to systems of weak vector quasi-equilibrium problems. The results presented in the paper improve and extend many results from the literature. Some examples are given to illustrate our results.
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1 Introduction and preliminaries

Variational relation problems were first introduced and studied by Luc in [1]. These problems include as special cases variational inclusion problems, vector equilibrium problems, vector variational inequality problems and vector optimization problems, *etc.* Later, the results of many authors had been extended and studied as regards the existence and stability of solutions in different models; see for example [2–11] and the references therein.

In 1950, Fort [12] first introduced the notion of essential fixed points of a continuous mapping from a compact metric space into itself and proved that any mapping can be approximately closed by a mapping whose fixed points are all essential. Later, Kinoshita [13] introduced the notion of essential components of the set of fixed points of a single-valued map and proved that there exists at least one essential component of the set of its fixed points. Recently, the essential components of the solution set have been studied for vector equilibrium problems [14, 15], vector variational inequality problems [16], *etc.* Very recently, Yang and Pu [17] introduced and studied the system of strong vector quasi-equilibrium problems (in short, (SSVQEP)) and also obtained the existence of essential components for these problems.

Motivated by the research works mentioned above, in this paper, we introduce the system of generalized quasi-variational relation problems. Then we establish some existence theorems of solution sets for this problem. Moreover, we also obtain an existence theorem for essentially connected components of the set of solutions for a system of generalized quasi-variational relation problems. These results are then applied to systems of quasi-variational inclusion problems and systems of weak vector quasi-equilibrium problems.

Now, we pass to our problem setting. Let $I = \{1, \dots, n\}$ be an index set. For each $i \in I$, let X_i and Y_i be two real locally convex Hausdorff topological vector spaces and K_i a nonempty convex compact subset of X_i. Denote

$$K_{\hat{i}} = \prod_{j\in I, j\neq i} K_j, \qquad K = \prod_{i\in I} K_i = K_i \times K_{\hat{i}}, \qquad X = \prod_{i\in I} X_i.$$

For each $x \in K$, we can write $x = (x_i, x_{\hat{i}})$. For each $i \in I$, let $S_i, T_i : K \to 2^{K_i}$ be set-valued mappings, and let $R_i(x_i, x_{\hat{i}}, y_i)$ be a relation linking $x_i \in K_i$, $x_{\hat{i}} \in K_{\hat{i}}$ and $y_i \in K_i$. We consider the following system of generalized quasi-variational relation problems (in short, (SQVR)).

(SQVR): Find $(\bar{x}_i, \bar{x}_{\hat{i}}) \in K$ such that, for each $i \in I$, $\bar{x}_i \in S_i(\bar{x})$ and

$$R_i(\bar{x}_i, \bar{x}_{\hat{i}}, y_i) \text{ holds}, \quad \forall y_i \in T_i(\bar{x}),$$

where $\bar{x}$ is a solution of (SQVR). We denote by $\Sigma(R)$ the solution set of (SQVR).

Next, we recall some basic definitions and some of their properties.

Let X, Y be two topological vector spaces; let A be a nonempty subset of X and $F : A \to 2^Y$ be a multifunction.

(i) F is said to be *lower semicontinuous* (*lsc*) *at* $x_0 \in A$ if $F(x_0) \cap U \neq \emptyset$ for some open set $U \subseteq Y$ implies the existence of a neighborhood N of x_0 such that $F(x) \cap U \neq \emptyset$, $\forall x \in N$.

(ii) F is said to be *upper semicontinuous* (*usc*) *at* $x_0 \in A$ if, for each open set $U \supseteq G(x_0)$, there is a neighborhood N of x_0 such that $U \supseteq F(x)$, $\forall x \in N$.

(iii) F is said to be *continuous at* $x_0 \in A$ if it is both lsc and usc at $x_0 \in A$.

(iv) F is said to be *closed* if $\mathrm{Graph}(F) = \{(x, y) : x \in A, y \in F(x)\}$ is a closed subset in $A \times Y$.

Let A, B, C be convex sets in topological vector spaces and $R(x, y, z)$ be a relation between elements of the three sets. The relation R is said to be *closed* if the set $\{(x, y, z) \in A \times B \times C : R(x, y, z) \text{ holds}\}$ is closed. The relation R is said to be *convex* in the first variable if whenever $R(x_1, y, z)$ holds and $R(x_2, y, z)$ holds, then $R(\lambda x_1 + (1 - \lambda)x_2, y, z)$ holds, for all $x_1, x_2 \in A$, $\lambda \in [0, 1]$.

Definition 1.1 ([18]) Let X, Y be two topological vector spaces, A is a nonempty subset of X, and $F : A \to 2^Y$ be a multifunction; and $C \subset Y$ is a nonempty, closed, and convex cone. F is called *upper C-continuous at* $x_0 \in A$, if, for any neighborhood U of the origin in Y, there is a neighborhood V of x_0 such that

$$F(x) \subset F(x_0) + U + C, \quad \forall x \in V.$$

Definition 1.2 ([18]) Let X and Y be two topological vector spaces and A be a nonempty convex subset of X. A set-valued mapping $F : A \to 2^Y$ is said to be *properly C-quasiconvex* if, for any $x, y \in A$ and $\lambda \in [0, 1]$, we have

$$\text{either} \quad F(x) \subset F\bigl(tx + (1 - t)y\bigr) + C$$

$$\text{or} \quad F(y) \subset F\bigl(tx + (1 - t)y\bigr) + C.$$

Lemma 1.1 ([18]) *Let X, Y be two Hausdorff topological vector spaces and $F : X \to 2^Y$ be a multivalued map.*

(i) *If F is upper semicontinuous with closed values, then F is closed.*

(ii) *If F is closed and Y is compact, then F is upper semicontinuous.*

Lemma 1.2 ([19]) *Let X, Y be two Hausdorff topological vector spaces and $F : X \to 2^Y$ be a set-valued mapping with compact values. Then F is upper semicontinuous $x_0 \in X$ if and only if, for each net $\{x_\alpha\} \subseteq X$ which converges to $x_0 \in X$ and for each net $\{y_\alpha\} \subseteq F(x_\alpha)$, there exist $y_0 \in F(x_0)$ and a subnet $\{y_\beta\}$ of $\{y_\alpha\}$ such that $y_\beta \to y_0$.*

Lemma 1.3 ([20]) *Let A be a nonempty compact convex subset of a Hausdorff topological vector space X. Suppose that $M : A \to 2^A \cup \{\emptyset\}$ be a set-valued map with the following conditions*:

(i) *for each at $x \in A$, $M(x)$ is convex*;

(ii) *for each at $x \in A$, $x \notin M(x)$*;

(iii) *for each at $y \in A$, $M^{-1}(y) = \{x \in A : y \in M(x)\}$ is open in A.*

Then there exists $x_0 \in A$ such that $M(x_0) = \emptyset$.

Lemma 1.4 (Kakutani-Fan-Glicksberg [21]) *Let A be a nonempty compact convex subset of a locally convex Hausdorff vector topological space X. If $F : A \to 2^A$ is upper semicontinuous and for any $x \in A$, $F(x)$ is nonempty, convex, and closed, then there exists an $x^* \in A$ such that $x^* \in F(x^*)$.*

2 Existence of solutions

In this section, we establish an existence theorem of solutions for system of generalized quasi-variational relation problems.

Theorem 2.1 *For each $i \in I$, assume that*

(i) *S_i is upper semicontinuous in K with nonempty compact convex values*;

(ii) *T_i is lower semicontinuous in K with nonempty convex values*;

(iii) *for all $(x_i, x_{\hat{i}}) \in K_i \times K_{\hat{i}}$, $R_i(x_i, x_{\hat{i}}, x_i)$ holds*;

(iv) *the set $\{(x_{\hat{i}}, y_i) \in K_{\hat{i}} \times K_i : R_i(\cdot, x_{\hat{i}}, y_i)$ does not hold$\}$ is convex in K_i*;

(v) *the relation R_i is convex in the first variable and closed.*

Then the (SQVR) *has a solution, i.e., there exists $(\bar{x}_i, \bar{x}_{\hat{i}}) \in K$ such that, for each $i \in I$, $\bar{x}_i \in S_i(\bar{x})$ and*

$$R_i(\bar{x}_i, \bar{x}_{\hat{i}}, y_i) \text{ holds}, \quad \forall y_i \in T_i(\bar{x}).$$

Moreover, the solution set of the (SQVR) *is closed.*

Proof We define a set-valued map: $\Phi : K \to 2^K$ by $\Phi(x) = \prod_{i \in I} \Phi_i(x)$, where

$$\Phi_i(x) = \{a_i \in S_i(x) : R_i(a_i, x_{\hat{i}}, y_i) \text{ holds}, \forall y_i \in T_i(x)\}, \quad \forall x \in K.$$

(The map Φ is called the best-reply map; see [14, 17].)

For each $i \in I$:

(I) For any $x \in K$, we show that $\Phi_i(x) \neq \emptyset$ is nonempty.

Indeed, for all $x \in K$, we define a set-valued map $M_i : S_i(x) \to 2^{S_i(x)} \cup \{\emptyset\}$ by

$$M_i(x) = \{y_i \in T_i(x) : R_i(x_i, x_{\hat{i}}, y_i) \text{ does not hold}\}, \quad \text{for each } x_i \in S_i(x).$$

(a) For each $x_i \in S_i(x)$, by condition (iv), $M_i(x)$ is a convex set.
(b) For each $x_i \in S_i(x)$, by condition (iii), $x_i \notin M_i(x_i)$.
(c) For each $y_i \in T_i(x)$, by condition (v), the set
$M_i^{-1}(y_i) = \{x_i \in S_i(x) : y_i \in M_i(x)\} = \{x_i \in S_i(x) : R_i(x_i, x_{\hat{i}}, y_i) \text{ does not hold}\}$ is open in $S_i(x)$.

By Lemma 1.3, there exists $\bar{x}_i \in S_i(x)$ such that $M_i(\bar{x}_i) = \emptyset$, *i.e.*, $\Phi_i(x) \neq \emptyset$.

(II) We show that $\Phi_i(x)$ is convex.

Let $a_i^1, a_i^2 \in \Phi_i(x)$ and $\lambda \in [0,1]$ and put $a_i = \lambda a_i^1 + (1-\lambda)a_i^2$. Since $a_i^1, a_i^2 \in S_i(x)$ and $S_i(x)$ is convex, we have $a_i \in S_i(x)$. Thus, for $a_i^1, a_i^2 \in \Phi_i(x)$, it follows that

$$R_i(a_i, x_{\hat{i}}, y_i) \text{ holds}, \quad \forall y_i \in T_i(x).$$

By (v), R_i is convex in the first variable, we have

$$R_i\left(\lambda a_i^1 + (1-\lambda)a_i^2, x_{\hat{i}}, y_i\right) \text{ holds}, \quad \forall \lambda \in [0,1], \forall y_i \in T_i(x),$$

i.e., $a_i \in \Phi_i(x)$. Therefore, $\Phi_i(x)$ is convex.

(III) We will prove that Φ_i is upper semicontinuous in K with nonempty compact values.

Since K is a compact set, it suffices to show that Φ_i is a closed mapping. Indeed, let $\{(a_i^\alpha, x^\alpha)\}_{\alpha \in \Lambda}$ be any a net in Graph(Φ_i) such that $(a_i^\alpha, x^\alpha) \to (a_i^0, x^0)$. Now, we need only prove that $a_i^0 \in \Phi_i(x^0)$. Since $a_i^\alpha \in S_i(x^\alpha)$ and S_i is upper semicontinuous at $x^0 \in K$ with nonempty compact values, we have S_i is closed at $x^0 \in K$, thus, $a_i^0 \in S_i(x^0)$. Suppose to the contrary $a_i^0 \notin \Phi_i(x^0)$. Then $\exists y_i^0 \in T_i(x^0)$ such that

$$R_i\left(a_i^0, x_{\hat{i}}, y_i^0\right) \text{ does not hold}. \tag{2.1}$$

By the lower semicontinuity of T_i, there is a net $\{y_i^\alpha\}$ with $y_i^\alpha \in T_i(x^\alpha)$ such that $y_i^\alpha \to y_i^0$. Since $a_i^\alpha \in \Phi_i(x^\alpha)$,

$$R_i\left(a_i^\alpha, x_{\hat{i}}^\alpha, y_i^\alpha\right) \text{ holds}. \tag{2.2}$$

By condition (v) and (2.2),

$$R\left(a_i^0, x_{\hat{i}}, y_i^0\right) \text{ holds}. \tag{2.3}$$

This is a contradiction between (2.1) and (2.3). Thus, $a_i^0 \in \Phi_i(x^0)$. Hence, Φ_i is upper semicontinuous in K with nonempty compact values.

By the definition of the mapping Φ is upper semicontinuous with nonempty compact values. By Lemma 1.4, there exists $(\bar{x}_i, \bar{x}_{\hat{i}}) \in K$ such that, for each $i \in I$, $\bar{x}_i \in S_i(\bar{x})$ and

$$R_i(\bar{x}_i, \bar{x}_{\hat{i}}, y_i) \text{ holds}, \quad \forall y_i \in T_i(\bar{x}).$$

(IV) Now we prove that $\Sigma(R)$ is closed.

Let a net $\{x_\alpha, \alpha \in I\} \in \Sigma(R)$: $x_\alpha \to x_0$. We need to prove that $x_0 \in \Sigma(R)$. Indeed, by the lower semicontinuity of T_i, for any $y_i^0 \in T_i(x^0)$, there exists $y_i^\alpha \in T_i(x^\alpha)$ such that $y_i^\alpha \to y_i^0$. As $x^\alpha \in \Sigma(R)$,

$$R_i\big(x_i^\alpha, x_{\hat{i}}^\alpha, y_i^\alpha\big) \text{ holds.}$$

Since S_i is upper semicontinuous with nonempty and closed values, by Lemma 1.1(i), we find that S_i is closed. Thus, $x^0 \in S_i(x^0)$. By condition (v),

$$R_i\big(x_i^0, x_{\hat{i}}^0, y_i^0\big) \text{ holds.}$$

This means that $x^0 \in \Sigma(R)$. Thus $\Sigma(R)$ is a closed set. □

If we let $S_i(x) = T_i(x) = K_i$ for each $i \in I$ and $x \in X$, then (SQVR) becomes the following system of variational relation problems (in short, (SVR)). So, we obtain following result.

Corollary 2.1 *For each $i \in I$, assume that*

(i) *for all $(x_i, x_{\hat{i}}) \in K_i \times K_{\hat{i}}$, $R_i(x_i, x_{\hat{i}}, x_i)$ holds*;
(ii) *the set $\{(x_{\hat{i}}, y_i) \in K_{\hat{i}} \times K_i : R_i(\cdot, x_{\hat{i}}, y_i)$ does not hold$\}$ is convex in K_i*;
(iii) *the relation R_i is convex in the first variable and closed.*

Then the (SVR) *has a solution, i.e., there exists $(\bar{x}_i, \bar{x}_{\hat{i}}) \in K$ such that, for each $i \in I$,*

$$R_i(\bar{x}_i, \bar{x}_{\hat{i}}, y_i) \textit{ holds}, \quad \forall y_i \in K_i.$$

Moreover, the solution set of the (SVR) *is closed.*

If I is a singleton, $S_i(x) = S(x)$, $T_i(x) = T(x)$, then (SQVR) reduces to the following quasi-variational relation problem (in short, (QVR)). So, we also obtain the following result.

Corollary 2.2 *Assume that*

(i) *S is upper semicontinuous in K with nonempty compact convex values*;
(ii) *T is lower semicontinuous in K with nonempty convex values*;
(iii) *for all $x \in K$, $R(x, x)$ holds*;
(iv) *the set $\{y \in K : R(\cdot, y)$ does not hold$\}$ is convex in K*;
(v) *the relation R is convex in the first variable and closed.*

Then the (QVR) *has a solution, i.e., there exists $\bar{x} \in K$ such that $\bar{x} \in S(\bar{x})$ and*

$$R(\bar{x}, y) \textit{ holds}, \quad \forall y \in T(\bar{x}).$$

Moreover, the solution set of the (QVR) *is closed.*

Remark 2.1

(i) For each $i \in I$, let X_i, Y_i and K_i, $K_{\hat{i}}$, K, $T_i = S_i$ be as in (SQVR). Let $F_i : K_i \times K_{\hat{i}} \times K_i \to 2^{Y_i}$ be a set-valued mapping, $C_i \subset Y_i$ be a nonempty, closed, and convex cone, and the relation R_i be defined as follows:

$$R_i(x_i, x_{\hat{i}}, y_i) \text{ holds} \quad \text{iff} \quad F_i(x_i, x_{\hat{i}}, y_i) \subset C_i.$$

Then (SQVR) becomes the system of strong vector quasi-equilibrium problem (in short, (SSVQEP)) studied in [17].

(ii) Let $S_i(x) = T_i(x) = K_i$ for each $i \in I$ and $x \in X$ and let F_i, C_i as in Remark 2.1(i). Then (SVR) becomes the system of strong vector equilibrium problems (in short, (SSVEP)) studied in [17].

(iii) If I is a singleton, $S_i(x) = S(x)$, $T_i(x) = T(x)$, and let $F : K \times K \to 2^Y$ be a set-valued mapping, $C \subset Y$ be a nonempty, closed, and convex cone, and the relation R be defined as follows:

$$R(x,y) \text{ holds} \quad \text{iff} \quad F(x,y) \subset C.$$

Then (QVR) becomes the strong vector quasi-equilibrium problem (in short, (SQVEP)) studied in [17].

(iv) Yang and Pu [17] obtained some existence results for the system of strong vector quasi-equilibrium problems. However, the assumptions of theorems in [17] are different from the assumptions of Theorem 2.1, Corollary 2.1, and Corollary 2.2.

(v) Corollary 2.2 is a particular case of Theorems 3.1 and 3.3 in [2]. However, the assumptions and proof methods in Theorems 3.1 and 3.3 in [2] are different from the assumptions and proof methods of Corollary 2.2.

In the next example all assumptions of Corollary 2.2 are satisfied, but Theorem 3.3 in [17] is not applicable. The reason is that F is neither upper C-continuous nor properly C-quasiconvex.

Example 2.1 Let $I = \{1\}$, $X_i = Y_i = \mathbb{R}$, $K_i = [0,1]$, $C = \mathbb{R}_+$, and let $S_i, T_i : K_i \times K_i \to 2^{K_i}$ and $F : K_i \times K_i \to 2^{Y_i}$ be defined by

$$S_i(x,y) = T_i(x,y) = [0,1],$$

$$F_i(x_i, x_{\hat{i}}, y_i) = F(x,y) = \begin{cases} [1,2] & \text{if } x_0 = \frac{1}{2}, \\ [\frac{1}{2},1] & \text{otherwise.} \end{cases}$$

We let the relation R_i be defined by $R_i(x_i, x_{\hat{i}}, y_i)$ holding iff $F_i(x_i, x_{\hat{i}}, y_i) = F(x,y) \subseteq \mathbb{R}_+$. We show that all assumptions of Corollary 2.2 are satisfied. However, F is neither upper C-continuous nor properly C-quasiconvex at $x_0 = \frac{1}{2}$.

Firstly, we prove that F is not upper C-continuous at $x_0 = \frac{1}{2}$. Indeed, we let a neighborhood $U = [-\frac{1}{6}, \frac{1}{6}]$ of the origin in Z, then for any neighborhood $V = [\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon]$ of $x_0 = \frac{1}{2}$, where $\varepsilon > 0$, we choose $\frac{1}{2} \neq x^* \in V$ and $y = \frac{1}{2}$. Then

$$\begin{aligned} F(x^*, y) = F\left(x^*, \frac{1}{2}\right) = \left[\frac{1}{2}, 1\right] &\not\subset F(x_0, y) + U + C \\ &= F\left(\frac{1}{2}, \frac{1}{2}\right) + \left[-\frac{1}{6}, \frac{1}{6}\right] + \mathbb{R}_+ \\ &= [1,2] + \left[-\frac{1}{6}, \frac{1}{6}\right] + \mathbb{R}_+ = \left[\frac{5}{6}, \frac{13}{6}\right] + \mathbb{R}_+. \end{aligned}$$

Next, we show that F is not properly C-quasiconvex at $x_0 = \frac{1}{2}$. Indeed, we let $y = \frac{1}{2}$, $\lambda = \frac{1}{2}$, and $x_1 = 0$, $x_2 = 1$. Then

$$\begin{aligned}
F(x_1, y) &= F\left(0, \frac{1}{2}\right) = \left[\frac{1}{2}, 1\right] \not\subset F\big(x_1\lambda + (1-\lambda)x_2, y\big) + C \\
&= F\left(\frac{1}{2}, \frac{1}{2}\right) + \mathbb{R}_+ = [1, 2] + \mathbb{R}_+, \\
F(x_2, y) &= F\left(1, \frac{1}{2}\right) = \left[\frac{1}{2}, 1\right] \not\subset F\big(x_1\lambda + (1-\lambda)x_2, y\big) + C \\
&= F\left(\frac{1}{2}, \frac{1}{2}\right) + \mathbb{R}_+ = [1, 2] + \mathbb{R}_+.
\end{aligned}$$

Thus, it gives also the case where Corollary 2.2 can be applied, but Theorem 3.3 in [17] does not work.

3 Essential components

In this section, we discuss the existence of essential components for (SQVR).

First, we recall some notions; see [17, 22, 23]. Let A be a nonempty and compact subset of a linear normed X. Denote by M the set of all upper semicontinuous maps $R : A \to 2^A$ with nonempty convex compact values. For any $R, P \in M$, we define $\xi(R, P) = \sup_{x \in A} H(R(x), P(x))$, where H is the Hausdorff metric defined in A. It is easy to verify that (M, ξ) is a metric space. For each $R \in M$, we denote by $\Xi(R)$ the set of all fixed points of R. By Kakutani-Fan-Glicksberg's fixed point theorem, $\Xi(R)$ is a nonempty compact set.

For each $R \in M$, the connected component of a point $x \in \Xi(R)$ is the union of all the connected subsets of $\Xi(R)$ containing x. Note that the connected components are connected closed subsets of $\Xi(R)$, and, since A is compact, thus, all the connected components are connected compact. It is easy to see that the connected components of two distinct points of $\Xi(R)$ either coincide or are disjoint, so that all connected components constitute a decomposition of $\Xi(R)$ into connected pairwise disjoint compact subsets, *i.e.*,

$$\Xi(R) = \bigcup_{\alpha \in \Lambda} \Xi_\alpha(R),$$

where Λ is an index set. For each $\alpha \in \Lambda$, $\Xi_\alpha(R)$ is a nonempty connected compact subset of $\Xi(R)$, and for any $\alpha, \beta \in \Lambda$ $(\alpha \neq \beta)$, $\Xi_\alpha(R) \cap \Xi_\beta(R) = \emptyset$.

Definition 3.1 ([22]) Let $R \in M$ and E be a nonempty and closed subset of $\Xi(R)$. E is said to be an essential set of $\Xi(R)$ if, for each open set $O \supset E$, there exists an open neighborhood U of R in M such that for any $R' \in U$ with $\Xi(R') \cap O \neq \emptyset$. If a connected component $\Xi_\alpha(R)$ of $\Xi(R)$ is an essential set with respect to M, then $\Xi_\alpha(R)$ is said be an essential component of $\Xi(R)$ with respect to M.

Lemma 3.1 ([23]) *For any $R \in M$, there is at least one essential component of $\Xi(R)$ with respect to M.*

Next, we discuss the existence of essential components for (SQVR).

Let Ω be the collection of all (SQVR) satisfying the conditions of Theorem 2.1. For each $\omega \in \Omega$, denote by $\Psi(\omega)$ the solution set of ω. It is easy to see that $\Psi(\omega) = \Xi(\Phi)$, where Φ is the best-reply map of ω. By the proof of Theorem 2.1, we know $\Phi \in M$.

Definition 3.2 Let $\omega \in \Omega$ and Ξ_α a connected component of $\Psi(\omega)$. Ξ_α is said to be essential if it, as a connected component of $\Xi(\Phi)$, is an essential component of $\Xi(\Phi)$ with respect to M, where Φ is the best-reply map of ω.

By Lemma 3.1, we obtain the following result.

Theorem 3.1 *For any $\omega \in \Omega$, there is at least one essential component of $\Psi(\omega)$.*

Remark 3.1

(i) In the special case of Remark 2.1, Theorem 3.1 improves and extends Theorems 4.1 and 4.2 in [17].

(ii) In 2011, Khanh and Quan [24] studied the existence of essential components for generalized KKM points. Moreover, Khanh and Quan also applied these results to optimization-related problems. However, the assumptions and proof methods are very different from the assumptions and proof methods in Theorem 3.1.

(iii) If I is a singleton, then (SQVR) becomes the variational relation problem (in short, (VR)) studied by Khanh and Luc in [3]. However, Khanh and Luc discussed some kinds of semicontinuous sets as outer-continuous, inner-continuous, inner-open, and outer-open solution sets for (VR), while our Theorem 3.1 discusses the existence of essential components for (SQVR) by using Kakutani-Fan-Glicksberg's fixed point theorem.

4 Applications (I): system of quasi-variational inclusion problems

For each $i \in I$, let X_i, Y_i and K_i, $K_{\hat{i}}$, K, $T_i = S_i$ be as in (SQVR). Let $F_i : K_i \times K_{\hat{i}} \times K_i \to 2^{Y_i}$ be a set-valued mapping. We consider the following system of quasi-variational inclusion problems (in short, (SQIP)): Find $(\bar{x}_i, \bar{x}_{\hat{i}}) \in K$ such that, for each $i \in I$, $\bar{x}_i \in S_i(\bar{x})$ and

$$0 \in F_i(\bar{x}_i, \bar{x}_{\hat{i}}, y_i), \quad \forall y_i \in S_i(\bar{x}),$$

where $\bar{x}$ is a solution of (SQIP).

Definition 4.1 Let X, Y, Z be topological vector spaces. Suppose $F : X \times Y \times X \to 2^Z$ is a multifunction. F is said to be *generalized* $\{0\}$*-quasiconvex* (in the first variable) in a convex set $A \subset X$, if, whenever $0 \in F(x_1, y, z)$ and $0 \in F(x_2, y, z)$, then $0 \in F(\lambda x_1 + (1 - \lambda)x_2, y, z)$, $\forall x_1, x_2 \in A$, $\forall \lambda \in [0, 1]$.

Theorem 4.1 *For each $i \in I$, assume that*

(i) *S_i is continuous in K with nonempty compact convex values*;

(ii) *for all $(x_i, x_{\hat{i}}) \in K_i \times K_{\hat{i}}$, $0 \in F_i(x_i, x_{\hat{i}}, x_i)$*;

(iii) *the set $\{(x_{\hat{i}}, y_i) \in K_{\hat{i}} \times K_i : 0 \notin F_i(\cdot, x_{\hat{i}}, y_i)\}$ is convex in K_i*;

(iv) *for all $(x_{\hat{i}}, y_i) \in K_{\hat{i}} \times K_i$, $F_i(\cdot, x_{\hat{i}}, y_i)$ is generalized $\{0\}$-quasiconvex in K_i*;

(v) *the set $\{(x_i, x_{\hat{i}}, y_i) \in K_i \times K_{\hat{i}} \times K_i : 0 \in F_i(x_i, x_{\hat{i}}, y_i)\}$ is closed.*

Then the (SQIP) *has a solution, i.e., there exists* $(\bar{x}_i, \bar{x}_{\hat{i}}) \in K$ *such that, for each* $i \in I$, $\bar{x}_i \in S_i(\bar{x})$ *and*

$$0 \in F_i(\bar{x}_i, \bar{x}_{\hat{i}}, y_i), \quad \forall y_i \in S_i(\bar{x}).$$

Moreover, the solution set of the (SQIP) *is closed.*

Proof Let the relation R_i be defined as follows:

$$R_i(x_i, x_{\hat{i}}, y_i) \text{ holds} \quad \text{iff} \quad 0 \in F_i(x_i, x_{\hat{i}}, y_i).$$

Then the problem (SQIP) becomes a particular case of (SQVR) and Theorem 4.1 is a direct consequence of Theorem 2.1. □

Next, we discuss the existence of essential components for (SQIP).

Let Θ be the collection of all (SQIP) satisfying the conditions of Theorem 4.1. For each $\theta \in \Theta$, denote by $\Delta(\theta)$ the solution set of θ. It is easy to see that $\Delta(\theta) = \Xi(\Phi)$, where Φ is the best-reply map of θ. By the proof of Theorem 4.1, we know $\Phi \in M$.

Definition 4.2 Let $\theta \in \Theta$ and Ξ_α be a connected component of $\Delta(\theta)$. Ξ_α is said to be *essential* if it, as a connected component of $\Xi(\Phi)$, is an essential component of $\Xi(\Phi)$ with respect to M, where Φ is the best-reply map of θ.

By Lemma 3.1, we also obtain the following result.

Theorem 4.2 *For any* $\theta \in \Theta$, *there is at least one essential component of* $\Delta(\theta)$.

5 Applications (II): system of weak vector quasi-equilibrium problems

For each $i \in I$, let X_i, Y_i and $K_i, K_{\hat{i}}, K$, $T_i = S_i$ be as in (SQVR). Let $f_i : K_i \times K_{\hat{i}} \times K_i \to Y_i$ be a vector function and $C_i \subset Y_i$ be a nonempty, closed, and convex cone. We consider the following weak system of quasi-equilibrium problems (in short, (WSQVEP)).

(WSQVEP): Find $(\bar{x}_i, \bar{x}_{\hat{i}}) \in K$ such that, for each $i \in I$, $\bar{x}_i \in S_i(\bar{x})$ and

$$f_i(\bar{x}_i, \bar{x}_{\hat{i}}, y_i) \notin -\operatorname{int} C_i, \quad \forall y_i \in S_i(\bar{x}).$$

Definition 5.1 Let X, Y, Z be topological vector spaces and $C \subset Z$ be a nonempty, closed, and convex cone. Suppose $f : X \times Y \times X \to Z$ is a vector function. f is said to be *weakly C-quasiconvex* (in the first variable) in a convex set $A \subset X$, if whenever $f(x_1, y, z) \notin -\operatorname{int} C$ and $f(x_2, y, z) \notin -\operatorname{int} C$, then $f(\lambda x_1 + (1-\lambda)x_2, y, z) \notin -\operatorname{int} C$, $\forall x_1, x_2 \in A$, $\forall \lambda \in [0,1]$.

Theorem 5.1 *For each* $i \in I$, *assume that*

(i) S_i *is continuous in* K *with nonempty compact convex values*;
(ii) *for all* $(x_i, x_{\hat{i}}) \in K_i \times K_{\hat{i}}$, $f_i(x_i, x_{\hat{i}}, x_i) \notin -\operatorname{int} C_i$;
(iii) *the set* $\{(x_{\hat{i}}, y_i) \in K_{\hat{i}} \times K_i : f_i(\cdot, x_{\hat{i}}, y_i) \in -\operatorname{int} C_i\}$ *is convex in* K_i;
(iv) *for all* $(x_{\hat{i}}, y_i) \in K_{\hat{i}} \times K_i$, $f_i(\cdot, x_{\hat{i}}, y_i)$ *is weakly* C_i*-quasiconvex in* K_i;
(v) *the set* $\{(x_i, x_{\hat{i}}, y_i) \in K_i \times K_{\hat{i}} \times K_i : f_i(x_i, x_{\hat{i}}, y_i) \notin -\operatorname{int} C_i\}$ *is closed.*

Then the (WSQVEP) *has a solution, i.e., there exists* $(\bar{x}_i, \bar{x}_{\hat{i}}) \in K$ *such that, for each* $i \in I$, $\bar{x}_i \in S_i(\bar{x})$ *and*

$$f_i(\bar{x}_i, \bar{x}_{\hat{i}}, y_i) \notin -\operatorname{int} C_i, \quad \forall y_i \in S_i(\bar{x}).$$

Moreover, the solution set of the (WSQVEP) *is closed.*

Proof Let the relation R_i be defined as follows:

$$R_i(x_i, x_{\hat{i}}, y_i) \text{ holds} \quad \text{iff} \quad f_i(x_i, x_{\hat{i}}, y_i) \notin -\operatorname{int} C_i.$$

Then the problem (WSQVEP) becomes a particular case of (SQVR) and Theorem 5.1 is a direct consequence of Theorem 2.1. □

Definition 5.2 ([14]) Let X and Z be two topological vector spaces and $A \subseteq X$ be nonempty convex set, and $C \subset Z$ be a nonempty, closed, and convex cone. Suppose $f : A \to Z$ is a vector function. f is called *C-continuous at* $x_0 \in A$ if, for any open neighborhood V of the zero element θ in Z, there exists an open neighborhood U of x_0 in A such that

$$f(x) \in f(x_0) + V + C, \quad \forall x \in U;$$

and C-continuous in A if it is C-continuous at every point of A.

Remark 5.1 Lin *et al.* [14] obtained some existence results of (WSVQEP). However, the assumptions of Theorems 3.1, 3.3, and 3.4 in [14] are different from the assumptions in Theorem 5.1. Example 5.1 shows that all the assumptions of Theorem 5.1 are satisfied. But Theorem 3.1 in [14] does not work. The reason is that f_i is not-C_i-continuous.

Example 5.1 Let $I = \{1\}$, $X_i = Y_i = Z_i = \mathbb{R}$, $K_i = [0,1]$, $C_i = \mathbb{R}_+$, and let $S_i : K_i \to 2^{K_i}$ and $f : K_i \to \mathbb{R}$ be defined by

$$S_i(x) = [0,1],$$

$$f(x,y) = f(x) = \begin{cases} [0, \frac{1}{2}] & \text{if } x_0 = \frac{1}{6}, \\ [1,2] & \text{otherwise.} \end{cases}$$

We show that all assumptions of Theorem 5.1 are satisfied. However, f is not-C-continuous at $x_0 = \frac{1}{6}$. Thus, it gives the case where Theorem 5.1 can be applied but Theorems 3.1, 3.3, and 3.4 in [14] do not work.

Remark 5.2 If we let $S_i(x) = K_i$ for each $i \in I$. Then (WSQVEP) becomes the system of vector equilibrium problem (in short, (SVEP)). If we let $I = \{1\}$, then (WSQVEP) becomes the strong vector equilibrium problem (in short, (VEP)). The problems (SVEP) and (VEP) are studied in [14].

Definition 5.3 ([14]) Let X and Z be two topological vector spaces and $A \subseteq X$ be a nonempty convex set, and $C \subset Z$ be a nonempty, closed, and convex cone. Suppose

$f : A \to Z$ is a vector function. f is called *C-quasiconvex-like* if, for any $x_i, x_2 \in A$ and each $\lambda \in [0,1]$, we have

$$\text{either} \quad f\bigl(\lambda x_1 + (1-\lambda)x_2\bigr) \in f(x_1) - C$$
$$\text{or} \quad f\bigl(\lambda x_1 + (1-\lambda)x_2\bigr) \in f(x_2) - C.$$

Example 5.2 shows that in the special cases of Remarks 5.1 and 5.2, all the assumptions of Theorem 5.1 are satisfied. But Theorems 3.1, 3.3, and 3.4 in [14] do not work. The reason is that f_i is not-C_i-quasiconvex-like.

Example 5.2 Let X_i, Y_i, Z_i, K_i, C_i be as in Example 5.1, and let $S_i : [0,1] \to 2^{K_i}$ and $f : [0,1] \to \mathbb{R}$ be defined by

$$S_i(x) = [0,1],$$
$$f(x) = \begin{cases} [\frac{1}{3}, \frac{1}{2}] & \text{if } x = \frac{1}{6}, \\ [2,5] & \text{otherwise.} \end{cases}$$

We show that all assumptions of Theorem 5.1 are satisfied. However, f is not-C-quasiconvex-like at $x_0 = \frac{1}{6}$. Indeed, let $\lambda = \frac{1}{2}$ and $x_1 = 0$, $x_2 = \frac{1}{3}$. Then

$$\begin{aligned} f\bigl(\lambda x_1 + (1-\lambda)x_2\bigr) &= f\left(\frac{1}{6}\right) = \left[\frac{1}{3}, \frac{1}{2}\right] \not\subset f(x_1) + C = f(0) + C \\ &= [2,5] + [0,+\infty), \\ f\bigl(\lambda x_1 + (1-\lambda)x_2\bigr) &= f\left(\frac{1}{6}\right) = \left[\frac{1}{3}, \frac{1}{2}\right] \not\subset f(x_2) + C = f\left(\frac{1}{3}\right) + C \\ &= [2,5] + [0,+\infty). \end{aligned}$$

Thus, Theorem 5.1 can be applied but Theorems 3.1, 3.3, and 3.4 in [14] do not work.

Theorem 5.2 *Assume for the problem* (WSQVEP) *assumptions* (i), (ii), (iii), *and* (iv) *are as in Theorem* 5.1 *and replace* (v) *by* (v′)

(v′) *f_i is continuous in $K_i \times K_{\hat{i}} \times K_i$.*

Then the (WSQVEP) *has a solution. Moreover, the solution set of the* (WSQVEP) *is closed.*

Proof We omit the proof since the technique is similar to that for Theorem 5.1 with suitable modifications. □

Next, we discuss the existence of essential components for (WSQVEP).

Let Υ be the collection of all (WSQVEP) satisfying the conditions of Theorem 5.1. For each $\gamma \in \Upsilon$, denote by $\Pi(\gamma)$ the solution set of γ. It is easy to see that $\Pi(\gamma) = \Xi(\Phi)$, where Φ is the best-reply map of γ. By the proof of Theorem 5.1, we know that $\Phi \in M$.

Definition 5.4 Let $\gamma \in \Upsilon$ and Ξ_α a connected component of $\Pi(\gamma)$. Ξ_α is said to be *essential* if it, as a connected component of $\Xi(\Phi)$, is an essential component of $\Xi(\Phi)$ with respect to M, where Φ is the best-reply map of γ.

By Lemma 3.1, we also obtain the following result.

Theorem 5.3 *For any $\gamma \in \Upsilon$, there is at least one essential component of $\Pi(\gamma)$.*

Remark 5.3 In the special case of Remarks 5.1 and 5.2, Theorem 5.3 improves and extends Theorems 4.1 and 4.4 in [14].

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The authors declare that they have no competing interests.

Authors' contributions
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