Journal of Inequalities and Applications
a SpringerOpen Journal

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Higher-order expansions for distributions of extremes from general error distribution

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Abstract
In this short note, with optimal normalizing constants, the higher-order expansion for a distribution of normalized partial maximum from the general error distribution is derived, by which one deduces the associate convergence rate of the distribution of the extreme to the Gumbel extreme value distribution.
MSC: Primary 62E20; 60E05; secondary 60F15; 60G15

Keywords: expansion; extreme; general error distribution

1 Introduction

Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed random variables with marginal cumulative distribution function (cdf) F_ν following the general error distribution ($F_\nu \sim \mathrm{GED}(\nu)$ for short), and let $M_n = \max_{1\leq k\leq n} X_k$ denote the partial maximum of $\{X_n, n \geq 1\}$. The probability density function (pdf) of GED(ν) is given by

$$f_\nu(x) = \frac{\nu \exp(-(1/2)|x/\lambda|^\nu)}{\lambda 2^{1+1/\nu}\Gamma(1/\nu)}, \quad x \in \mathbb{R},$$

where $\nu > 0$ is the shape parameter, $\lambda = [2^{-2/\nu}\Gamma(1/\nu)/\Gamma(3/\nu)]^{1/2}$ and $\Gamma(\cdot)$ denotes the Gamma function (Nelson [1]). Note that GED(2) reduces to the standard normal distribution.

Recently, several contributions investigated asymptotic behaviors of normalized maxima from the GED(ν). It is well known that the limiting distribution of extremes from the GED(2), *i.e.*, the normal distribution, is a Gumbel extreme value distribution, see Leadbetter *et al.* [2] and Resnick [3]. Peng *et al.* [4] established the Mills type ratio of GED(ν) and proved that there exist normalizing constants $a_n > 0$ and $b_n \in \mathbb{R}$ such that

$$\lim_{n\to\infty} \mathbb{P}(M_n \leq a_n x + b_n) = \lim_{n\to\infty} F_\nu^n(a_n x + b_n) = \Lambda(x) = \exp\bigl(-\exp(-x)\bigr), \quad x \in \mathbb{R},$$

i.e., F_ν is in the domain of attraction of Λ, which we denote by $F_\nu \in D(\Lambda)$. For the uniform convergence rate of normalized maxima from the GED(ν), Hall [5] established the optimal uniform convergence rate as $\nu = 2$, *i.e.*, the normal case; Peng *et al.* [6] extended the result to the case of $\nu > 1$. Both studies show that the optimal convergence rate of extremes from the GED(ν) is proportional to $1/\log n$.

For more informative studies of extremes from the GED, Nair [7] considered higher-order expansions for distribution and moments of normalized maxima from the GED(2) under optimal normalizing constants. Let $\Phi(x)$ denote the distribution function of the standard normal distribution GED(2), Nair [7] proved that

$$\tilde{b}_n^2\big[\tilde{b}_n^2\big(\Phi^n(\tilde{a}_n x + \tilde{b}_n) - \Lambda(x)\big) - \tilde{k}(x)\Lambda(x)\big] \to \left(\tilde{w}(x) + \frac{1}{2}\tilde{k}^2(x)\right)\Lambda(x) \tag{1.1}$$

as $n \to \infty$, where the optimal normalizing constants $\tilde{a}_n$ and $\tilde{b}_n$ are given by

$$1 - \Phi(\tilde{b}_n) = n^{-1}, \qquad \tilde{a}_n = \tilde{b}_n^{-1}.$$

Here, $\tilde{k}(x)$ and $\tilde{w}(x)$ are, respectively, of the following form:

$$\tilde{k}(x) = 2^{-1}\big(x^2 + 2x\big)e^{-x}$$

and

$$\tilde{w}(x) = -8^{-1}\big(x^4 + 4x^3 + 8x^2 + 16x\big)e^{-x}.$$

In this short note, the aim is to establish a higher-order expansion for the distribution of normalized maxima from the GED(ν) for $\nu > 0$. For some recent related work on uniform convergence rates and higher-order expansions of extremes for given distributions, see Liao and Peng [8] for the log-normal distribution, and Liao *et al.* [9, 10] for skew distributions.

In order to derive the higher-order expansions of extremes from the GED(ν), we cite some results from Peng *et al.* [4, 6]. The following Mills ratio of the GED(ν) is due to Peng *et al.* [4]:

$$\frac{1 - F_\nu(x)}{f_\nu(x)} \sim \frac{2\lambda^\nu}{\nu} x^{1-\nu} \quad \text{as } x \to \infty, \tag{1.2}$$

which deduces the following distributional tail representation of GED(ν):

$$1 - F_\nu(x) = c(x)\exp\left(-\int_\lambda^x \frac{g(t)}{f(t)}\,dt\right)$$

for large $x > 0$, where

$$c(x) \to \frac{\exp(-1/2)}{2^{1/\nu}\Gamma(1/\nu)} \quad \text{as } x \to \infty$$

and

$$f(t) = 2\nu^{-1}\lambda^\nu t^{1-\nu}, \qquad g(t) = 1 + 2(\nu - 1)\nu^{-1}\lambda^\nu t^{-\nu}. \tag{1.3}$$

Noting that $f'(t) \to 0$ and $g(t) \to 1$, we may choose normalizing constants a_n and b_n satisfying the following equations:

$$1 - F_\nu(b_n) = n^{-1}, \qquad a_n = f(b_n). \tag{1.4}$$

Under these normalizing constants, we have

$$\lim_{n\to\infty} F_\nu^n(a_n x + b_n) = \Lambda(x).$$

This paper is organized as follows. Section 2 provides the main results. Some auxiliary results and the proofs of the main results are given in Section 3.

2 Main result

In this section, we provide asymptotic expansions of a distribution for the partial maximum of the GED with normalizing constants a_n and b_n given by (1.4).

Theorem 1 *Let $F_\nu(x)$ denote the cdf of* GED(ν) *with $\nu > 0$. Then*:

(i) *For $\nu \neq 1$, with normalizing constants a_n and b_n given by* (1.4), *we have*

$$b_n^\nu\bigl[b_n^\nu\bigl(F_\nu^n(a_n x + b_n) - \Lambda(x)\bigr) - k_\nu(x)\Lambda(x)\bigr] \to \left(w_\nu(x) + \frac{k_\nu^2(x)}{2}\right)\Lambda(x) \tag{2.1}$$

as $n \to \infty$, where $k_\nu(x)$ and $w_\nu(x)$ are, respectively, given by

$$k_\nu(x) = \bigl(1 - \nu^{-1}\bigr)\lambda^\nu\bigl(x^2 + 2x\bigr)e^{-x}$$

and

$$w_\nu(x) = \bigl(\nu^{-1} - 1\bigr)\lambda^{2\nu}\left[4x + 2x^2 + \frac{2}{3}\bigl(2 - \nu^{-1}\bigr)x^3 + \frac{1}{2}\bigl(1 - \nu^{-1}\bigr)x^4\right]e^{-x}.$$

(ii) *For $\nu = 1$, with normalizing constants $a_n = 2^{-1/2}$ and $b_n = 2^{-1/2}(\log n - \log 2)$, we have*

$$e^{\sqrt{2}b_n}\bigl[e^{\sqrt{2}b_n}\bigl(F_1^n(a_n x + b_n) - \Lambda(x)\bigr) - k_1(x)\Lambda(x)\bigr] \to \left(w_1(x) + \frac{k_1^2(x)}{2}\right)\Lambda(x) \tag{2.2}$$

as $n \to \infty$, where $k_1(x)$ and $w_1(x)$ are, respectively, given by

$$k_1(x) = -\frac{1}{4}e^{-2x}, \qquad w_1(x) = -\frac{1}{12}e^{-3x}.$$

Remark 1 The main result coincides with (1.1) as the GED reduces to the standard normal distribution GED(2).

Remark 2 From (1.2) and (1.4), it is easy to check that $b_n^\nu = O(\log n)$. Hence, for $\nu \neq 1$, Theorem 1(i) shows that the convergence rate of $F_\nu^n(a_n x + b_n)$ to its ultimate extreme value distribution $\Lambda(x)$ is proportional to $1/\log n$, while for the case of $\nu = 1$, Theorem 1(ii) shows that the convergence rate is proportional to $1/n$.

3 The proofs

In order to prove the main results, we need some auxiliary lemmas. The first lemma deals with a decomposition of the distributional tail representation of GED(ν).

Lemma 1 *Let $F_\nu(x)$ and $f_\nu(x)$, respectively, denote the cdf and pdf of* GED(ν) *with $\nu \neq 1$; for large $x > 0$, we have*

$$\begin{aligned}1 - F_\nu(x) = {}& \frac{\exp(-1/2)}{2^{1/\nu}\Gamma(1/\nu)}\bigl[1 + 2\bigl(\nu^{-1} - 1\bigr)\lambda^\nu x^{-\nu} \\ &+ 4\bigl(\nu^{-1} - 1\bigr)\bigl(\nu^{-1} - 2\bigr)\lambda^{2\nu}x^{-2\nu} + O\bigl(x^{-3\nu}\bigr)\bigr]\exp\biggl(-\int_\lambda^x \frac{g(t)}{f(t)}\,dt\biggr)\end{aligned} \tag{3.1}$$

with $f(t)$ and $g(t)$ given by (1.3).

Proof Using integration by parts we have

$$\begin{aligned}&1 - F_\nu(x) \\ &\quad= \frac{\nu}{2^{1+1/\nu}\Gamma(1/\nu)}\int_{x/\lambda}^\infty \exp\biggl(-\frac{t^\nu}{2}\biggr)\,dt \\ &\quad= f_\nu(x)\frac{2\lambda^\nu}{\nu}x^{1-\nu}\bigl[1 + 2\bigl(\nu^{-1} - 1\bigr)\lambda^\nu x^{-\nu} \\ &\qquad+ 4\bigl(\nu^{-1} - 1\bigr)\bigl(\nu^{-1} - 2\bigr)\lambda^{2\nu}x^{-2\nu} + 8\bigl(\nu^{-1} - 1\bigr)\bigl(\nu^{-1} - 2\bigr)\bigl(\nu^{-1} - 3\bigr)\lambda^{3\nu}x^{-3\nu}\bigr] \\ &\qquad+ \frac{16\nu}{2^{1+1/\nu}\Gamma(1/\nu)}\bigl(\nu^{-1} - 1\bigr)\bigl(\nu^{-1} - 2\bigr)\bigl(\nu^{-1} - 3\bigr)\bigl(\nu^{-1} - 4\bigr)\int_{x/\lambda}^\infty \exp\biggl(-\frac{t^\nu}{2}\biggr)t^{-4\nu}\,dt.\end{aligned} \tag{3.2}$$

An application of L'Hospital's rule shows that

$$\lim_{x\to\infty}\frac{\int_{x/\lambda}^\infty \exp(-\frac{t^\nu}{2})t^{-4\nu}\,dt}{\exp(-\frac{x^\nu}{2\lambda^\nu})x^{1-4\nu}} = 0. \tag{3.3}$$

Combining the latter with (1.2), (3.2), and (3.3), for large x we have

$$\begin{aligned}1 - F_\nu(x) &= f_\nu(x)\frac{2\lambda^\nu}{\nu}x^{1-\nu}\bigl[1 + 2\bigl(\nu^{-1} - 1\bigr)\lambda^\nu x^{-\nu} \\ &\qquad+ 4\bigl(\nu^{-1} - 1\bigr)\bigl(\nu^{-1} - 2\bigr)\lambda^{2\nu}x^{-2\nu} + O\bigl(x^{-3\nu}\bigr)\bigr] \\ &= \frac{\exp(-1/2)}{2^{1/\nu}\Gamma(1/\nu)}\bigl[1 + 2\bigl(\nu^{-1} - 1\bigr)\lambda^\nu x^{-\nu} \\ &\qquad+ 4\bigl(\nu^{-1} - 1\bigr)\bigl(\nu^{-1} - 2\bigr)\lambda^{2\nu}x^{-2\nu} + O\bigl(x^{-3\nu}\bigr)\bigr]\exp\biggl(-\int_\lambda^x \frac{g(t)}{f(t)}\,dt\biggr),\end{aligned}$$

which is the desired result. □

Lemma 2 *Let $h_\nu(b_n; x) = n\log F_\nu(a_n x + b_n) + e^{-x}$ with normalizing constants a_n and b_n given by* (1.4), *then for $\nu \neq 1$ we have*

$$\lim_{n\to\infty} b_n^\nu\bigl(b_n^\nu h_\nu(b_n; x) - k_\nu(x)\bigr) = w_\nu(x), \tag{3.4}$$

where $k_\nu(x)$ and $w_\nu(x)$ are given by Theorem 1.

Proof It is well known that $n(1-F_\nu(a_nx+b_n))\to e^{-x}$ as $n\to\infty$. By $1-F_\nu(b_n)=n^{-1}$, we know that $b_n\to\infty$ if and only if $n\to\infty$. The following fact holds by (1.2):

$$\lim_{n\to\infty}\frac{1-F_\nu(a_nx+b_n)}{b_n^{-m\nu}}=0\quad\text{for } m=1,2. \tag{3.5}$$

Let

$$A_\nu(n,x)=\frac{1+\frac{2(1-\nu)}{\nu\lambda^{-\nu}}b_n^{-\nu}+\frac{4(1-\nu)(1-2\nu)}{\nu^2\lambda^{-2\nu}}b_n^{-2\nu}+O(b_n^{-3\nu})}{1+\frac{2(1-\nu)}{\nu\lambda^{-\nu}}(a_nx+b_n)^{-\nu}+\frac{4(1-\nu)(1-2\nu)}{\nu^2\lambda^{-2\nu}}(a_nx+b_n)^{-2\nu}+O((a_nx+b_n)^{-3\nu})}.$$

It is easy to check that $\lim_{n\to\infty}A_\nu(n,x)=1$ and

$$A_\nu(n,x)-1=\bigl(1+o(1)\bigr)\left[4\frac{(1-\nu)}{\nu}\lambda^{2\nu}b_n^{-2\nu}x+16\frac{(1-\nu)(1-2\nu)}{\nu^2}\lambda^{3\nu}b_n^{-3\nu}x+O\bigl(b_n^{-3\nu}\bigr)\right].$$

Hence,

$$\lim_{n\to\infty}\frac{A_\nu(n,x)-1}{b_n^{-\nu}}=0 \tag{3.6}$$

and

$$\lim_{n\to\infty}\frac{A_\nu(n,x)-1}{b_n^{-2\nu}}=4\bigl(\nu^{-1}-1\bigr)\lambda^{2\nu}x. \tag{3.7}$$

By (3.1) we have

$$\begin{aligned}
&\frac{1-F_\nu(b_n)}{1-F_\nu(a_nx+b_n)}e^{-x}\\
&\quad=A_\nu(n,x)\exp\left[\int_0^x\left(\frac{(\nu-1)a_n}{b_n+a_nt}+\frac{\nu a_n(b_n+a_nt)^{\nu-1}}{2\lambda^\nu}-1\right)dt\right]\\
&\quad=A_\nu(n,x)\left\{1+\int_0^x\left(\frac{(\nu-1)a_n}{b_n+a_nt}+\frac{\nu a_n(b_n+a_nt)^{\nu-1}}{2\lambda^\nu}-1\right)dt\right.\\
&\qquad\left.+\frac{1}{2}\left[\int_0^x\left(\frac{(\nu-1)a_n}{b_n+a_nt}+\frac{\nu a_n(b_n+a_nt)^{\nu-1}}{2\lambda^\nu}-1\right)dt\right]^2\bigl(1+o(1)\bigr)\right\}.
\end{aligned} \tag{3.8}$$

It follows from (3.5)-(3.8) that

$$\begin{aligned}
&\lim_{n\to\infty}b_n^\nu h_\nu(b_n;x)\\
&\quad=\lim_{n\to\infty}\frac{\log F_\nu(a_nx+b_n)+n^{-1}e^{-x}}{n^{-1}b_n^{-\nu}}\\
&\quad=\lim_{n\to\infty}\left[\frac{-(1-F_\nu(a_nx+b_n))-\frac{1}{2}(1-F_\nu(a_nx+b_n))^2(1+o(1))}{n^{-1}b_n^{-\nu}}+\frac{(1-F_\nu(b_n))e^{-x}}{n^{-1}b_n^{-\nu}}\right]\\
&\quad=\lim_{n\to\infty}\frac{(1-F_\nu(a_nx+b_n))}{n^{-1}}\frac{(\frac{1-F_\nu(b_n)}{1-F_\nu(a_nx+b_n)}e^{-x}-1)}{b_n^{-\nu}}\\
&\quad=e^{-x}\lim_{n\to\infty}\left[\frac{A_\nu(n,x)[\int_0^x(\frac{(\nu-1)a_n}{b_n+a_nt}+\frac{\nu a_n(b_n+a_nt)^{\nu-1}}{2\lambda^\nu}-1)\,dt](1+o(1))}{b_n^{-\nu}}+\frac{A_\nu(n,x)-1}{b_n^{-\nu}}\right]
\end{aligned}$$

$$
\begin{aligned}
&= e^{-x}\lim_{n\to\infty}\int_0^x b_n^{\nu}\left(\frac{(\nu-1)a_n}{b_n+a_nt}+\frac{\nu a_n(b_n+a_nt)^{\nu-1}}{2\lambda^{\nu}}-1\right)dt \\
&= \left(1-\nu^{-1}\right)\lambda^{\nu}\left(x^2+2x\right)e^{-x} = k_{\nu}(x),
\end{aligned}
\tag{3.9}
$$

where the last step is due to the dominated convergence theorem since

$$
\lim_{n\to\infty} b_n^{\nu}\left(\frac{\nu a_n(b_n+a_nt)^{\nu-1}}{2\lambda^{\nu}}-1\right) = 2\left(1-\nu^{-1}\right)\lambda^{\nu}t \tag{3.10}
$$

and

$$
\lim_{n\to\infty}\frac{(\nu-1)a_nb_n^{\nu}}{b_n+a_nt} = 2\left(1-\nu^{-1}\right)\lambda^{\nu}. \tag{3.11}
$$

By arguments similar to (3.9), we have

$$
\begin{aligned}
&\lim_{n\to\infty} b_n^{\nu}\left(b_n^{\nu}h_{\nu}(b_n;x)-k_{\nu}(x)\right) \\
&\quad= \lim_{n\to\infty}\frac{\log F_{\nu}(a_nx+b_n)+n^{-1}e^{-x}-n^{-1}b_n^{-\nu}k_{\nu}(x)}{n^{-1}b_n^{-2\nu}} \\
&\quad= \lim_{n\to\infty}\left[\frac{-(1-F_{\nu}(a_nx+b_n))-\frac{1}{2}(1-F_{\nu}(a_nx+b_n))^2(1+o(1))}{n^{-1}b_n^{-2\nu}}\right. \\
&\qquad\left.+\frac{n^{-1}e^{-x}-n^{-1}b_n^{-\nu}k_{\nu}(x)}{n^{-1}b_n^{-2\nu}}\right] \\
&\quad= \lim_{n\to\infty}\frac{-(1-F_{\nu}(a_nx+b_n))+n^{-1}e^{-x}(1-k_{\nu}(x)e^{x}b_n^{-\nu})}{n^{-1}b_n^{-2\nu}} \\
&\quad= \lim_{n\to\infty}\frac{1-F_{\nu}(a_nx+b_n)}{n^{-1}}\,\frac{\frac{1-F_{\nu}(b_n)}{1-F_{\nu}(a_nx+b_n)}e^{-x}(1-k_{\nu}(x)e^{x}b_n^{-\nu})-1}{b_n^{-2\nu}} \\
&\quad= e^{-x}\lim_{n\to\infty}\left[A_{\nu}(n,x)b_n^{2\nu}\left(\int_0^x\left(\frac{(\nu-1)a_n}{b_n+a_nt}+\frac{\nu a_n(b_n+a_nt)^{\nu-1}}{2\lambda^{\nu}}-1\right)dt-k_{\nu}(x)e^{x}b_n^{-\nu}\right)\right. \\
&\qquad -k_{\nu}(x)e^{x}A_{\nu}(n,x)b_n^{\nu}\int_0^x\left(\frac{(\nu-1)a_n}{b_n+a_nt}+\frac{\nu a_n(b_n+a_nt)^{\nu-1}}{2\lambda^{\nu}}-1\right)dt \\
&\qquad +\frac{1}{2}\left(1+o(1)\right)A_{\nu}(n,x)b_n^{2\nu}\left(\int_0^x\left(\frac{(\nu-1)a_n}{b_n+a_nt}+\frac{\nu a_n(b_n+a_nt)^{\nu-1}}{2\lambda^{\nu}}-1\right)dt\right)^2 \\
&\qquad\left.+\frac{A_{\nu}(n,x)-1}{b_n^{-2\nu}}\right] \\
&\quad= \left(\nu^{-1}-1\right)\lambda^{2\nu}\left[4x+2x^2+\frac{2}{3}\left(2-\nu^{-1}\right)x^3+\frac{1}{2}\left(1-\nu^{-1}\right)x^4\right]e^{-x} \\
&\quad= w_{\nu}(x).
\end{aligned}
$$

The proof is complete. □

For $\nu = 1$, noting that the GED(1) is the Laplace distribution with pdf given by

$$
f_1(x) = 2^{-1/2}\exp\left(-2^{1/2}|x|\right),\quad x\in\mathbb{R}, \tag{3.12}
$$

and the Laplace distributional tail can be written by

$$1 - F_1(x) = 2^{-1/2} f_1(x) = 2^{-1} \exp\bigl(-2^{1/2}\bigr) \exp\left(-\int_1^x \frac{1}{f(t)}\,dt\right), \quad x > 0, \tag{3.13}$$

with $f(t) = 2^{-1/2}$. For the Laplace distribution, we have the following result.

Lemma 3 *For $v = 1$, let $h_1(b_n;x) = n \log F_1(a_n x + b_n) + e^{-x}$ with normalizing constants $a_n = 2^{-1/2}$ and $b_n = 2^{-1/2}(\log n - \log 2)$. Then*

$$\lim_{n\to\infty} e^{\sqrt{2}b_n}\bigl(e^{\sqrt{2}b_n} h_1(b_n;x) - k_1(x)\bigr) = w_1(x), \tag{3.14}$$

where $k_1(x)$ and $w_1(x)$ are those given by Theorem 1.

Proof Noting that for GED(1), *i.e.*, the Laplace distribution with pdf $f_1(x) = 2^{-1/2} \times \exp(-2^{1/2}|x|)$, we have

$$\lim_{n\to\infty} F_1^n(a_n x + b_n) = \Lambda(x)$$

with normalizing constants $a_n = 2^{-1/2}$ and $b_n = 2^{-1/2}(\log n - \log 2)$. So, by (1.4) and (3.13), we have

$$\begin{aligned}
&\lim_{n\to\infty} e^{\sqrt{2}b_n} h_1(b_n;x) \\
&\quad = \lim_{n\to\infty} \frac{\log F_1(a_n x + b_n) + n^{-1}e^{-x}}{n^{-1}e^{-\sqrt{2}b_n}} \\
&\quad = \lim_{n\to\infty} \left[\frac{-(1 - F_1(a_n x + b_n)) - \frac{1}{2}(1 - F_1(a_n x + b_n))^2(1 + o(1))}{n^{-1}e^{-\sqrt{2}b_n}} + \frac{(1 - F_1(b_n))e^{-x}}{n^{-1}e^{-\sqrt{2}b_n}}\right] \\
&\quad = \lim_{n\to\infty} \frac{-\frac{1}{2}(1 - F_1(a_n x + b_n))^2(1 + o(1))}{n^{-1}e^{-\sqrt{2}b_n}} \\
&\quad = -\frac{1}{4}e^{-2x} = k_1(x)
\end{aligned} \tag{3.15}$$

and

$$\begin{aligned}
&\lim_{n\to\infty} e^{\sqrt{2}b_n}\bigl(e^{\sqrt{2}b_n} h_1(b_n;x) - k_1(x)\bigr) \\
&\quad = \lim_{n\to\infty} \frac{\log F_1(a_n x + b_n) + n^{-1}e^{-x}}{n^{-1}e^{-2\sqrt{2}b_n}} - k_1(x)e^{\sqrt{2}b_n} \\
&\quad = \lim_{n\to\infty} \left[\frac{-(1 - F_1(a_n x + b_n)) - \frac{1}{2}(1 - F_1(a_n x + b_n))^2 - \frac{1}{3}(1 - F_1(a_n x + b_n))^3(1 + o(1))}{n^{-1}e^{-2\sqrt{2}b_n}} \right. \\
&\qquad \left. + \frac{(1 - F_1(b_n))e^{-x}}{n^{-1}e^{-2\sqrt{2}b_n}} - k_1(x)e^{\sqrt{2}b_n}\right] \\
&\quad = \lim_{n\to\infty} \frac{-\frac{1}{3}(1 - F_1(a_n x + b_n))^3(1 + o(1))}{n^{-1}e^{-2\sqrt{2}b_n}} \\
&\quad = -\frac{1}{12}e^{-3x} = w_1(x).
\end{aligned} \tag{3.16}$$

The proof is complete. □

Proof of Theorem 1 By (3.9) and (3.15), we have

$$h_\nu(b_n;x) \to 0 \quad \text{and} \quad \left|\sum_{i=3}^{\infty} \frac{h_\nu^{i-3}(b_n;x)}{i!}\right| < \exp\big(\big|h_\nu(b_n;x)\big|\big) \to 1 \tag{3.17}$$

as $n \to \infty$. For the case of $\nu \neq 1$, by Lemma 2 and (3.17), we have

$$\begin{aligned} & b_n^\nu\big[b_n^\nu\big(F_\nu^n(a_n x + b_n) - \Lambda(x)\big) - k_\nu(x)\Lambda(x)\big] \\ &\quad = b_n^\nu\big[b_n^\nu\big(\exp\big(h_\nu(b_n;x)\big) - 1\big) - k_\nu(x)\big]\Lambda(x) \\ &\quad = \left[b_n^\nu\big(b_n^\nu h_\nu(b_n;x) - k_\nu(x)\big) + b_n^{2\nu} h_\nu^2(b_n;x)\left(\frac{1}{2} + h_\nu(b_n;x)\sum_{i=3}^{\infty}\frac{h_\nu^{i-3}(b_n;x)}{i!}\right)\right]\Lambda(x) \\ &\quad \to \left(w_\nu(x) + \frac{1}{2}k_\nu^2(x)\right)\Lambda(x) \end{aligned}$$

as $n \to \infty$. Similarly, by Lemma 3 and (3.17), we get

$$e^{\sqrt{2}b_n}\big[e^{\sqrt{2}b_n}\big(F_1^n(a_n x + b_n) - \Lambda(x)\big) - k_1(x)\Lambda(x)\big] \to \left(w_1(x) + \frac{1}{2}k_1^2(x)\right)\Lambda(x)$$

as $n \to \infty$.

The proof is complete. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
PJ obtained the theorem and completed the proof. TL corrected and improved the final version. Both authors read and approved the final manuscript.

Acknowledgements
This work was supported by the National Natural Science Foundation of China (11171275), the Natural Science Foundation Project of CQ (cstc2012jjA00029) and the Fundamental Research Funds for the Central Universities (XDJK2013C021, XDJK2014D020).

Received: 2 January 2014 Accepted: 14 May 2014 Published: 27 May 2014

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10.1186/1029-242X-2014-213
Cite this article as: Jia and Li: **Higher-order expansions for distributions of extremes from general error distribution.** *Journal of Inequalities and Applications* 2014, 2014:213