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A new iteration method for variational inequalities on the set of common fixed points for a finite family of quasi-pseudocontractions in Hilbert spaces

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Abstract
In this paper, we propose a new iteration method based on the hybrid steepest descent method and Ishikawa-type method for seeking a solution of a variational inequality involving a Lipschitz continuous and strongly monotone mapping on the set of common fixed points for a finite family of Lipschitz continuous and quasi-pseudocontractive mappings in a real Hilbert space.
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1 Introduction and preliminaries

Let C be a nonempty closed and convex subset of a real Hilbert space H with the inner product and induced norm $\|\cdot\|$. A mapping F of C into H is said to be monotone if

$$\langle Fx - Fy, x - y\rangle \geq 0 \tag{1.1}$$

for all $x, y \in C$.

The variational inequality problem with respect to F and C is to find a point $z \in C$ such that

$$\langle Fz, y - z\rangle \geq 0 \quad \text{for all } y \in C. \tag{1.2}$$

Variational inequalities were initially investigated by Kinderlehrer and Stampacchia in [1], and have been widely studied by many authors ever since, due to the fact that they cover as diverse disciplines as partial differential equations, optimization, optimal control, mathematical programming, mechanics and finance (see [1–3]).

We know that if F is a k-Lipschitz continuous and η-strongly monotone mapping, *i.e.*, F enjoys the following properties:

$$\|Fx - Fy\| \leq k\|x - y\| \quad \text{and} \quad \langle Fx - Fy, x - y\rangle \geq \eta\|x - y\|^2$$

for all $x, y \in C$, where k and η are fixed positive numbers, then (1.2) has a unique solution. It is well known that (1.2) is equivalent to the fixed point equation

$$z = P_C\big[(I - \mu F)z\big], \tag{1.3}$$

where P_C stands for the metric projection from H onto C and μ is an arbitrarily positive number. Consequently, the well-known iterative procedure, the projected gradient method (PGM) [3–6], can be used to solve (1.2). PGM generates an iterative sequence by the recursion

$$x_1 \in C \quad \text{and} \quad x_{n+1} = P_C\big[(I - \mu F)x_n\big], \quad n \geq 1. \tag{1.4}$$

When F is a k-Lipschitz continuous and η-strongly monotone mapping, as $\mu \in (0, \frac{2\eta}{k^2})$, the sequence $\{x_n\}$ generated by (1.4) converges strongly to a unique solution of (1.2).

The projected gradient method (1.4) involves the metric projection P_C. In order to reduce the complexity caused by P_C, Yamada [7] introduced a hybrid steepest descent method (HSDM) for solving (1.2). By assuming that $C = \bigcap_{i=1}^N \operatorname{Fix}(T_i) \neq \emptyset$, where $\operatorname{Fix}(T_i) = \{x \in H : x = T_i x\}$ and T_i is a nonexpansive mapping on H, *i.e.*,

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in H$, Yamada proposed the following iterative algorithm:

$$x_1 \in H, \quad x_{n+1} = (I - \lambda_n \mu F)T_{[n]}x_n, \tag{1.5}$$

where $T_{[n]} = T_{n \bmod N}$, taking values in $\{1, 2, \dots, N\}$, $\mu \in (0, \frac{2\eta}{k^2})$ and $\{\lambda_n\}$ is a sequence of real numbers in $(0, 1)$, and proved that, under the following conditions:

(L1) $\lim_{n\to\infty} \lambda_n = 0$,
(L2) $\sum_{n=1}^\infty \lambda_n = \infty$,
(L3) $\sum_{n=1}^\infty |\lambda_n - \lambda_{n+N}| < \infty$, and
(L4) $C = \operatorname{Fix}(T_1 T_2 \cdots T_{N-1} T_N) = \operatorname{Fix}(T_N T_1 \cdots T_{N-2} T_{N-1}) = \cdots = \operatorname{Fix}(T_2 T_3 \cdots T_N T_1)$,

the sequence $\{x_n\}$ generated by (1.5) converges strongly to a unique solution of (1.2). The algorithms and convergence results of Yamada in [7] have been improved and extended to a finite or an infinite family of nonexpansive mappings; see, for example, Xu and Kim [8], Zeng [9], Liu and Cai [10], and Iemoto and Takahashi [11]. However, all such improvements and extensions are confined to a finite or an infinite family of nonexpansive mappings.

In this paper, we propose a new iterative algorithm based on a combination of the projected gradient method for variational inequalities with the Ishikawa-type method for fixed point problems to solve (1.2) with $C = \bigcap_{i=1}^N \operatorname{Fix}(T_i)$, where $\{T_i\}_{i=1}^N$ is a finite family of L_i-Lipschitz continuous and quasi-pseudocontractive mappings on Ω, where Ω is a nonempty closed and convex subset of H, while $F : \Omega \to H$ is a k-Lipschitz continuous and η-strongly monotone mapping.

Given a stating point $x_1 \in \Omega$, the iteration is generated by

$$\begin{cases} x_1 \in \Omega, \\ y_{ni} = (1 - \alpha_{ni})x_n + \alpha_{ni} T_i x_n, \quad i = 1, 2, \dots, N, \\ x_{n+1} = P_\Omega[(I - \lambda_n \mu F)(\beta_{n0} x_n + \sum_{i=1}^N \beta_{ni} T_i y_{ni})], \quad n \geq 1, \end{cases} \tag{1.6}$$

where $\{\beta_{ni} : i = 0,1,2,\dots,N\} \subset (a,b) \subset (0,1)$, $\{\lambda_n\} \subset (0,1)$ satisfy the following conditions:

(i) $\sum_{i=0}^{N} \beta_{ni} = 1$;

(ii) $\lim_{n\to\infty} \lambda_n = 0$, $\sum_{n=1}^{\infty} \lambda_n = \infty$;

(iii) $\sum_{i=1}^{N} \beta_{ni} \le \min\{\alpha_{ni} : i = 1,2,\dots,N\} \le \max\{\alpha_{ni} : i = 1,2\} \le \alpha < \frac{1}{\sqrt{1+L^2}+1}$, $\forall n \ge 1$,

where $L := \max\{L_i : i = 1,2,\dots,N\}$, while μ is a fixed constant satisfying $\mu \in (0, \frac{2\eta}{k^2})$.

By virtue of new analysis techniques, we prove that the sequence generated by (1.6) converges strongly to a unique solution of (1.2) with $C = \bigcap_{i=1}^{N} \mathrm{Fix}(T_i)$.

In order to reach our goal, we need the following conceptions and facts.

Let D be a nonempty subset of a real Hilbert space H. A mapping $T : D \to H$ is called κ-strictly pseudocontractive if and only if there exists a constant $\kappa \in [0,1)$ such that

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + \kappa \big\|(I - T)x - (I - T)y\big\|^2 \tag{1.7}$$

for all $x, y \in D$. When $\kappa = 1$, T is said to be pseudocontractive.

T is said to be quasi-pseudocontractive if and only if $\mathrm{Fix}(T) \neq \emptyset$ and

$$\|Tx - y\|^2 \le \|x - y\|^2 + \big\|(I - T)x\big\|^2 \tag{1.8}$$

for all $x \in D$ but $y \in \mathrm{Fix}(T)$.

We remark that inequalities (1.7) and (1.8) are equivalent to the inequalities

$$\langle Tx - Ty, x - y\rangle \le \|x - y\|^2 - \frac{1-\kappa}{2} \big\|(I - T)x - (I - T)y\big\|^2 \quad \text{for all } x, y \in D \tag{1.9}$$

and

$$\langle Tx - y, x - y\rangle \le \|x - y\|^2 \tag{1.10}$$

for all $x \in D$ but $y \in \mathrm{Fix}(T)$, respectively.

We note that if T is κ-strictly pseudocontractive, then it is Lipschitz continuous and pseudocontractive; if T is a pseudocontraction with a fixed point, then T is a quasi-pseudocontraction; however, the converse may be not true.

Recall that the metric (nearest point) projection from H onto a nonempty closed convex subset E of H is defined as follows: for each point $x \in H$, there exists a unique point $P_E x \in E$ with the property

$$\|x - P_E x\| \le \|x - y\| \quad \text{for all } y \in E,$$

that is, for any point $x \in H$, $\bar{x} = P_E x$ if and only if $\bar{x} \in E$ and $\|x - \bar{x}\| = \inf\{\|x - y\| : y \in E\}$.

Lemma 1.1 [12, 13] *Let $P_E : H \to E$ be a metric projection from H on a nonempty closed convex subset E of H. Then the following conclusions hold true*:

(p_1) *Given $x \in H$ and $z \in E$. Then $z = P_E x$ if and only if there holds the inequality*

$$\langle x - z, y - z\rangle \le 0, \quad \forall y \in E. \tag{1.11}$$

(p_2)

$$\langle P_E x - P_E y, x - y\rangle \geq \|P_E x - P_E y\|^2, \quad \forall x, y \in H, \tag{1.12}$$

in particular, one has

$$\|P_E x - P_E y\| \leq \|x - y\|, \quad \forall x, y \in H. \tag{1.13}$$

Lemma 1.2 [14, 15]

(i) $\|x \pm y\|^2 = \|x\|^2 \pm 2\langle x, y\rangle + \|y\|^2$ *for all* $x, y \in H$;

(ii) $\|(1-t)x + ty\|^2 = (1-t)\|x\|^2 + t\|y\|^2 - t(1-t)\|x - y\|^2$ *for all* $x, y \in H$ *and* $t \in \mathbb{R}$;

(iii) *for all* $x_i \in H$ *and* $\alpha_i \in [0,1]$ $(i = 1, 2, \ldots, n)$ *such that* $\sum_{i=1}^n \alpha_i = 1$, *the following equality holds*:

$$\left\|\sum_{i=1}^n \alpha_i x_i\right\|^2 = \sum_{i=1}^n \alpha_i \|x_i\|^2 - \sum_{1 \leq i < j \leq n} \alpha_i \alpha_j \|x_i - x_j\|^2.$$

Lemma 1.3 [7] *Let* Ω *be a nonempty subset of* H *and* $F : \Omega \to H$ *be a* k*-Lipschitz continuous and* η*-strongly monotone mapping. For each* $\lambda \in (0,1]$ *and* $\mu \in (0, \frac{2\eta}{k^2})$, *write* $T^\lambda := (I - \lambda\mu F)$ *and* $\tau := 1 - \sqrt{1 - \mu(2\eta - \mu k^2)} \in (0,1)$. *Then we have*

$$\|T^\lambda x - T^\lambda y\| \leq (1 - \lambda\tau)\|x - y\|$$

for all $x, y \in \Omega$.

Lemma 1.4 [16] *Let* E *be a nonempty closed convex subset of a real Hilbert space* H *and* $T : E \to E$ *be* L*-Lipschitz continuous and quasi-pseudocontractive. Then* $\mathrm{Fix}(T)$ *is a nonempty closed convex subset of* E, *and therefore* $P_{\mathrm{Fix}(T)}x$ *is well defined for each* $x \in H$.

Lemma 1.5 [17] *Let* E *be a nonempty closed convex subset of a real Hilbert space* H *and* $T : E \to E$ *be a demicontinuous pseudocontraction from* E *into itself. Then* $\mathrm{Fix}(T)$ *is a closed convex subset of* E *and* $I - T$ *is demiclosed at zero.*

Lemma 1.6 [18] *Let* $\{a_n\}$ *be a sequence of real numbers such that there exists a subsequence* $\{n_i\}$ *of* $\{n\}$ *such that* $a_{n_i} < a_{n_i+1}$ *for all* $i \in \mathbb{N}$. *Then there exists a nondecreasing sequence* $\{m_k\} \subset \mathbb{N}$ *such that* $m_k \to \infty$ *and the following properties are fulfilled*:

$$a_{m_k} \leq a_{m_k+1} \quad \text{and} \quad a_k \leq a_{m_k+1}$$

for all sufficiently large numbers $k \in \mathbb{N}$.

Lemma 1.7 [19] *Let* $\{s_n\}$ *be a sequence of nonnegative real numbers satisfying the following relation*:

$$s_{n+1} \leq (1 - t_n)s_n + t_n\sigma_n, \quad n \geq n_0,$$

where $\{t_n\} \subset (0,1)$ *and* $\{\sigma_n\} \subset R$ *satisfy the following conditions*: $\lim_{n\to\infty} t_n = 0$, $\sum_{n=1}^\infty t_n = \infty$, *and* $\overline{\lim}_{n\to\infty}\sigma_n \leq 0$. *Then* $s_n \to 0$ *as* $n \to \infty$.

2 Main results

Theorem 2.1 *Let Ω be a nonempty, closed and convex subset of a real Hilbert space H. Let $T_1, T_2, \dots, T_N : \Omega \to \Omega$ be L_i-Lipschitz continuous and quasi-pseudocontractive with Lipschitz constants $L_1, L_2, \dots, L_N$, respectively. Let $F : \Omega \to H$ be a k-Lipschitz continuous and η-strongly monotone mapping. Assume that $\mathcal{F} = \bigcap_{i=1}^N \mathrm{Fix}(T_i) \neq \emptyset$ and $I - T_i$ are demi-closed at zero for $i = 1, 2, \dots, N$. Let $\{x_n\}$ be defined by* (1.6). *Then $\{x_n\}$ converges strongly to a unique solution x^* of* (1.2), *where $x^* = P_{\mathcal{F}}(I - \mu F)x^*$.*

Proof First of all, we show that $P_{\mathcal{F}}x$ is well defined for each $x \in H$. Indeed, in view of Lemma 1.4, we know that $\mathrm{Fix}(T_i)$ are closed convex for $i = 1, 2, \dots, N$, and hence $\mathcal{F}$ is also nonempty, closed and convex; consequently, $P_{\mathcal{F}}x$ is well defined for any $x \in H$. Secondly, we show that there exists a unique $x^* \in \mathcal{F}$ such that

$$x^* = P_{\mathcal{F}}(I - \mu F)x^*. \tag{2.1}$$

Indeed, in view of Lemma 1.3, we know that $I - \mu F : \Omega \to H$ is a contraction, and hence $P_{\mathcal{F}}(I - \mu F) : \Omega \to \Omega$ is also a contraction on Ω. Then we use the Banach contraction mapping principle to deduce (2.1).

Write $u_n = \beta_{n0}x_n + \sum_{i=1}^N \beta_{ni} T_i y_{ni}$. Then, $\forall p \in \mathcal{F}$, by virtue of Lemma 1.2, (1.6) and (1.8), we have that

$$\begin{aligned}
&\|y_{ni} - p\|^2 \\
&\quad = \big\|(1 - \alpha_{ni})(x_n - p) + \alpha_{ni}(T_i x_n - p)\big\|^2 \\
&\quad = (1 - \alpha_{ni})\|x_n - p\|^2 + \alpha_{ni}\|T_i x_n - p\|^2 - \alpha_{ni}(1 - \alpha_{ni})\|x_n - T_i x_n\|^2 \\
&\quad \leq (1 - \alpha_{ni})\|x_n - p\|^2 + \alpha_{ni}\big[\|x_n - p\|^2 + \|x_n - T_i x_n\|^2\big] \\
&\qquad - \alpha_{ni}(1 - \alpha_{ni})\|x_n - T_i x_n\|^2 \\
&\quad = \|x_n - p\|^2 + \alpha_{ni}^2\|x_n - T_i x_n\|^2
\end{aligned} \tag{2.2}$$

for $i = 1, 2, \dots, N$ and all $n \geq 1$.

Furthermore, from (1.6) and Lemma 1.2, we get that

$$\begin{aligned}
&\|y_{ni} - T_i y_{ni}\|^2 \\
&\quad = \big\|(1 - \alpha_{ni})(x_n - T_i y_{ni}) + \alpha_{ni}(T_i x_n - T_i y_{ni})\big\|^2 \\
&\quad = (1 - \alpha_{ni})\|x_n - T_i y_{ni}\|^2 + \alpha_{ni}\|T_i x_n - T_i y_{ni}\|^2 - \alpha_{ni}(1 - \alpha_{ni})\|x_n - T_i x_n\|^2 \\
&\quad \leq (1 - \alpha_{ni})\|x_n - T_i y_{ni}\|^2 + \alpha_{ni}L^2\|x_n - y_{ni}\|^2 - \alpha_{ni}(1 - \alpha_{ni})\|x_n - T_i x_n\|^2 \\
&\quad = (1 - \alpha_{ni})\|x_n - T_i y_{ni}\|^2 + \alpha_{ni}^3 L^2\|x_n - T_i x_n\|^2 - \alpha_{ni}(1 - \alpha_{ni})\|x_n - T_i x_n\|^2 \\
&\quad = (1 - \alpha_{ni})\|x_n - T_i y_{ni}\|^2 - \alpha_{ni}\big(1 - \alpha_{ni} - L^2\alpha_{ni}^2\big)\|x_n - T_i x_n\|^2
\end{aligned} \tag{2.3}$$

for $i = 1, 2, \dots, N$ and all $n \geq 1$.

At this point, we can estimate $\|u_n - p\|^2$. In fact, from Lemma 1.2, (2.2), (2.3) and conditions (i) and (iii) in (1.6), we have

$$\begin{aligned}
&\|u_n - p\|^2 \\
&\quad= \left\| \beta_{n0}(x_n - p) + \sum_{i=1}^{N} \beta_{ni}(T_i y_{ni} - p) \right\|^2 \\
&\quad= \beta_{n0}\|x_n - p\|^2 + \sum_{i=1}^{N} \beta_{ni}\left\|(T_i y_{ni} - p)\right\|^2 - \sum_{i=1}^{N} \beta_{n0}\beta_{ni}\|x_n - T_i y_{ni}\|^2 \\
&\qquad - \sum_{1\le i<j\le N} \beta_{ni}\beta_{n_j}\|T_i y_{ni} - T_j y_{n_j}\|^2 \\
&\quad\le \beta_{n0}\|x_n - p\|^2 + \sum_{i=1}^{N} \beta_{ni}\left[\|y_{ni} - p\|^2 + \|y_{ni} - T_i y_{ni}\|^2\right] \\
&\qquad - \sum_{i=1}^{N} \beta_{n0}\beta_{ni}\|x_n - T_i y_{ni}\|^2 - \sum_{1\le i<j\le N} \beta_{ni}\beta_{n_j}\|T_i y_{ni} - T_j y_{n_j}\|^2 \\
&\quad\le \beta_{n0}\|x_n - p\|^2 + \sum_{i=1}^{N} \beta_{ni}\|x_n - p\|^2 + \sum_{i=1}^{N} \beta_{ni}(1 - \alpha_{ni})\|x_n - T_i y_{ni}\|^2 \\
&\qquad - \sum_{i=1}^{N} \beta_{ni}\alpha_{ni}\left(1 - \alpha_{ni} - L^2\alpha_{ni}^2\right)\|x_n - T_i x_n\|^2 - \sum_{i=1}^{N} \beta_{n0}\beta_{ni}\|x_n - T_i y_{ni}\|^2 \\
&\quad= \|x_n - p\|^2 - \sum_{i=1}^{N} \beta_{ni}\alpha_{ni}\left(1 - \alpha_{ni} - L^2\alpha_{ni}^2\right)\|x_n - T_i x_n\|^2 \\
&\qquad + \sum_{i=1}^{N} \beta_{ni}(1 - \alpha_{ni} - \beta_{n0})\|x_n - T_i y_{ni}\|^2 \\
&\quad\le \|x_n - p\|^2 - \sum_{i=1}^{N} \beta_{ni}\alpha_{ni}\left(1 - \alpha_{ni} - L^2\alpha_{ni}^2\right)\|x_n - T_i x_n\|^2 \\
&\quad\le \|x_n - p\|^2 - \left(\sum_{i=1}^{N} \beta_{ni}\right)^2\left(1 - \alpha - L^2\alpha^2\right)\|x_n - T_i x_n\|^2 \\
&\quad\le \|x_n - p\|^2 - (Na)^2\left(1 - \alpha - L^2\alpha^2\right)\|x_n - T_i x_n\|^2
\end{aligned} \tag{2.4}$$

for all $i = 1, 2, \ldots, N$ and all $n \ge 1$.

Note that $1 - \alpha - L^2\alpha^2 > 0$, it follows from (2.4) that

$$\|u_n - p\| \le \|x_n - p\| \quad \text{for all } n \ge 1. \tag{2.5}$$

In particular, for $x^* = P_{\mathcal{F}}(I - \mu F)x^* \in \mathcal{F}$, we have

$$\left\|u_n - x^*\right\| \le \left\|x_n - x^*\right\| \quad \text{for all } n \ge 1. \tag{2.6}$$

From Lemmas 1.1, 1.3 and (2.6), we can prove that $\{x_n\}$ is bounded. Indeed, we have

$$\begin{aligned}
\|x_{n+1} - x^*\| &= \|P_\Omega[(I - \lambda_n\mu F)u_n] - x^*\| = \|P_\Omega[(I - \lambda_n\mu F)u_n] - P_\Omega x^*\| \\
&\le \|(I - \lambda_n\mu F)u_n - x^*\| \\
&= \|(I - \lambda_n\mu F)u_n - (I - \lambda_n\mu F)x^* - \lambda_n\mu F x^*\| \\
&= \|(I - \lambda_n\mu F)u_n - (I - \lambda_n\mu F)x^*\| + \lambda_n\mu\|Fx^*\| \\
&\le (1 - \tau\lambda_n)\|u_n - x^*\| + \mu\lambda_n\|Fx^*\| \\
&\le (1 - \tau\lambda_n)\|u_n - x^*\| + \tau\lambda_n\frac{\mu\|Fx^*\|}{\tau} \\
&\le (1 - \tau\lambda_n)\|x_n - x^*\| + \tau\lambda_n\frac{\mu\|Fx^*\|}{\tau} \\
&\le \max\left\{\|x_1 - x^*\|, \frac{\mu\|Fx^*\|}{\tau}\right\} := M
\end{aligned}$$

for all $n \ge 1$, and therefore $\{x_n\}$ is bounded; consequently, $\{y_{ni}\}$, $\{u_n\}$ and $\{Fu_n\}$ are all bounded.

We next show that $x_n \to x^*$ $(n \to \infty)$.

By virtue of Lemmas 1.1-1.3, (1.6) and (2.4), we have that

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|P_\Omega[(I - \lambda_n\mu F)u_n] - x^*\|^2 \le \|(I - \lambda_n\mu F)u_n - x^*\|^2 \\
&= \|(I - \lambda_n\mu F)u_n - (I - \lambda_n\mu F)x^* - \lambda_n\mu Fx^*\|^2 \\
&= \|(I - \lambda_n\mu F)u_n - (I - \lambda_n\mu F)x^*\|^2 + \lambda_n^2\mu^2\|Fx^*\|^2 \\
&\quad + 2\mu\lambda_n\langle Fx^*, (I - \lambda_n\mu F)u_n - (I - \lambda_n\mu F)x^*\rangle \\
&\le (1 - \tau\lambda_n)\|u_n - x^*\|^2 + \lambda_n^2\mu^2\|Fx^*\|^2 \\
&\quad + 2\mu\lambda_n\langle Fx^*, x^* - u_n\rangle + 2\mu^2\lambda_n^2\langle Fx^*, Fu_n - Fx^*\rangle \\
&= (1 - \tau\lambda_n)\left[\|x_n - x^*\|^2 - (Na)^2(1 - \alpha - L^2\alpha^2)\|x_n - T_i x_n\|^2\right] \\
&\quad + 2\mu\lambda_n\langle Fx^*, x^* - u_n\rangle + 2\mu^2\lambda_n^2\langle Fx^*, Fu_n\rangle - \lambda_n^2\mu^2\|Fx^*\|^2 \\
&\le (1 - \tau\lambda_n)\|x_n - x^*\|^2 - (1 - \tau\lambda_n)C_1\|x_n - T_i x_n\|^2 \\
&\quad + 2\mu\lambda_n\langle Fx^*, x^* - u_n\rangle + 2\mu^2\|Fx^*\|\lambda_n^2\|Fu_n\| \\
&\le (1 - \tau\lambda_n)\|x_n - x^*\|^2 - (1 - \tau\lambda_n)C_1\|x_n - T_i x_n\|^2 \\
&\quad + 2\mu\lambda_n\langle Fx^*, x^* - u_n\rangle + C_2\lambda_n^2
\end{aligned} \tag{2.7}$$

for $i = 1, 2, \ldots, N$ and all $n \ge 1$, where $C_1 = (Na)^2(1 - \alpha - L^2\alpha^2)$ and $C_2 = 2\mu^2\|Fx^*\| \times \sup\{\|Fu_n\| : n \ge 1\}$ are fixed positive constants.

Set $s_n = \|x_n - x^*\|^2$. Then (2.7) reduces to

$$s_{n+1} - s_n + \tau\lambda_n s_n + (1 - \tau\lambda_n)C_1\|x_n - T_i x_n\|^2 \le 2\mu\lambda_n\langle Fx^*, x^* - u_n\rangle + C_2\lambda_n^2 \tag{2.8}$$

for $i = 1, 2, \ldots, N$ and all $n \ge 1$.

Now we consider two possible cases.

Case 1. $\{s_n\}$ is decreasing eventually, that is, there exists some integer $n_0 \geq 1$ such that

$$s_{n+1} \leq s_n \quad \text{for all } n \geq n_0.$$

In this case, we have $\lim_{n\to\infty} s_n$ exists.

Taking the limit in (2.8), noting that $\lambda_n \to 0$ as $n \to \infty$, we get that

$$x_n - T_i x_n \to 0 \quad \text{as } n \to \infty \tag{2.9}$$

for $i = 1, 2, \ldots, N$. It follows from (1.6) that

$$x_n - y_{ni} \to 0 \quad \text{as } n \to \infty \tag{2.10}$$

for $i = 1, 2, \ldots, N$. Since T_i is L_i-Lipschitz continuous, we have that

$$\|T_i x_n - T_i y_{ni}\| \leq L_i \|x_n - y_{ni}\| \tag{2.11}$$

for $i = 1, 2, \ldots, N$. Consequently, from (2.9)∼(2.11) we get that

$$\begin{aligned}
\|u_n - x_n\| &= \left\| \sum_{i=1}^{N} \beta_{ni}(x_n - T_i y_{ni}) \right\| \\
&\leq \sum_{i=1}^{N} \beta_{ni} \|x_n - T_i y_{ni}\| \\
&\leq \sum_{i=1}^{N} \beta_{ni} \|x_n - T_i x_n\| + \sum_{i=1}^{N} \beta_{ni} \|T_i x_n - T_i y_{ni}\| \\
&\leq b \sum_{i=1}^{N} \|x_n - T_i x_n\| + Lb \sum_{i=1}^{N} \|x_n - y_{ni}\| \to 0
\end{aligned}$$

as $n \to \infty$, which derives that

$$\varlimsup_{n\to\infty} \langle Fx^*, x^* - u_n \rangle = \varlimsup_{n\to\infty} \langle Fx^*, x^* - x_n \rangle. \tag{2.12}$$

Assume that

$$\varlimsup_{n\to\infty} \langle Fx^*, x^* - x_n \rangle = \lim_{k\to\infty} \langle Fx^*, x^* - x_{nk} \rangle. \tag{2.13}$$

Without loss of generality, we can assume that $x_{nk} \to \hat{x}$ weakly as $k \to \infty$; then $\hat{x} = T_i \hat{x}$ for $i = 1, 2, \ldots, N$, by virtue of (2.9) and our assumption, and hence $\hat{x} \in \mathcal{F}$. It follows from (2.13) and (1.2) that

$$\varlimsup_{n\to\infty} \langle Fx^*, x^* - u_n \rangle = \langle Fx^*, x^* - \hat{x} \rangle \leq 0. \tag{2.14}$$

Set $t_n = \tau \lambda_n$ and $\sigma_n = \frac{2\mu}{\tau} \langle Fx^*, x^* - u_n \rangle + \frac{C_2}{\tau} \lambda_n$. Then (2.8) reduces to

$$s_{n+1} \leq (1 - t_n) s_n + t_n \sigma_n,$$

where $\varlimsup_{n\to\infty} \sigma_n \leq 0$. Now Lemma 1.7 can be used to deduce $s_n \to 0$ as $n \to \infty$.

Case 2. $\{s_n\}$ is not decreasing eventually, that is, there exists a subsequence $\{ni\}$ of $\{n\}$ such that $s_{ni} \le s_{ni+1}$ for all $i \in \mathbb{N}$. By virtue of Lemma 1.6, we know that there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$, $s_{m_k} \le s_{m_k+1}$ and $s_k \le s_{m_k+1}$ for all sufficiently large $k \in \mathbb{N}$. In this case, we have $s_{m_k+1} - s_{m_k} \ge 0$ for large enough $k \in \mathbb{N}$. It follows from (2.8) that

$$\lim_{k\to\infty}(s_{m_k+1} - s_{m_k}) = 0, \tag{2.15}$$

$$\lim_{k\to\infty}(x_{m_k} - T_i x_{m_k}) = 0 \quad \text{for } i = 1,2,\dots,N, \tag{2.16}$$

and

$$\overline{\lim_{k\to\infty}} s_{m_k} \le 2\mu \overline{\lim_{k\to\infty}} \langle Fx^*, x^* - u_{m_k}\rangle. \tag{2.17}$$

By using a reasoning similar to case 1, we can obtain that $\overline{\lim}_{k\to\infty}\langle Fx^*, x^* - u_{m_k}\rangle \le 0$, and hence $\overline{\lim}_{k\to\infty} s_{m_k} \le 0$ by (2.17), *i.e.*, $s_{m_k} \to 0$ as $k \to \infty$, which derives $s_{m_k+1} \to 0$ as $k \to \infty$; consequently, $s_k \to 0$ as $k \to \infty$, since $s_k \le s_{m_k+1}$ for sufficiently large $k \in \mathbb{N}$. This completes the proof. □

Remark 2.1 When $\Omega = H$, P_Ω in (1.6) can be dropped.

Corollary 2.1 *Let Ω be a nonempty, closed and convex subset of a real Hilbert space H. Let $T_1, T_2, \dots, T_N : \Omega \to \Omega$ be N L_i-Lipschitz continuous and strongly pseudocontractive with Lipschitz constants $L_1, L_2, \dots, L_N$, respectively. Let F, $\mathcal{F}$ and $\{x_n\}$ be the same as in Theorem* 2.1. *Then $\{x_n\}$ converges strongly to a unique solution x^* of* (1.2), *where $x^* = P_{\mathcal{F}}(I - \mu F)x^*$.*

Proof By virtue of Lemma 1.5, we know that $\mathrm{Fix}(T_i)$ are closed convex for $i = 1,2,\dots,N$ and hence $\mathcal{F} = \bigcap_{i=1}^N \mathrm{Fix}(T_i)$ is nonempty, closed and convex. Lemma 1.5 also ensures that $I - T_i$ are demiclosed at zero for $i = 1,2,\dots,N$ and hence the conclusion of Corollary 2.1 follows exactly from Theorem 2.1. □

Corollary 2.2 *Let Ω be a nonempty, closed and convex subset of a real Hilbert space H. Let $T_1, T_2, \dots, T_N : \Omega \to \Omega$ be N strict pseudocontractions, respectively. Let F, $\mathcal{F}$ and $\{x_n\}$ be the same as in Theorem* 2.1. *Then $\{x_n\}$ converges strongly to a unique solution x^* of* (1.2), *where $x^* = P_{\mathcal{F}}(I - \mu F)x^*$.*

Proof Since every strictly pseudocontractive mapping is Lipschitz continuous and pseudocontractive, we have the desired conclusion. □

Corollary 2.3 *Let Ω be a nonempty, closed and convex subset of a real Hilbert space H. Let $T_1, T_2, \dots, T_N : \Omega \to \Omega$ be N nonexpansive mappings, respectively. Let F, $\mathcal{F}$ and $\{x_n\}$ be the same as in Theorem* 2.1. *Then $\{x_n\}$ converges strongly to a unique solution x^* of* (1.2), *where $x^* = P_{\mathcal{F}}(I - \mu F)x^*$.*

Proof Since any nonexpansive mapping is 1-Lipschitz continuous and pseudocontractive, we have the desired conclusion by Corollary 2.2. □

Table 1 The results of the algorithm in [20]

n	0	500	1,000	5,000	10,000	12,000	14,000	17,000
x_1	2	1.8602	1.8441	1.8074	1.7919	1.7878	1.7843	1.7800
x_2	3	2.7902	2.7662	2.7112	2.6878	2.6817	2.6765	2.6700

Table 2 The results of algorithm (1.6)

n	0	500	1,000	5,000	10,000	12,000	14,000	17,000
x_1	2	1.4957	1.4490	1.3332	1.2878	1.2761	1.2663	1.2541
x_2	3	2.2435	2.1673	1.9998	1.9317	1.9142	1.8995	1.8812

Remark 2.2 When $F = I$, we have the following strong convergence theorem.

Corollary 2.4 *Let Ω be a nonempty, closed and convex subset of a real Hilbert space H. Let $T_1, T_2, \dots, T_N : \Omega \to \Omega$ be N L_i-Lipschitz continuous and quasi-pseudocontractive with Lipschitz constants $L_1, L_2, \dots, L_N$, respectively. Let $\mathcal{F}$ and $I - T_i$ be the same as in Theorem* 2.1. *Let $\{x_n\}$ be defined by* (1.6) *with $F = I$. Then $\{x_n\}$ converges strongly to the minimum-norm fixed point of the family $\{T_i\}_{i=1}^N$.*

3 Numerical example

Example 3.1 [20] Consider the following optimization problem: find an element

$$x^* \in C : \varphi(x^*) = \min_{x\in C} \varphi(x), \tag{3.1}$$

where $\varphi(x) = \frac{1}{2}\|x\|^2$, $x = (x_1, x_2) \in \mathbb{R}^2$, a Euclid space, and $C = C_1 \cap C_2$, defined by

$$C_1 = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 - 2x_2 + 1 \le 0\},$$
$$C_2 = \{(x_1, x_2) \in \mathbb{R}^2 : 4x_1 - x_2 - 3 \ge 0\}$$

(3.1) has a unique solution $x^* = (1,1)$ and $F = \nabla\varphi = I$ is 1-Lipschitz continuous and $\frac{1}{2}$-strongly monotone. Starting with the point $x^1 = (x_1, x_2) = (2,3)$, $\mu = \frac{1}{20} \in (0,1)$ and $\lambda_n = \frac{1}{n+1}$, set $\alpha_{ni} = \frac{1}{100} + \frac{1}{n+100}$, $\beta_{n_1} = \beta_{n_2} = \frac{1}{2}\alpha_{ni}$, $\beta_{n0} = 1 - \alpha_{ni}$ for $i = 1, 2$, Table 1 shows the results of algorithm in [20], we obtained the results of algorithm (1.6) in Table 2. Obviously, the results in Table 2 are better.

Example 3.2 [21] Let $H = \mathbb{R}$ with absolute value norm. Let $\Omega = [-2, 1]$ and $T_1, T_2 : \Omega \to \Omega$ be defined by

$$T_1 x = \begin{cases} x + x^2, & x \in [-2, 0], \\ x, & x \in (0, 1], \end{cases}$$

and

$$T_2 x := \begin{cases} x, & x \in [-2, \frac{1}{2}], \\ x - 2(x - \frac{1}{2})^2, & x \in (\frac{1}{2}, 1]. \end{cases}$$

Table 3 Values of $\{x_n\}$ with initial values $x_0 = -1$ and $x_0 = 0.8$

n	0	500	1,000	5,000	10,000	12,000	14,000	17,000
x_n	–1	–0.0739	–0.0447	–0.0127	–0.0071	–0.0060	–0.0053	–0.0045
x_n	0.8	0.1130	0.0803	0.0361	0.0255	0.0233	0.0216	0.0196

Then $\mathcal{F} = \mathrm{Fix}(T_1) \cap \mathrm{Fix}(T_2) = [0,1] \cap [-2, \frac{1}{2}] = [0, \frac{1}{2}]$, $T_1 : \Omega \to \Omega$ is 5-Lipschitz continuous and pseudocontractive and $T_2 : \Omega \to \Omega$ is 10-Lipschitz continuous and pseudocontractive. We find the point $x^* \in \mathcal{F}$ with the minimum-norm. To do so, set $F = I$.

Now, taking $\lambda_n = \frac{1}{n+10}$, $\alpha_{ni} = \frac{1}{100} + \frac{1}{n+100}$, $\beta_{n1} = \beta_{n2} = \frac{1}{2}\alpha_{ni}$, $\beta_{n0} = 1 - \alpha_{ni}$ for $i = 1, 2$, and $\mu = \frac{1}{2} \in (0, \frac{2\eta}{k^2}) = (0, 1)$, we see that the conditions of Corollary 2.1 are fulfilled and algorithm (1.6) provides the data in Table 3. Our result is better.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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