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Boundedness for Riesz transform associated with Schrödinger operators and its commutator on weighted Morrey spaces related to certain nonnegative potentials

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Abstract

Let $L = -\Delta + V$ be a Schrödinger operator, where Δ is the Laplacian on $\mathbb{R}^n$ and the nonnegative potential V belongs to the reverse Hölder class B_q for $q \geq n/2$. The Riesz transform associated with the operator L is denoted by $T = \nabla(-\Delta + V)^{-\frac{1}{2}}$ and the dual Riesz transform is denoted by $T^* = (-\Delta + V)^{-\frac{1}{2}}\nabla$. In this paper, we establish the boundedness for the operator T^* and its commutator on the weighted Morrey spaces $L^{p,\lambda}_{\alpha,V,\omega}(\mathbb{R}^n)$ related to certain nonnegative potentials belonging to the reverse Hölder class B_q for $n/2 \leq q < n$, where $p_0' < p < \infty$ and $\frac{1}{p_0} = \frac{1}{q} - \frac{1}{n}$.

Keywords: Morrey spaces; commutator; reverse Hölder class; Schrödinger operator; Riesz transform

1 Introduction

In this paper, we consider the Schrödinger operator

$$L = -\Delta + V(x) \quad \text{on } \mathbb{R}^n, n \geq 3,$$

where $V(x)$ is a nonnegative potential belonging to the reverse Hölder class B_q for $q \geq n/2$. The Riesz transform associated with the Schrödinger operator L is defined by $T = \nabla L^{-\frac{1}{2}}$ and the commutator operator

$$[b, T](f)(x) = T(bf)(x) - b(x)Tf(x), \quad x \in \mathbb{R}^n, \tag{1}$$

where f is a suitable integral function. Also, the dual Riesz transform associated with the Schrödinger operator L is defined by $T^* = L^{-\frac{1}{2}}\nabla$ and the commutator operator

$$\left[b, T^*\right](f)(x) = T^*(bf)(x) - b(x)T^*f(x), \quad x \in \mathbb{R}^n. \tag{2}$$

First, Tang and Dong established the boundedness of some Schrödinger type operators on the Morrey spaces related to the nonnegative potential V belonging to the reverse Hölder class in [1]. Furthermore, Liu and Wang investigated the boundedness of the dual

Riesz transforms and its commutators on the Morrey spaces related to the nonnegative potential V belonging to the reverse Hölder class in [2]. Recently, Pan and Tang established the boundedness of some Schrödinger type operators on weighted Morrey spaces related to the nonnegative potential V belonging to the reverse Hölder class in [3]. Motivated by [3], our aim is to establish the boundedness for the dual Riesz transform associated with Schrödinger operators and its commutators on weighted Morrey spaces related to the certain nonnegative potentials, where the condition on the potential is weaker than that in [3]. Our result is a nontrivial generalization of the main results in [3].

A nonnegative locally L^q integrable function $V(x)$ on $\mathbb{R}^n$ is said to belong to B_q $(1 < q < \infty)$ if there exists $C > 0$ such that the reverse Hölder inequality

$$\left(\frac{1}{|B|}\int_B V(x)^q\,dx\right)^{1/q} \le C\left(\frac{1}{|B|}\int_B V(x)\,dx\right) \tag{3}$$

holds for every ball B in $\mathbb{R}^n$.

It is important that the B_q class has a property of 'self improvement'; that is, if $V \in B_q$, then $V \in B_{q+\varepsilon}$ for some $\varepsilon > 0$ (see [4]).

We assume the potential $V \in B_q$ for $q \ge n/2$ throughout the paper. We introduce the auxiliary function $\rho(x, V) = \rho(x)$ defined by

$$\rho(x) = \sup_{r>0}\left\{r : \frac{1}{r^{n-2}}\int_{B(x,r)} V(y)\,dy \le 1\right\}, \quad x \in \mathbb{R}^n.$$

It is well known that $0 < \rho(x) < \infty$ for any $x \in \mathbb{R}^n$ (*cf.* Lemma 1 in Section 2).

A kind of new Morrey spaces is established by Tang and Dong in [1]. Furthermore, the weighted Morrey space is introduced by Pan and Tang in [3]. Let $p \in [1,\infty)$, $\alpha \in (-\infty,\infty)$, and $\lambda \in [0,1)$. For $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ and $V \in B_q$ $(q > 1)$, we say $f \in L^{p,\lambda}_{\alpha,V,\omega}(\mathbb{R}^n)$ (weighted Morrey spaces related to the nonnegative potential V) provided that

$$\|f\|^p_{L^{p,\lambda}_{\alpha,V,\omega}(\mathbb{R}^n)} = \sup_{B(x_0,r)\subset\mathbb{R}^n}\left(1 + \frac{r}{\rho(x_0)}\right)^{\alpha}\omega\big(B(x_0,2r)\big)^{-\lambda}\int_{B(x_0,r)}\big|f(x)\big|^p\omega(x)\,dx < \infty,$$

where $B = B(x_0, r)$ denotes a ball with centered at x_0 and radius r, and the weight functions $\omega \in A^{\rho,\infty}_p$ (see Section 2).

Now we are in a position to give the main results in this paper.

Theorem 1 *Suppose $V \in B_q$ for $n/2 \le q < n$, $\alpha \in (-\infty,\infty)$, $\lambda \in (0,1)$, and $1/p_0 = 1/q - 1/n$. Then, for $p_0' \le p < \infty$ and $\omega \in A^{\rho,\infty}_{p/p_0'}$,*

$$\big\|T^*f\big\|_{L^{p,\lambda}_{\alpha,V,\omega}(\mathbb{R}^n)} \le C\|f\|_{L^{p,\lambda}_{\alpha,V,\omega}(\mathbb{R}^n)},$$

where C is independent of f.

Theorem 2 *Suppose $V \in B_q$ for $n/2 \le q < n$, $b \in BMO_\rho$, $\alpha \in (-\infty,\infty)$, $\lambda \in (0,1)$, and p_0 so that $1/p_0 = 1/q - 1/n$. Then, for $p_0' \le p < \infty$ and $\omega \in A^{\rho,\infty}_{p/p_0'}$,*

$$\big\|\big[b, T^*\big]f\big\|_{L^{p,\lambda}_{\alpha,V,\omega}(\mathbb{R}^n)} \le C\|f\|_{L^{p,\lambda}_{\alpha,V,\omega}(\mathbb{R}^n)},$$

where C is independent of f.

We will use C to denote a positive constant, which is not necessarily same at each occurrence and even is different in the same line, and may depend on the dimension n and the constant in (3). By $A \sim B$, we mean that there exists a constant C such that $1/C \le A/B \le C$.

2 Some lemmas

In this section, we collect some known results proved in [4] in order to prove the main results in this paper.

Lemma 1 *There exist constants $C, k_0 > 0$ such that*

$$\frac{1}{C}\left(1 + \frac{|x-y|}{\rho(x)}\right)^{-k_0} \le \frac{\rho(y)}{\rho(x)} \le C\left(1 + \frac{|x-y|}{\rho(x)}\right)^{k_0/(k_0+1)}.$$

In particular, $\rho(y) \sim \rho(x)$ if $|x-y| < C\rho(x)$.

Lemma 2 (1) *For $0 < r < R < \infty$,*

$$\frac{1}{r^{n-2}}\int_{B(x,r)} V(y)\,dy \le C\left(\frac{r}{R}\right)^{2-\frac{n}{q}} \frac{1}{R^{n-2}}\int_{B(x,R)} V(y)\,dy$$

and

$$\frac{1}{r^{n-2}}\int_{B(x,r)} V(y)\,dy \sim 1 \quad \text{if and only if} \quad r \sim \rho(x).$$

(2) *There exist $C > 0$ and $l_0 > 0$ such that*

$$\frac{1}{R^{n-2}}\int_{B(x,R)} V(y)\,dy \le C\left(1 + \frac{R}{\rho(x)}\right)^{l_0}.$$

Let $\mathcal{K}$ be the kernel of T and $\mathcal{K}^*$ be the kernel of T^*.

Lemma 3 *If $V \in B_q$ for $q \ge n/2$, then for every N there exists a constant $C_N > 0$ such that*

$$\left|\mathcal{K}^*(x,z)\right| \le \frac{C_N}{(1 + \frac{|x-z|}{\rho(x)})^N} \frac{1}{|x-z|^{n-1}}\left(\int_{B(z,|x-z|/4)} \frac{V(u)}{|u-z|^{n-1}}\,du + \frac{1}{|x-z|}\right). \tag{4}$$

Moreover, the last inequality also holds with $\rho(x)$ replaced by $\rho(z)$.

In this paper, we always write $\Psi_\theta(B) = (1 + r/\rho(x_0))^\theta$, where $\theta > 0$; x_0 and r denote the center and radius of B, respectively.

A weight will always mean a nonnegative function which is locally integrable. As in [5], we say that a weight ω belongs to the class $A_p^{\rho,\theta}$ for $1 < p < \infty$, if there is a positive constant C such that for the whole ball $B = B(x,r)$

$$\left(\frac{1}{\Psi_\theta(B)|B|}\int_B \omega(y)\,dy\right)\left(\frac{1}{\Psi_\theta(B)|B|}\int_B \omega^{-\frac{1}{p-1}}(y)\,dy\right)^{p-1} \le C.$$

We also say that a nonnegative function ω satisfies the $A_1^{\rho,\theta}$ condition if there exists a positive constant C, for all balls B

$$M_V^\theta(\omega)(x) \le C\omega(x), \quad \text{a.e. } x \in \mathbb{R}^n,$$

where

$$M_V^\theta f(x) = \sup_{x\in B} \frac{1}{\Psi_\theta(B)|B|}\int_B |f(y)|\,dy.$$

Since $\Psi_\theta(B) \geq 1$, obviously, $A_p \subset A_p^{\rho,\theta}$ for $1 \leq p < \infty$, where A_p denote the classical Muckenhoupt weights (see [6]). It follows from [7] that $A_p \subset\subset A_p^{\rho,\theta}$ for $1 \leq p < \infty$. For convenience, we always assume that $\Psi(B)$ denotes $\Psi_\theta(B)$, $A_p^{\rho,\infty} = \bigcup_{\theta>0} A_p^{\rho,\theta}$, and $A_\infty^{\rho,\infty} = \bigcup_{p\geq 1} A_p^{\rho,\infty}$.

Lemma 4 ([7]) *Let $0 < \theta < \infty$, then*:
- (i) *If $1 \leq p_1 < p_2 < \infty$, then $A_{p_1}^{\rho,\theta} \subset A_{p_2}^{\rho,\theta}$.*
- (ii) *$\omega \in A_p^{\rho,\theta}$ if and only if $\omega^{-\frac{1}{p-1}} \in A_{p'}^{\rho,\theta}$, where $1/p + 1/p' = 1$.*
- (iii) *If $\omega \in A_p^{\rho,\theta}$ for $1 \leq p < \infty$, then there exists a constant $C > 0$ such that for any $\lambda > 1$*

$$\omega(\lambda B(x_0,r)) \leq C\left(1 + \frac{\lambda r}{\rho(x_0)}\right)^{(k_0+1)\theta} \omega(B(x_0,r)).$$

Lemma 5 ([8]) *Let $0 < \theta < \infty$, $1 \leq p < \infty$. If $\omega \in A_p^{\rho,\theta}$, then there exist positive constants δ, η, and C such that*

$$\left(\frac{1}{|B|}\int_B \omega(y)^{1+\delta}\,dy\right)^{1/(1+\delta)} \leq C\frac{1}{|B|}\int_B \omega(y)\,dy\left(1 + \frac{r}{\rho(x_0)}\right)^{\eta}$$

for all ball $B(x_0,r)$.

As a consequence of Lemma 5, we have the following result.

Corollary 1 ([8]) *Let $0 < \theta < \infty$, $1 \leq p < \infty$. If $\omega \in A_p^{\rho,\theta}$, then there exist positive constants $q > 1$, η, and C such that*

$$\frac{\omega(E)}{\omega(B)} \leq C\left(\frac{|E|}{|B|}\right)^{1/q}\left(1 + \frac{r}{\rho(x_0)}\right)^{\eta}$$

for any measurable subset E of a ball $B(x_0,r)$.

Bongioanni *et al.* [9] introduced a new space $BMO_\theta(\rho)$ defined by

$$\|f\|_{BMO_\theta(\rho)} = \sup_{B\subset\mathbb{R}^n} \frac{1}{\Psi_\theta(B)|B|}\int_B |f(x) - f_B|\,dx < \infty,$$

where $f_B = \frac{1}{|B|}\int_B f(y)\,dy$ and $\Psi_\theta(B) = (1 + r/\rho(x_0))^\theta$, $B = B(x_0,r)$, and $\theta > 0$.

In particularly, Bongioanni *et al.* [9] proved the following results for $BMO_\theta(\rho)$.

Proposition 1 *Let $\theta > 0$ and $1 \leq s < \infty$. If $b \in BMO_\theta(\rho)$, then*

$$\left(\frac{1}{|B|}\int_B |b(x) - b_B|^s\,dx\right)^{1/s} \leq C\|b\|_{BMO_\theta(\rho)}\left(1 + \frac{r}{\rho(x_0)}\right)^{\theta'}$$

holds for all $B = B(x_0,r)$, with $x_0 \in \mathbb{R}^n$ and $r > 0$, where $\theta' = (k_0+1)\theta$ and k_0 is the constant appearing in Lemma 1.

Proposition 2 *Let $b \in BMO_\theta(\rho)$, $B = B(x_0, r)$, and $1 \le s < \infty$. Then*

$$\left(\frac{1}{|2^kB|}\int_{2^kB}\left|b(y) - b_B\right|^s dy\right)^{\frac{1}{s}} \le C\|b\|_{BMO_\theta(\rho)} k\left(1 + \frac{2^k r}{\rho(x_0)}\right)^{\theta'} \tag{5}$$

for all $k \in \mathbb{N}$, with $\theta' = (k_0 + 1)\theta$ and the constant k_0 is given as in Proposition 1.

Obviously, the classical *BMO* space is properly contained in $BMO_\theta(\rho)$; for more examples please see [9]. For convenience, we let $BMO_\rho = \bigcup_{\theta>0} BMO_\theta(\rho)$.

From Corollary 2.2 in [3], the following result holds true.

Corollary 2 *If $b \in BMO_\rho$ and $\omega \in A_\infty^{\rho,\infty}$, then there exist positive constants C and η such that for every ball $B = B(x, r)$, we have*

$$\frac{1}{\omega(B)}\int_B\left|b(x) - b_B\right|^p \omega(x)\,dx \le \left(1 + \frac{r}{\rho(x)}\right)^\eta \|b\|_{BMO_\rho}^p,$$

where $b_B = \frac{1}{|B|}\int_B b(y)\,dy$.

3 The proof of our main results

Proof of Theorem 1 Without loss of generality, we may assume that $\alpha < 0$ and $\omega \in A_{p/p_0'}^{\rho,\theta}$. Pick any ball $B = B(x_0, r)$, and write

$$f(x) = f_1(x) + f_2(x),$$

where $f_1 = \chi_{B(x_0,2r)} f$. Hence, we have

$$\begin{aligned}
&\left(\int_{B(x_0,r)}\left|T^*f(x)\right|^p \omega(x)\,dx\right)^{1/p} \\
&\quad\le \left(\int_{B(x_0,r)}\left|T^*f_1(x)\right|^p \omega(x)\,dx\right)^{1/p} + \left(\int_{B(x_0,r)}\left|T^*f_2(x)\right|^p \omega(x)\,dx\right)^{1/p}.
\end{aligned} \tag{6}$$

By the L_ω^p boundedness of T^* (see Theorem 3 in [5]), we obtain

$$\int_{B(x_0,r)}\left|T^*f_1(x)\right|^p \omega(x)\,dx \le C\left(1 + \frac{r}{\rho(x_0)}\right)^{-\alpha} \omega\big(B(x_0, 2r)\big)^\lambda \|f\|_{L_{\alpha,V,\omega}^{p,\lambda}(\mathbb{R}^n)}^p. \tag{7}$$

Now, for $x \in B(x_0, r)$ and using Lemma 3, we have

$$\begin{aligned}
\left|T^*f_2(x)\right| &= \left|\int_{|x_0-z|>2r} \mathcal{K}^*(x,z) f(z)\,dz\right| \\
&\le I_1(x) + I_2(x),
\end{aligned} \tag{8}$$

where

$$I_1(x) = C_N \int_{|x_0-z|>2r} \frac{|f(z)|}{|x-z|^n(1 + \frac{|x-z|}{\rho(x)})^N}\,dz$$

and

$$I_2(x) = C_N \int_{|x_0-z|>2r} \frac{|f(z)|}{|x-z|^{n-1}(1+\frac{|x-z|}{\rho(x)})^N} \int_{B(z,|x-z|/4)} \frac{V(u)}{|u-z|^{n-1}}\,du\,dz.$$

Then

$$\begin{aligned}&\left(\int_{B(x_0,r)} \left|T^*f_2(x)\right|^p \omega(x)\,dx\right)^{1/p} \\ &\quad\le \left(\int_{B(x_0,r)} \left(I_1(x)\right)^p \omega(x)\,dx\right)^{1/p} + \left(\int_{B(x_0,r)} \left(I_2(x)\right)^p \omega(x)\,dx\right)^{1/p}.\end{aligned} \tag{9}$$

By the proof of Theorem 1.1 in [3], we have

$$\int_{B(x_0,r)} \left(I_1(x)\right)^p \omega(x)\,dx \le C\left(1+\frac{r}{\rho(x_0)}\right)^{-\alpha} \omega\left(B(x_0,2r)\right)^{\lambda} \|f\|^p_{L^{p,\lambda}_{\alpha,V,\omega}(\mathbb{R}^n)}.$$

Next we deal with $I_2(x)$. For $x \in B(x_0,r)$, $\frac{|x_0-z|}{2} \le |x-z| \le \frac{3|x_0-z|}{2}$. We get

$$\begin{aligned}&\int_{B(x_0,r)} \left(I_2(x)\right)^p \omega(x)\,dx \\ &\quad= C_N \int_{B(x_0,r)} \left(\int_{|x_0-z|>2r} \frac{|f(z)|}{|x-z|^{n-1}(1+\frac{|x-z|}{\rho(x)})^N} \int_{B(z,|x-z|/4)} \frac{V(u)}{|u-z|^{n-1}}\,du\,dz\right)^p \omega(x)\,dx \\ &\quad\le \sum_{i=1}^{\infty} C_N \int_{B(x_0,r)} \left(\int_{B(x_0,2^{i+1}r)\setminus B(x_0,2^i r)} \frac{|f(z)|}{|x_0-z|^{n-1}(1+\frac{|x_0-z|}{\rho(x)})^N}\right. \\ &\qquad\times \left.\int_{B(x_0,2^{i+3}r)} \frac{V(u)}{|u-z|^{n-1}}\,du\,dz\right)^p \omega(x)\,dx \\ &\quad\le \sum_{i=1}^{\infty} C_N \int_{B(x_0,r)} \frac{1}{(1+\frac{2^i r}{\rho(x)})^{Np}} \left(2^i r\right)^{-(n-1)p} \left(\int_{B(x_0,2^i r)} \left|f(z)\right| \mathcal{I}_1(V\chi_{B(x_0,2^i r)})\,dz\right)^p \omega(x)\,dx.\end{aligned}$$

Let $p_0' \le p < \infty$. By simple computation, $\frac{p}{p_0'} = 1 + \frac{p}{\nu}$. By the definition of $A^{\rho,\theta}_{p/p_0'}$,

$$\begin{aligned}&\left(\frac{1}{\Psi(B(x_0,2^i r))|B(x_0,2^i r)|} \int_{B(x_0,2^i r)} \omega^{-\nu/p}(y)\,dy\right)^{1/\nu} \\ &\quad= \left(\frac{1}{\Psi(B(x_0,2^i r))|B(x_0,2^i r)|} \int_{B(x_0,2^i r)} \omega^{-\frac{1}{\frac{p}{\nu}+1-1}}(y)\,dy\right)^{(\frac{p}{\nu}+1-1)\frac{1}{\nu(\frac{p}{\nu}+1-1)}} \\ &\quad\le C\left(\frac{1}{\Psi(B(x_0,2^i r))|B(x_0,2^i r)|} \int_{B(x_0,2^i r)} \omega(y)\,dy\right)^{-\frac{1}{p}},\end{aligned} \tag{10}$$

where $1/q = 1/s + 1/n$ and $1/p + 1/\nu + 1/s = 1$.

Using Hölder's inequality, (10), and the boundedness of the fractional integral $\mathcal{I}_1 : L^q \to L^s$ with $1/q = 1/s + 1/n$, for $1/p + 1/\nu + 1/s = 1$, we have

$$\begin{aligned}&\int_{B(x_0,2^i r)} \left|f(x)\right| \mathcal{I}_1(V\chi_{B(x_0,2^i r)})\,dx \\ &\quad= \int_{B(x_0,2^i r)} \left|f(x)\right| \omega^{1/p}\omega^{-1/p} \mathcal{I}_1(V\chi_{B(x_0,2^i r)})\,dx\end{aligned}$$

$$
\begin{aligned}
&\leq \bigl(\Psi\bigl(B\bigl(x_0,2^ir\bigr)\bigr)\bigl|B\bigl(x_0,2^ir\bigr)\bigr|\bigr)^{\frac{1}{\nu}}\left(\int_{B(x_0,2^ir)}\bigl|f(x)\bigr|^p\omega(x)\,dx\right)^{1/p}\\
&\quad\times\left(\frac{1}{\Psi(B(x_0,2^ir))|B(x_0,2^ir)|}\int_{B(x_0,2^ir)}\omega(x)^{-\nu/p}\,dx\right)^{1/\nu}\\
&\quad\times\left(\int_{B(x_0,2^ir)}\bigl(\mathcal{I}_1(V\chi_{B(x_0,2^ir)})\bigr)^s\,dx\right)^{1/s}\\
&\leq C\bigl(\Psi\bigl(B\bigl(x_0,2^ir\bigr)\bigr)\bigl|B\bigl(x_0,2^ir\bigr)\bigr|\bigr)^{\frac{1}{\nu}}\left(\int_{B(x_0,2^ir)}\bigl|f(x)\bigr|^p\omega(x)\,dx\right)^{1/p}\\
&\quad\times\left(\frac{1}{\Psi(B(x_0,2^ir))|B(x_0,2^ir)|}\int_{B(x_0,2^ir)}\omega(x)\,dx\right)^{-1/p}\\
&\quad\times\left(\int_{B(x_0,2^ir)}\bigl(\mathcal{I}_1(V\chi_{B(x_0,2^ir)})\bigr)^s\,dx\right)^{1/s}\\
&\leq C\bigl(\Psi\bigl(B\bigl(x_0,2^ir\bigr)\bigr)\bigl|B\bigl(x_0,2^ir\bigr)\bigr|\bigr)^{1/p+1/\nu}\left(\int_{B(x_0,2^ir)}\bigl|f(x)\bigr|^p\omega(x)\,dx\right)^{1/p}\\
&\quad\times\omega\bigl(B\bigl(x_0,2^ir\bigr)\bigr)^{-1/p}\bigl\|\mathcal{I}_1(V\chi_{B(x_0,2^ir)})\bigr\|_s\\
&\leq C\bigl(\Psi\bigl(B\bigl(x_0,2^ir\bigr)\bigr)\bigl|B\bigl(x_0,2^ir\bigr)\bigr|\bigr)^{1/p+1/\nu}\left(\int_{B(x_0,2^ir)}\bigl|f(x)\bigr|^p\omega(x)\,dx\right)^{1/p}\\
&\quad\times\omega\bigl(B\bigl(x_0,2^ir\bigr)\bigr)^{-1/p}\|V\chi_{B(x_0,2^ir)}\|_q.
\end{aligned}
\tag{11}
$$

For $V\in B_q$, using Lemma 2, we get

$$
\begin{aligned}
\|V\chi_{B(x_0,2^ir)}\|_q &\leq C\bigl(2^ir\bigr)^{-n/q'}\int_{B(x_0,2^ir)}V(x)\,dx\\
&\leq C\bigl(2^ir\bigr)^{-n/q'+n-2}\bigl(2^ir\bigr)^{-n+2}\int_{B(x_0,2^ir)}V(x)\,dx\\
&\leq C\bigl(2^ir\bigr)^{-n/q'+n-2}\left(1+\frac{2^ir}{\rho(x_0)}\right)^{l_0}.
\end{aligned}
\tag{12}
$$

It is easy to check that $-(n-1)p-\frac{pn}{q'}+(n-2)p+n+\frac{pn}{\nu}=0$. Furthermore, using Corollary 1, we have

$$
\frac{\omega(B(x_0,2r))}{\omega(B(x_0,2^{i+1}r))}\leq C\bigl(2^i\bigr)^{-\frac{n}{q}}\left(1+\frac{2^ir}{\rho(x_0)}\right)^{\eta}.
\tag{13}
$$

Therefore, by (13),

$$
\begin{aligned}
&\int_{B(x_0,r)}\bigl(I_2(x)\bigr)^p\omega(x)\,dx\\
&\quad= C_N\int_{B(x_0,r)}\left(\int_{|x_0-z|>2r}\frac{|f(z)|}{|x-z|^{n-1}(1+\frac{|x-z|}{\rho(x)})^N}\int_{B(z,|x-z|/4)}\frac{V(u)}{|u-z|^{n-1}}\,du\,dz\right)^p\omega(x)\,dx\\
&\quad\leq\sum_{i=1}^{\infty}C_N\bigl(2^ir\bigr)^{-(n-1)p-\frac{pn}{q'}+(n-2)p+n+\frac{pn}{\nu}}
\end{aligned}
$$

$$\begin{aligned}
&\times \int_{B(x_0,r)} \frac{(1+\frac{2^i r}{\rho(x_0)})^{(1+\frac{p}{\nu})\theta+l_0 p}}{\omega(B(x_0,2^i r))(1+\frac{2^i r}{\rho(x)})^{Np}} \int_{B(x_0,2^i r)} \bigl|f(y)\bigr|^p \omega(y)\,dy\,\omega(x)\,dx \\
&\le \sum_{i=1}^{\infty} C_N \omega\bigl(B\bigl(x_0,2^{i+1}r\bigr)\bigr)^{\lambda-1} \|f\|^p_{L^{p,\lambda}_{\alpha,V,\omega}(\mathbb{R}^n)} \int_{B(x_0,r)} \frac{(1+\frac{2^i r}{\rho(x_0)})^{-\alpha+(1+\frac{p}{\nu})\theta+l_0 p}}{(1+\frac{2^i r}{\rho(x)})^{Np}} \omega(x)\,dx \\
&\le \sum_{i=1}^{\infty} C_N \omega\bigl(B\bigl(x_0,2^{i+1}r\bigr)\bigr)^{\lambda-1} \omega\bigl(B(x_0,r)\bigr) \frac{(1+\frac{2^i r}{\rho(x_0)})^{-\alpha+(1+\frac{p}{\nu})\theta+l_0 p}}{(1+\frac{2^i r}{\rho(x_0)})^{Np/(k_0+1)}} \|f\|^p_{L^{p,\lambda}_{\alpha,V,\omega}(\mathbb{R}^n)} \\
&\le \sum_{i=1}^{\infty} C_N \left(\frac{\omega(B(x_0,2r))}{\omega(B(x_0,2^{i+1}r))}\right)^{1-\lambda} \omega\bigl(B(x_0,2r)\bigr)^{\lambda} \frac{(1+\frac{2^i r}{\rho(x_0)})^{-\alpha+(1+\frac{p}{\nu})\theta+l_0 p}}{(1+\frac{2^i r}{\rho(x_0)})^{Np/(k_0+1)}} \|f\|^p_{L^{p,\lambda}_{\alpha,V,\omega}(\mathbb{R}^n)} \\
&\le \sum_{i=1}^{\infty} C_N 2^{-in(1-\lambda)/q} \omega\bigl(B(x_0,2r)\bigr)^{\lambda} \frac{(1+\frac{2^i r}{\rho(x_0)})^{-\alpha+(1+\frac{p}{\nu})\theta+l_0 p+\eta}}{(1+\frac{2^i r}{\rho(x_0)})^{Np/(k_0+1)}} \|f\|^p_{L^{p,\lambda}_{\alpha,V,\omega}(\mathbb{R}^n)} \\
&\le C\|f\|^p_{L^{p,\lambda}_{\alpha,V,\omega}(\mathbb{R}^n)},
\end{aligned} \tag{14}$$

where we choose N large enough so that the above series converges.

From (6)-(14), we obtain

$$\bigl\|T^* f\bigr\|_{L^{p,\lambda}_{\alpha,V,\omega}(\mathbb{R}^n)} \le C\|f\|_{L^{p,\lambda}_{\alpha,V,\omega}(\mathbb{R}^n)}.$$

Thus, Theorem 1 is proved. □

Proof of Theorem 2 During the proof of Theorem 2, we always denote $\theta' = (k_0+1)\theta$. Without loss of generality, we may assume that $\alpha < 0$, $b \in BMO_\theta(\rho)$, and $\omega \in A^{\rho,\theta}_{p/p_0'}$. Pick any ball $B = B(x_0, r)$, and write

$$f(x) = f_1(x) + f_2(x),$$

where $f_1 = \chi_{B(x_0,2r)} f$. Hence, we have

$$\begin{aligned}
&\left(\int_{B(x_0,r)} \bigl|\bigl[b,T^*\bigr]f(x)\bigr|^p \omega(x)\,dx\right)^{1/p} \\
&\le \left(\int_{B(x_0,r)} \bigl|\bigl[b,T^*\bigr]f_1(x)\bigr|^p \omega(x)\,dx\right)^{1/p} + \left(\int_{B(x_0,r)} \bigl|\bigl[b,T^*\bigr]f_2(x)\bigr|^p \omega(x)\,dx\right)^{1/p}.
\end{aligned} \tag{15}$$

By the L^p_ω boundedness of $[b, T^*]$ (see Theorem 2 in [8]), we obtain

$$\int_{B(x_0,r)} \bigl|\bigl[b,T^*\bigr]f_1(x)\bigr|^p \omega(x)\,dx \le C\left(1+\frac{r}{\rho(x_0)}\right)^{-\alpha} \omega\bigl(B(x_0,2r)\bigr)^{\lambda} \|f\|^p_{L^{p,\lambda}_{\alpha,V,\omega}(\mathbb{R}^n)}. \tag{16}$$

Set $b_B = \frac{1}{|B(x_0,r)|}\int_{B(x_0,r)} b(x)\,dx$. Write $[b, T^*]f_2 = (b - b_B)T^* f_2 - T^*(f_2(b - b_B))$. Then

$$\begin{aligned}
&\left(\int_{B(x_0,r)} \bigl|\bigl[b,T^*\bigr]f_2(x)\bigr|^p \omega(x)\,dx\right)^{1/p} \\
&\le \left(\int_{B(x_0,r)} \bigl|(b-b_B)T^* f_2\bigr|^p \omega(x)\,dx\right)^{1/p} + \left(\int_{B(x_0,r)} \bigl|T^*\bigl(f_2(b-b_B)\bigr)\bigr|^p \omega(x)\,dx\right)^{1/p}.
\end{aligned} \tag{17}$$

By (8) in the proof of Theorem 1, we obtain

$$\begin{aligned}
&\int_{B(x_0,r)} \left|(b-b_B)T^*f_2\right|^p \omega(x)\,dx \\
&\quad\le 2^{p-1}\left(\int_{B(x_0,r)} |b-b_B|^p \big(I_1(x)\big)^p \omega(x)\,dx + \int_{B(x_0,r)} |b-b_B|^p \big(I_2(x)\big)^p \omega(x)\,dx\right).
\end{aligned}$$

Let $p'_0 \le p < \infty$. By simple computation, $\frac{p}{p'_0} < 1 + \frac{p}{p'}$. By Lemma 4, $A^{\rho,\theta}_{p/p'_0} \subseteq A^{\rho,\theta}_{1+p/p'}$. Then

$$\begin{aligned}
&\left(\frac{1}{\Psi(B(x_0,2^i r))|B(x_0,2^i r)|}\int_{B(x_0,2^i r)} \omega^{-p'/p}(y)\,dy\right)^{p/p'} \\
&\quad=\left(\frac{1}{\Psi(B(x_0,2^i r))|B(x_0,2^i r)|}\int_{B(x_0,2^i r)} \omega^{-\frac{1}{1+\frac{p}{p'}-1}}(y)\,dy\right)^{(1+\frac{p}{p'}-1)} \\
&\quad\le C\left(\frac{1}{\Psi(B(x_0,2^i r))|B(x_0,2^i r)|}\int_{B(x_0,2^i r)} \omega(y)\,dy\right)^{-1}.
\end{aligned} \tag{18}$$

By Lemma 1 and Corollary 2, as well as Lemma 3, we have

$$\begin{aligned}
&\int_{B(x_0,r)} |b-b_B|^p \big(I_1(x)\big)^p \omega(x)\,dx \\
&\quad\le C_N \int_{B(x_0,r)} |b-b_B|^p \left(\int_{|x_0-z|>2r} \frac{|f(z)|}{|x-z|^n(1+\frac{|x-z|}{\rho(x)})^N}\,dz\right)^p \omega(x)\,dx \\
&\quad\le \sum_{i=1}^{\infty} C_N \int_{B(x_0,r)} |b-b_B|^p \left(\int_{B(x_0,2^{i+1}r)\setminus B(x_0,2^i r)} \frac{|f(z)|}{|x_0-z|^n(1+\frac{|x_0-z|}{\rho(x)})^N}\,dz\right)^p \omega(x)\,dx \\
&\quad\le \sum_{i=1}^{\infty} C_N \int_{B(x_0,r)} |b-b_B|^p \frac{1}{(1+\frac{2^i r}{\rho(x)})^{Np}} \big(2^i r\big)^{-np} \left(\int_{B(x_0,2^i r)} \big|f(z)\big|\,dz\right)^p \omega(x)\,dx \\
&\quad\le \sum_{i=1}^{\infty} C_N \int_{B(x_0,r)} |b-b_B|^p \frac{1}{(1+\frac{2^i r}{\rho(x)})^{Np}} \big(2^i r\big)^{-np} \\
&\qquad\times \left(\int_{B(x_0,2^i r)} \big|f(z)\big|\omega(z)^{1/p}\omega(z)^{-1/p}\,dz\right)^p \omega(x)\,dx \\
&\quad\le \sum_{i=1}^{\infty} C_N \int_{B(x_0,r)} |b-b_B|^p \frac{1}{(1+\frac{2^i r}{\rho(x)})^{Np}} \big(2^i r\big)^{-np} \left(\int_{B(x_0,2^i r)} \big|f(z)\big|^p \omega(z)\,dz\right) \\
&\qquad\times \left(\int_{B(x_0,2^i r)} \omega(z)^{-\frac{p'}{p}}\,dz\right)^{\frac{p}{p'}} \omega(x)\,dx \\
&\quad\le \sum_{i=1}^{\infty} C_N \int_{B(x_0,r)} |b-b_B|^p \frac{1}{(1+\frac{2^i r}{\rho(x)})^{Np}} \big(2^i r\big)^{-np} \left(\int_{B(x_0,2^i r)} \big|f(z)\big|^p \omega(z)\,dz\right) \\
&\qquad\times \left(1+\frac{2^i r}{\rho(x_0)}\right)^{(1+\frac{p}{p'})\theta} \big|B\big(x_0,2^i r\big)\big|^{1+\frac{p}{p'}} \omega\big(B\big(x_0,2^i r\big)\big)^{-1} \omega(x)\,dx \\
&\quad\le \sum_{i=1}^{\infty} C_N \int_{B(x_0,r)} |b-b_B|^p \frac{(1+\frac{2^i r}{\rho(x_0)})^{p\theta}}{(1+\frac{2^i r}{\rho(x)})^{Np}} \omega\big(B\big(x_0,2^i r\big)\big)^{-1}
\end{aligned}$$

$$
\begin{aligned}
&\times \int_{B(x_0,2^i r)} \big|f(z)\big|^p \omega(z)\,dz\omega(x)\,dx \\
&\le \sum_{i=1}^{\infty} C_N \omega\big(B\big(x_0,2^{i+1}r\big)\big)^{\lambda-1} \|f\|^p_{L^{p,\lambda}_{\alpha,V,\omega}(\mathbb{R}^n)} \int_{B(x_0,r)} |b-b_B|^p \frac{(1+\frac{2^i r}{\rho(x_0)})^{-\alpha+p\theta}}{(1+\frac{2^i r}{\rho(x)})^{Np}} \omega(x)\,dx \\
&\le \sum_{i=1}^{\infty} C_N \omega\big(B\big(x_0,2^{i+1}r\big)\big)^{\lambda-1} \omega\big(B(x_0,r)\big) \frac{(1+\frac{2^i r}{\rho(x_0)})^{-\alpha+p\theta+\eta}}{(1+\frac{2^i r}{\rho(x_0)})^{Np/(k_0+1)}} \|b\|^p_{BMO_\rho} \|f\|^p_{L^{p,\lambda}_{\alpha,V,\omega}(\mathbb{R}^n)} \\
&\le \sum_{i=1}^{\infty} C_N \left(\frac{\omega(B(x_0,2r))}{\omega(B(x_0,2^{i+1}r))} \right)^{1-\lambda} \omega\big(B(x_0,2r)\big)^{\lambda} \\
&\quad \times \frac{(1+\frac{2^i r}{\rho(x_0)})^{-\alpha+p\theta+\eta}}{(1+\frac{2^i r}{\rho(x_0)})^{Np/(k_0+1)}} \|b\|^p_{BMO_\rho} \|f\|^p_{L^{p,\lambda}_{\alpha,V,\omega}(\mathbb{R}^n)} \\
&\le \sum_{i=1}^{\infty} C_N 2^{-in(1-\lambda)/q} \omega\big(B(x_0,2r)\big)^{\lambda} \frac{(1+\frac{2^i r}{\rho(x_0)})^{-\alpha+p\theta+2\eta}}{(1+\frac{2^i r}{\rho(x_0)})^{Np/(k_0+1)}} \|b\|^p_{BMO_\rho} \|f\|^p_{L^{p,\lambda}_{\alpha,V,\omega}(\mathbb{R}^n)},
\end{aligned}
\tag{19}
$$

where we choose N large enough so that the above series converges.

For $I_2(x)$, we assume $n/2 < q < n$ due to Lemma 3. Then, since $x \in B(x_0,r)$, we also have $\frac{|x_0-z|}{2} \le |x-z| \le \frac{3|x_0-z|}{2}$. Then

$$
\begin{aligned}
&\int_{B(x_0,r)} |b-b_B|^p \big(I_2(x)\big)^p \omega(x)\,dx \\
&\le C_N \int_{B(x_0,r)} |b-b_B|^p \bigg(\int_{|x_0-z|>2r} \frac{|f(z)|}{|x-z|^{n-1}(1+\frac{|x-z|}{\rho(x)})^N} \\
&\quad \times \int_{B(z,|x-z|/4)} \frac{V(u)}{|u-z|^{n-1}}\,du\,dz \bigg)^p \omega(x)\,dx \\
&\le \sum_{i=1}^{\infty} C_N \int_{B(x_0,r)} |b-b_B|^p \bigg(\int_{B(x_0,2^{i+1}r)\setminus B(x_0,2^i r)} \frac{|f(z)|}{|x_0-z|^{n-1}(1+\frac{|x_0-z|}{\rho(x)})^N} \\
&\quad \times \int_{B(x_0,2^{i+3}r)} \frac{V(u)}{|u-z|^{n-1}}\,du\,dz \bigg)^p \omega(x)\,dx \\
&\le \sum_{i=1}^{\infty} C_N \int_{B(x_0,r)} \frac{1}{(1+\frac{2^i r}{\rho(x)})^{Np}} \big(2^i r\big)^{-(n-1)p} |b-b_B|^p \\
&\quad \times \bigg(\int_{B(x_0,2^i r)} \big|f(z)\big| \mathcal{I}_1(V\chi_{B(x_0,2^i r)})\,dz \bigg)^p \omega(x)\,dx.
\end{aligned}
$$

By (11) and (12) in the proof of Theorem 1, we obtain

$$
\begin{aligned}
&\int_{B(x_0,r)} |b-b_B|^p \big(I_2(x)\big)^p \omega(x)\,dx \\
&\le C_N \int_{B(x_0,r)} |b-b_B|^p \bigg(\int_{|x_0-z|>2r} \frac{|f(z)|}{|x-z|^{n-1}(1+\frac{|x-z|}{\rho(x)})^N} \\
&\quad \times \int_{B(z,|x-z|/4)} \frac{V(u)}{|u-z|^{n-1}}\,du\,dz \bigg)^p \omega(x)\,dx
\end{aligned}
$$

$$
\begin{aligned}
&\le \sum_{i=1}^{\infty} C_N \omega\bigl(B\bigl(x_0, 2^{i+1}r\bigr)\bigr)^{\lambda-1} \|f\|^p_{L^{p,\lambda}_{\alpha,V,\omega}(\mathbb{R}^n)} \\
&\quad \times \int_{B(x_0,r)} |b-b_B|^p \frac{(1+\frac{2^i r}{\rho(x_0)})^{-\alpha+(1+p/\nu)\theta+l_0 p}}{(1+\frac{2^i r}{\rho(x)})^{Np}} \omega(x)\,dx \\
&\le \sum_{i=1}^{\infty} C_N \omega\bigl(B\bigl(x_0, 2^{i+1}r\bigr)\bigr)^{\lambda-1} \omega\bigl(B(x_0,r)\bigr) \\
&\quad \times \frac{(1+\frac{2^i r}{\rho(x_0)})^{-\alpha+(1+p/\nu)\theta+l_0 p+\eta}}{(1+\frac{2^i r}{\rho(x_0)})^{Np/(k_0+1)}} \|b\|^p_{BMO_\rho} \|f\|^p_{L^{p,\lambda}_{\alpha,V,\omega}(\mathbb{R}^n)} \\
&\le \sum_{i=1}^{\infty} C_N \left(\frac{\omega(B(x_0,2r))}{\omega(B(x_0,2^{i+1}r))}\right)^{1-\lambda} \omega\bigl(B(x_0,2r)\bigr)^{\lambda} \\
&\quad \times \frac{(1+\frac{2^i r}{\rho(x_0)})^{-\alpha+(1+p/\nu)\theta+l_0 p+\eta}}{(1+\frac{2^i r}{\rho(x_0)})^{Np/(k_0+1)}} \|b\|^p_{BMO_\rho} \|f\|^p_{L^{p,\lambda}_{\alpha,V,\omega}(\mathbb{R}^n)} \\
&\le \sum_{i=1}^{\infty} C_N 2^{-in(1-\lambda)/q} \omega\bigl(B(x_0,2r)\bigr)^{\lambda} \\
&\quad \times \frac{(1+\frac{2^i r}{\rho(x_0)})^{-\alpha+(1+p/\nu)\theta+l_0 p+2\eta}}{(1+\frac{2^i r}{\rho(x_0)})^{Np/(k_0+1)}} \|b\|^p_{BMO_\rho} \|f\|^p_{L^{p,\lambda}_{\alpha,V,\omega}(\mathbb{R}^n)} \qquad (20)
\end{aligned}
$$

if we choose N large enough.

Now, for $x \in B(x_0, r)$ and using Lemma 3, we have

$$
\begin{aligned}
\bigl|T^*\bigl(f_2(b-b_B)\bigr)\bigr| &= \left|\int_{|x_0-z|>2r} \mathcal{K}^*(x,z) f(z)(b-b_B)\,dz\right| \\
&\le \tilde{I}_1(x) + \tilde{I}_2(x), \qquad (21)
\end{aligned}
$$

where

$$
\tilde{I}_1(x) = C_N \int_{|x_0-z|>2r} \frac{|f(z)(b-b_B)|}{|x-z|^n(1+\frac{|x-z|}{\rho(x)})^N}\,dz
$$

and

$$
\tilde{I}_2(x) = C_N \int_{|x_0-z|>2r} \frac{|f(z)(b-b_B)|}{|x-z|^{n-1}(1+\frac{|x-z|}{\rho(x)})^N} \int_{B(z,|x-z|/4)} \frac{V(u)}{|u-z|^{n-1}}\,du\,dz.
$$

Then,

$$
\begin{aligned}
&\left(\int_{B(x_0,r)} \bigl|T^*\bigl(f_2(b-b_B)\bigr)\bigr|^p \omega(x)\,dx\right)^{1/p} \\
&\quad \le \left(\int_{B(x_0,r)} \bigl(\tilde{I}_1(x)\bigr)^p \omega(x)\,dx\right)^{1/p} + \left(\int_{B(x_0,r)} \bigl(\tilde{I}_2(x)\bigr)^p \omega(x)\,dx\right)^{1/p}. \qquad (22)
\end{aligned}
$$

Firstly, we consider $\tilde{I}_1(x)$. By Proposition 2 and (10), for $1/p + 1/\nu + 1/s = 1$, we have

$$\begin{aligned}
&\int_{B(x_0,2^i r)} \bigl|f(x)\bigl(b(x)-b_B\bigr)\bigr|\,dx \\
&\quad\le \int_{B(x_0,2^i r)} \bigl|f(x)\bigr|\omega^{1/p}\omega^{-1/p}\bigl|b(x)-b_B\bigr|\,dx \\
&\quad\le \Psi\bigl(B\bigl(x_0,2^i r\bigr)\bigr)\bigl|B\bigl(x_0,2^i r\bigr)\bigr| \\
&\qquad\times \biggl(\frac{1}{\Psi(B(x_0,2^i r))|B(x_0,2^i r)|}\int_{B(x_0,2^i r)} \bigl|f(x)\bigr|^p\omega(x)\,dx\biggr)^{1/p} \\
&\qquad\times \biggl(\frac{1}{\Psi(B(x_0,2^i r))|B(x_0,2^i r)|}\int_{B(x_0,2^i r)} \omega(x)^{-\nu/p}\,dx\biggr)^{1/\nu} \\
&\qquad\times \biggl(\frac{1}{\Psi(B(x_0,2^i r))|B(x_0,2^i r)|}\int_{B(x_0,2^i r)} \bigl|b(x)-b_B\bigr|^s\,dx\biggr)^{1/s} \\
&\quad\le C\Psi\bigl(B\bigl(x_0,2^i r\bigr)\bigr)\bigl|B\bigl(x_0,2^i r\bigr)\bigr| \\
&\qquad\times \biggl(\frac{1}{\Psi(B(x_0,2^i r))|B(x_0,2^i r)|}\int_{B(x_0,2^i r)} \bigl|f(x)\bigr|^p\omega(x)\,dx\biggr)^{1/p} \\
&\qquad\times \biggl(\frac{1}{\Psi(B(x_0,2^i r))|B(x_0,2^i r)|}\int_{B(x_0,2^i r)} \omega(x)\,dx\biggr)^{-1/p} \\
&\qquad\times \biggl(\frac{1}{\Psi(B(x_0,2^i r))|B(x_0,2^i r)|}\int_{B(x_0,2^i r)} \bigl|b(x)-b_B\bigr|^s\,dx\biggr)^{1/s} \\
&\quad\le C\bigl(\Psi\bigl(B\bigl(x_0,2^i r\bigr)\bigr)\bigr)^{1/p+1/\nu}\bigl|B\bigl(x_0,2^i r\bigr)\bigr| \\
&\qquad\times \biggl(\int_{B(x_0,2^i r)} \bigl|f(x)\bigr|^p\omega(x)\,dx\biggr)^{1/p}\omega\bigl(B\bigl(x_0,2^i r\bigr)\bigr)^{-1/p} \\
&\qquad\times \biggl(\frac{1}{|B(x_0,2^i r)|}\int_{B(x_0,2^i r)} \bigl|b(x)-b_B\bigr|^s\,dx\biggr)^{1/s} \\
&\quad\le Ci\biggl(1+\frac{2^i r}{\rho(x_0)}\biggr)^{(1/p+1/\nu)\theta+\theta'}\bigl|B\bigl(x_0,2^i r\bigr)\bigr|\biggl(\int_{B(x_0,2^i r)} \bigl|f(x)\bigr|^p\omega(x)\,dx\biggr)^{1/p} \\
&\qquad\times \omega\bigl(B\bigl(x_0,2^i r\bigr)\bigr)^{-1/p}\|b\|_{BMO_\rho} \\
&\quad\le Ci\biggl(1+\frac{2^i r}{\rho(x_0)}\biggr)^{(1/p+1/\nu)\theta+\theta'}\bigl(2^i r\bigr)^n\biggl(\int_{B(x_0,2^i r)} \bigl|f(x)\bigr|^p\omega(x)\,dx\biggr)^{1/p} \\
&\qquad\times \omega\bigl(B\bigl(x_0,2^i r\bigr)\bigr)^{-1/p}\|b\|_{BMO_\rho}.
\end{aligned}\tag{23}$$

Then we get

$$\begin{aligned}
&\int_{B(x_0,r)} \bigl(\tilde{I}_1(x)\bigr)^p\omega(x)\,dx \\
&\quad= C_N\int_{B(x_0,r)} \biggl(\int_{|x_0-z|>2r} \frac{|f(z)(b-b_B)|}{|x-z|^n(1+\frac{|x-z|}{\rho(x)})^N}\,dz\biggr)^p\omega(x)\,dx \\
&\quad\le \sum_{i=1}^{\infty} C_N\int_{B(x_0,r)} \biggl(\int_{B(x_0,2^{i+1}r)\setminus B(x_0,2^i r)} \frac{|f(z)(b-b_B)|}{|x_0-z|^n(1+\frac{|x_0-z|}{\rho(x)})^N}\,dz\biggr)^p\omega(x)\,dx
\end{aligned}$$

$$
\begin{aligned}
&\le \sum_{i=1}^{\infty} C_N \int_{B(x_0,r)} \frac{1}{(1+\frac{2^i r}{\rho(x)})^{Np}} \big(2^i r\big)^{-np} \biggl(\int_{B(x_0,2^i r)} \big|f(z)(b-b_B)\big|\,dz\biggr)^p \omega(x)\,dx \\
&\le \sum_{i=1}^{\infty} C_N i^p \int_{B(x_0,r)} \frac{(1+\frac{2^i r}{\rho(x_0)})^{(1+p/\nu)\theta+p\theta'}}{(1+\frac{2^i r}{\rho(x)})^{Np}} \biggl(\int_{B(x_0,2^i r)} \big|f(z)\big|^p \omega(z)\,dz\biggr) \\
&\quad\times \omega\big(B\big(x_0,2^i r\big)\big)^{-1} \|b\|_{BMO_\rho}^p \omega(x)\,dx \\
&\le \sum_{i=1}^{\infty} C_N i^p \omega\big(B\big(x_0,2^{i+1} r\big)\big)^{\lambda-1} \|f\|^p_{L^{p,\lambda}_{\alpha,V,\omega}(\mathbb{R}^n)} \|b\|_{BMO_\rho}^p \\
&\quad\times \int_{B(x_0,r)} \frac{(1+\frac{2^i r}{\rho(x_0)})^{-\alpha+(1+p/\nu)\theta+p\theta'}}{(1+\frac{2^i r}{\rho(x)})^{Np}} \omega(x)\,dx \\
&\le \sum_{i=1}^{\infty} C_N i^p \omega\big(B\big(x_0,2^{i+1} r\big)\big)^{\lambda-1} \omega\big(B(x_0,r)\big) \\
&\quad\times \frac{(1+\frac{2^i r}{\rho(x_0)})^{-\alpha+(1+p/\nu)\theta+p\theta'}}{(1+\frac{2^i r}{\rho(x_0)})^{Np/(k_0+1)}} \|b\|_{BMO_\rho}^p \|f\|^p_{L^{p,\lambda}_{\alpha,V,\omega}(\mathbb{R}^n)} \\
&\le \sum_{i=1}^{\infty} C_N i^p \biggl(\frac{\omega(B(x_0,2r))}{\omega(B(x_0,2^{i+1}r))}\biggr)^{1-\lambda} \omega\big(B(x_0,2r)\big)^{\lambda} \\
&\quad\times \frac{(1+\frac{2^i r}{\rho(x_0)})^{-\alpha+(1+p/\nu)\theta+p\theta'}}{(1+\frac{2^i r}{\rho(x_0)})^{Np/(k_0+1)}} \|b\|_{BMO_\rho}^p \|f\|^p_{L^{p,\lambda}_{\alpha,V,\omega}(\mathbb{R}^n)} \\
&\le \sum_{i=1}^{\infty} C_N i^p 2^{-in(1-\lambda)/q} \omega\big(B(x_0,2r)\big)^{\lambda} \\
&\quad\times \frac{(1+\frac{2^i r}{\rho(x_0)})^{-\alpha+(1+p/\nu)\theta+p\theta'+\eta}}{(1+\frac{2^i r}{\rho(x_0)})^{Np/(k_0+1)}} \|b\|_{BMO_\rho}^p \|f\|^p_{L^{p,\lambda}_{\alpha,V,\omega}(\mathbb{R}^n)},
\end{aligned}
\tag{24}
$$

where we choose N large enough so that the above series converges.

For $V \in B_q$, then $V \in B_{q+\varepsilon}$ for $\varepsilon > 0$. Using Lemma 2, we get

$$
\begin{aligned}
\|V\chi_{B(x_0,2^i r)}\|_{q+\varepsilon} &\le C\big(2^i r\big)^{-n/(q+\varepsilon)'} \int_{B(x_0,2^i r)} V(x)\,dx \\
&\le C\big(2^i r\big)^{-n/(q+\varepsilon)'+n-2} \big(2^i r\big)^{-n+2} \int_{B(x_0,2^i r)} V(x)\,dx \\
&\le C\big(2^i r\big)^{-n/(q+\varepsilon)'+n-2} \biggl(1+\frac{2^i r}{\rho(x_0)}\biggr)^{l_0}.
\end{aligned}
\tag{25}
$$

Let $p_0' \le p < \infty$. We choose u such that $u = \frac{q(q+\varepsilon)}{\varepsilon}$ and $1/p + 1/\nu + 1/u + 1/s = 1$. Let $1/(q+\varepsilon) = 1/s + 1/n$. By simple computation,

$$
\begin{aligned}
\frac{p}{p_0'} &= p\biggl(1-\frac{1}{q}+\frac{1}{n}\biggr) = p\biggl(1-\frac{1}{q}+\frac{1}{n}+\frac{1}{q+\varepsilon}-\frac{1}{q+\varepsilon}\biggr) \\
&= p\biggl(1-\frac{1}{q(q+\varepsilon)}-\frac{1}{s}\biggr) = 1+\frac{p}{\nu}.
\end{aligned}
$$

Finally, we deal with $\tilde{I}_2(x)$. Using Hölder's inequality, (10), and the boundedness of the fractional integral $\mathcal{I}_1 : L^{q+\varepsilon} \to L^s$, for $1/p + 1/\nu + 1/u + 1/s = 1$, we have

$$\begin{aligned}
&\int_{B(x_0,2^ir)} \bigl|f(x)\bigl(b(x) - b_B\bigr)\bigr|\mathcal{I}_1(V\chi_{B(x_0,2^ir)})\,dx \\
&\quad\le \int_{B(x_0,2^ir)} \bigl|f(x)\bigr|\omega^{1/p}\omega^{-1/p}\bigl|b(x) - b_B\bigr|\mathcal{I}_1(V\chi_{B(x_0,2^ir)})\,dx \\
&\quad\le \bigl(\Psi\bigl(B\bigl(x_0,2^ir\bigr)\bigr)\bigr)^{1/\nu}\bigl(\bigl|B\bigl(x_0,2^ir\bigr)\bigr|\bigr)^{1/\nu+1/u}\biggl(\int_{B(x_0,2^ir)}\bigl|f(x)\bigr|^p\omega(x)\,dx\biggr)^{1/p} \\
&\qquad\times\biggl(\frac{1}{\Psi(B(x_0,2^ir))|B(x_0,2^ir)|}\int_{B(x_0,2^ir)}\omega(x)^{-\nu/p}\,dx\biggr)^{1/\nu} \\
&\qquad\times\biggl(\frac{1}{|B(x_0,2^ir)|}\int_{B(x_0,2^ir)}\bigl|b(x) - b_B\bigr|^u\,dx\biggr)^{1/u}\biggl(\int_{B(x_0,2^ir)}\bigl(\mathcal{I}_1(V\chi_{B(x_0,2^ir)})\bigr)^s\,dx\biggr)^{1/s} \\
&\quad\le C\bigl(\Psi\bigl(B\bigl(x_0,2^ir\bigr)\bigr)\bigr)^{1/\nu}\bigl(\bigl|B\bigl(x_0,2^ir\bigr)\bigr|\bigr)^{1/\nu+1/u}\biggl(\int_{B(x_0,2^ir)}\bigl|f(x)\bigr|^p\omega(x)\,dx\biggr)^{1/p} \\
&\qquad\times\biggl(\frac{1}{\Psi(B(x_0,2^ir))|B(x_0,2^ir)|}\int_{B(x_0,2^ir)}\omega(x)\,dx\biggr)^{-1/p} \\
&\qquad\times\biggl(\frac{1}{|B(x_0,2^ir)|}\int_{B(x_0,2^ir)}\bigl|b(x) - b_B\bigr|^u\,dx\biggr)^{1/u}\biggl(\int_{B(x_0,2^ir)}\bigl(\mathcal{I}_1(V\chi_{B(x_0,2^ir)})\bigr)^s\,dx\biggr)^{1/s} \\
&\quad\le C\bigl(\Psi\bigl(B\bigl(x_0,2^ir\bigr)\bigr)\bigr)^{1/p+1/\nu}\bigl|B\bigl(x_0,2^ir\bigr)\bigr|^{1/p+1/\nu+1/u} \\
&\qquad\times\biggl(\int_{B(x_0,2^ir)}\bigl|f(x)\bigr|^p\omega(x)\,dx\biggr)^{1/p}\omega\bigl(B\bigl(x_0,2^ir\bigr)\bigr)^{-1/p} \\
&\qquad\times\biggl(\frac{1}{|B(x_0,2^ir)|}\int_{B(x_0,2^ir)}\bigl|b(x) - b_B\bigr|^u\,dx\biggr)^{1/u}\bigl\|\mathcal{I}_1(V\chi_{B(x_0,2^ir)})\bigr\|_s \\
&\quad\le Ci\biggl(1 + \frac{2^ir}{\rho(x_0)}\biggr)^{(1/p+1/\nu)\theta+\theta'}\bigl|B\bigl(x_0,2^ir\bigr)\bigr|^{1/p+1/\nu+1/u}\biggl(\int_{B(x_0,2^ir)}\bigl|f(x)\bigr|^p\omega(x)\,dx\biggr)^{1/p} \\
&\qquad\times\omega\bigl(B\bigl(x_0,2^ir\bigr)\bigr)^{-1/p}\|b\|_{BMO_\rho}\|V\chi_{B(x_0,2^ir)}\|_{q+\varepsilon} \\
&\quad\le Ci\biggl(1 + \frac{2^ir}{\rho(x_0)}\biggr)^{(1/p+1/\nu)\theta+\theta'+l_0}\bigl(2^ir\bigr)^{(n-1)} \\
&\qquad\times\omega\bigl(B\bigl(x_0,2^ir\bigr)\bigr)^{-1/p}\biggl(\int_{B(x_0,2^ir)}\bigl|f(x)\bigr|^p\omega(x)\,dx\biggr)^{1/p}\|b\|_{BMO_\rho}.
\end{aligned}\tag{26}$$

Then

$$\begin{aligned}
&\int_{B(x_0,r)}\bigl(\tilde{I}_2(x)\bigr)^p\omega(x)\,dx \\
&\quad= C_N\int_{B(x_0,r)}\biggl(\int_{|x_0-z|>2r}\frac{|f(z)(b-b_B)|}{|x-z|^{n-1}(1+\frac{|x-z|}{\rho(x)})^N}\int_{B(z,|x-z|/4)}\frac{V(u)}{|u-z|^{n-1}}\,du\,dz\biggr)^p\omega(x)\,dx \\
&\quad\le\sum_{i=1}^{\infty}C_N\int_{B(x_0,r)}\biggl(\int_{B(x_0,2^{i+1}r)\setminus B(x_0,2^ir)}\frac{|f(z)(b-b_B)|}{|x_0-z|^{n-1}(1+\frac{|x_0-z|}{\rho(x)})^N}
\end{aligned}$$

$$
\begin{aligned}
&\times \int_{B(x_0,2^{i+3}r)} \frac{V(u)}{|u-z|^{n-1}}\,du\,dz\Bigg)^p \omega(x)\,dx \\
&\le \sum_{i=1}^{\infty} C_N \int_{B(x_0,r)} \frac{1}{(1+\frac{2^i r}{\rho(x)})^{Np}} \big(2^i r\big)^{-(n-1)p} \\
&\times \bigg(\int_{B(x_0,2^i r)} \big|f(z)(b-b_B)\big| \mathcal{I}_1(V\chi_{B(x_0,2^i r)})\,dz\bigg)^p \omega(x)\,dx \\
&\le \sum_{i=1}^{\infty} C_N i^p \omega\big(B\big(x_0,2^{i+1}r\big)\big)^{\lambda-1} \|f\|^p_{L^{p,\lambda}_{\alpha,V,\omega}(\mathbb{R}^n)} \|b\|^p_{BMO_\rho} \\
&\times \int_{B(x_0,r)} \frac{(1+\frac{2^i r}{\rho(x_0)})^{-\alpha+(1+p/\nu)\theta+p\theta'+l_0 p}}{(1+\frac{2^i r}{\rho(x)})^{Np}} \omega(x)\,dx \\
&\le \sum_{i=1}^{\infty} C_N i^p \omega\big(B\big(x_0,2^{i+1}r\big)\big)^{\lambda-1} \omega\big(B(x_0,r)\big) \\
&\times \frac{(1+\frac{2^i r}{\rho(x_0)})^{-\alpha+(1+p/\nu)\theta+p\theta'+l_0 p}}{(1+\frac{2^i r}{\rho(x_0)})^{Np/(k_0+1)}} \|b\|^p_{BMO_\rho} \|f\|^p_{L^{p,\lambda}_{\alpha,V,\omega}(\mathbb{R}^n)} \\
&\le \sum_{i=1}^{\infty} C_N i^p \bigg(\frac{\omega(B(x_0,2r))}{\omega(B(x_0,2^{i+1}r))}\bigg)^{1-\lambda} \omega\big(B(x_0,2r)\big)^{\lambda} \\
&\times \frac{(1+\frac{2^i r}{\rho(x_0)})^{-\alpha+(1+p/\nu)\theta+p\theta'+l_0 p}}{(1+\frac{2^i r}{\rho(x_0)})^{Np/(k_0+1)}} \|b\|^p_{BMO_\rho} \|f\|^p_{L^{p,\lambda}_{\alpha,V,\omega}(\mathbb{R}^n)} \\
&\le \sum_{i=1}^{\infty} C_N i^p 2^{-in(1-\lambda)/q} \omega\big(B(x_0,2r)\big)^{\lambda} \\
&\times \frac{(1+\frac{2^i r}{\rho(x_0)})^{-\alpha+(1+p/\nu)\theta+p\theta'+l_0 p+\eta}}{(1+\frac{2^i r}{\rho(x_0)})^{Np/(k_0+1)}} \|b\|^p_{BMO_\rho} \|f\|^p_{L^{p,\lambda}_{\alpha,V,\omega}(\mathbb{R}^n)},
\end{aligned}
\tag{27}
$$

where we choose N large enough so that the above series converges.

From (15)-(27), we obtain

$$\big\|\big[b,T^*\big]f\big\|_{L^{p,\lambda}_{\alpha,V,\omega}(\mathbb{R}^n)} \le C\|f\|_{L^{p,\lambda}_{\alpha,V,\omega}(\mathbb{R}^n)}.$$

Thus, we complete the proof of Theorem 2. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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