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Minkowski-type inequalities involving Hardy function and symmetric functions

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Abstract

The Hardy matrix $H_n(\mathbf{x},\boldsymbol{\alpha})$, the Hardy function $\operatorname{per} H_n(\mathbf{x},\boldsymbol{\alpha})$ and the generalized Vandermonde determinant $\det H_n(\mathbf{x},\boldsymbol{\alpha})$ are defined in this paper. By means of algebra and analysis theories together with proper hypotheses, we establish the following Minkowski-type inequality involving Hardy function:

$$\left[\operatorname{per} H_n(\mathbf{x}+\mathbf{y},\boldsymbol{\alpha})\right]^{\frac{1}{|\boldsymbol{\alpha}|}} \geq \left[\operatorname{per} H_n(\mathbf{x},\boldsymbol{\alpha})\right]^{\frac{1}{|\boldsymbol{\alpha}|}} + \left[\operatorname{per} H_n(\mathbf{y},\alpha)\right]^{\frac{1}{|\boldsymbol{\alpha}|}}.$$

As applications, our inequality is used to estimate the lower bounds of the increment of a symmetric function.

MSC: 26D15; 15A15

Keywords: Hardy matrix; Hardy function; Minkowski's inequality; homogeneous and symmetric polynomial; symmetric function

1 Introduction

We use the following notations throughout the paper (see [1]):

$$\mathbf{x} \triangleq (x_1,\ldots,x_n), \qquad \boldsymbol{\alpha} \triangleq (\alpha_1,\ldots,\alpha_n), \qquad |\boldsymbol{\alpha}| \triangleq \alpha_1+\cdots+\alpha_n,$$

$$\boldsymbol{\varepsilon} \triangleq (0,1,\ldots,n-1), \qquad \|\mathbf{x}\|_p \triangleq \left(\sum_{i=1}^n x_i^p\right)^{1/p}, \quad p>0.$$

Let $A=[a_{i,j}]_{n\times n}$ be an $n\times n$ matrix over a commutative ring. Then the *permanent* of the matrix A, written as $\operatorname{per} A$, is defined by

$$\operatorname{per} A \triangleq \sum_{\sigma\in S_n} a_{1,\sigma(1)}a_{2,\sigma(2)}\cdots a_{n,\sigma(n)},$$

where S_n is a symmetric group of n-order (see [2]). The matrix

$$H_n(\mathbf{x},\boldsymbol{\alpha}) \triangleq \left[x_j^{\alpha_i}\right]_{n\times n} = \begin{bmatrix} x_1^{\alpha_1} & x_2^{\alpha_1} & \cdots & x_n^{\alpha_1} \\ x_1^{\alpha_2} & x_2^{\alpha_2} & \cdots & x_n^{\alpha_2} \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{\alpha_n} & x_2^{\alpha_n} & \cdots & x_n^{\alpha_n} \end{bmatrix}$$

is called a *Hardy matrix*, the matrix functions $\operatorname{per} H_n(\mathbf{x}, \boldsymbol{\alpha})$ and $\det H_n(\mathbf{x}, \boldsymbol{\alpha})$ are called the *Hardy function* (see [2, 3]) and the *generalized Vandermonde determinant* (see [4, 5]), respectively.

Due to the facts that the symmetric polynomial and certain symmetric functions can be expressed by the Hardy function (see [6] and Remark 1), and that the interpolating quasi-polynomial can be expressed by the generalized Vandermonde determinant (see [4, 5]), the Hardy function and the generalized Vandermonde determinant are of great significance in mathematics.

Obviously, the Hardy function $\operatorname{per} H_n(\mathbf{x}, \boldsymbol{\alpha})$ is a symmetric function. For the Hardy function, we have the following well-known Hardy inequality (see[3, 7]): Let $\boldsymbol{\alpha}, \boldsymbol{\beta} \in (-\infty, \infty)^n$. Then the inequality

$$\operatorname{per} H_n(\mathbf{x}, \boldsymbol{\alpha}) \le \operatorname{per} H_n(\mathbf{x}, \boldsymbol{\beta}) \tag{1}$$

holds for any $\mathbf{x} \in (0, \infty)^n$ if and only if $\boldsymbol{\alpha} \prec \boldsymbol{\beta}$.

For the Hardy function $\operatorname{per} H_n(\mathbf{x}, \boldsymbol{\alpha})$, Wen and Wang in [2] (see Corollary 1 in [2]) obtained the following result: Let $\mathbf{x}, \mathbf{y} \in (0, \infty)^n$, $\boldsymbol{\alpha} \in (-\infty, \infty)^n$. If

$$x_1 \le x_2 \le \cdots \le x_n \quad \text{and} \quad y_1 \le y_2 \le \cdots \le y_n,$$

then

$$\frac{\operatorname{per} H_n(\mathbf{xy}, \boldsymbol{\alpha})}{n!} \ge \frac{\operatorname{per} H_n(\mathbf{x}, \boldsymbol{\alpha})}{n!} \times \frac{\operatorname{per} H_n(\mathbf{y}, \boldsymbol{\alpha})}{n!}, \tag{2}$$

where

$$\mathbf{xy} \triangleq (x_1 y_1, \ldots, x_n y_n).$$

For the generalized Vandermonde determinant $\det H_n(\mathbf{x}, \boldsymbol{\alpha})$, Wen and Cheng in [5] (see Lemma 3 in [5]) obtained the following result: Let $\mathbf{x}, \boldsymbol{\alpha} \in (0, \infty)^n$, $n \ge 2$. If

$$x_1 < x_2 < \cdots < x_n, \qquad \alpha_{j+1} - \alpha_j \ge 1, \quad j = 1, 2, \ldots, n-1,$$

then we have

$$\det H_n(\mathbf{x}, \boldsymbol{\alpha}) \le \left(\prod_{j=1}^{n-1} j! \right)^{-1} \det H_n(\boldsymbol{\alpha}, \boldsymbol{\varepsilon}) \det H_n(\mathbf{x}, \boldsymbol{\varepsilon}) \left(\frac{x_{n-1}^{d_n} + x_n^{d_n}}{2} \right)^{\frac{|\boldsymbol{\alpha}| - |\boldsymbol{\varepsilon}|}{d_n}}, \tag{3}$$

where

$$d_n \triangleq \max\{1, \alpha_n - \alpha_{n-1} - 1\}, \quad |\boldsymbol{\alpha}| > |\boldsymbol{\varepsilon}| = \frac{n(n-1)}{2}.$$

Famous Minkowski's inequality can be described as follows (see [8, 9]): If $0 < p < 1$, then for any $\mathbf{x}, \mathbf{y} \in (0, \infty)^n$, we have the inequality

$$\|\mathbf{x} + \mathbf{y}\|_p \ge \|\mathbf{x}\|_p + \|\mathbf{y}\|_p. \tag{4}$$

Inequality (4) is reversed if $p > 1$. Equality in (4) holds if and only if $\mathbf{x}$, $\mathbf{y}$ are linearly dependent.

Minkowski's inequality has a wide range of applications, especially in the algebraic geometry and space science (see [8–11]). In this paper, we establish the following Minkowski-type inequality (5) involving Hardy function.

Theorem 1 (Minkowski-type inequality) *Let* $\boldsymbol{\alpha} \in [0,1]^n$. *If* $0 < |\boldsymbol{\alpha}| \le 1$, *then for any* $\mathbf{x}, \mathbf{y} \in (0,\infty)^n$, *we have the following inequality*:

$$\left[\operatorname{per} H_n(\mathbf{x}+\mathbf{y},\boldsymbol{\alpha})\right]^{\frac{1}{|\boldsymbol{\alpha}|}} \ge \left[\operatorname{per} H_n(\mathbf{x},\boldsymbol{\alpha})\right]^{\frac{1}{|\boldsymbol{\alpha}|}} + \left[\operatorname{per} H_n(\mathbf{y},\boldsymbol{\alpha})\right]^{\frac{1}{|\boldsymbol{\alpha}|}}. \tag{5}$$

Equality in (5) *holds if* $\mathbf{x}$, $\mathbf{y}$ *are linearly dependent.*

In Section 3, we demonstrate the applications of Theorem 1. Our objective is to estimate the lower bounds of the increment of a symmetric function.

2 The proof of Theorem 1

In order to prove Theorem 1, we need the following lemmas.

Lemma 1 *If* $\boldsymbol{\alpha} \in [0,\infty)^2$, $|\boldsymbol{\alpha}| - (\alpha_1 - \alpha_2)^2 \ge 0$, $|\boldsymbol{\alpha}| > 0$, *then for any* $\mathbf{x}, \mathbf{y} \in (0,\infty)^2$, *we have the following Minkowski-type inequality*:

$$\left[\operatorname{per} H_2(\mathbf{x}+\mathbf{y},\boldsymbol{\alpha})\right]^{\frac{1}{|\boldsymbol{\alpha}|}} \ge \left[\operatorname{per} H_2(\mathbf{x},\boldsymbol{\alpha})\right]^{\frac{1}{|\boldsymbol{\alpha}|}} + \left[\operatorname{per} H_2(\mathbf{y},\boldsymbol{\alpha})\right]^{\frac{1}{|\boldsymbol{\alpha}|}}. \tag{6}$$

Equality in (6) *holds if* $\mathbf{x}$, $\mathbf{y}$ *are linearly dependent.*

Proof First of all, we consider the case

$$\boldsymbol{\alpha} \in (0,\infty)^2, \qquad |\boldsymbol{\alpha}| - (\alpha_1 - \alpha_2)^2 > 0.$$

Write $y_i/x_i = u_i$, $i = 1,2$. Then inequality (6) can be rewritten as

$$\begin{aligned}
&\left[x_1^{\alpha_1} x_2^{\alpha_2}(1+u_1)^{\alpha_1}(1+u_2)^{\alpha_2} + x_1^{\alpha_2} x_2^{\alpha_1}(1+u_1)^{\alpha_2}(1+u_2)^{\alpha_1}\right]^{\frac{1}{\alpha_1+\alpha_2}} \\
&\quad \ge \left(x_1^{\alpha_1} x_2^{\alpha_2} u_1^{\alpha_1} u_2^{\alpha_2} + x_1^{\alpha_2} x_2^{\alpha_1} u_1^{\alpha_2} u_2^{\alpha_1}\right)^{\frac{1}{\alpha_1+\alpha_2}} + \left(x_1^{\alpha_1} x_2^{\alpha_2} + x_1^{\alpha_2} x_2^{\alpha_1}\right)^{\frac{1}{\alpha_1+\alpha_2}}.
\end{aligned} \tag{7}$$

Without loss of generality, we can assume that

$$x_1^{\alpha_1} x_2^{\alpha_2} + x_1^{\alpha_2} x_2^{\alpha_1} = 1, \quad (x_1, x_2) \in (0,\infty)^2. \tag{8}$$

Indeed, if

$$x_1^{\alpha_1} x_2^{\alpha_2} + x_1^{\alpha_2} x_2^{\alpha_1} = C^{\alpha_1+\alpha_2}, \quad C > 0,$$

then

$$x_1^{*\alpha_1} x_2^{*\alpha_2} + x_1^{*\alpha_2} x_2^{*\alpha_1} = 1, \quad \left(x_1^*, x_2^*\right) \in (0,\infty)^2,$$

where

$$x_i^* = C^{-1}x_i, \quad i = 1,2.$$

Set

$$x_1^{\alpha_1}x_2^{\alpha_2}u_1^{\alpha_1}u_2^{\alpha_2} + x_1^{\alpha_2}x_2^{\alpha_1}u_1^{\alpha_2}u_2^{\alpha_1} = c^{\alpha_1+\alpha_2}, \quad c > 0, \tag{9}$$

and

$$F(u_1,u_2) \triangleq x_1^{\alpha_1}x_2^{\alpha_2}(1+u_1)^{\alpha_1}(1+u_2)^{\alpha_2} + x_1^{\alpha_2}x_2^{\alpha_1}(1+u_1)^{\alpha_2}(1+u_2)^{\alpha_1},$$
$$D \triangleq \left\{(u_1,u_2) \in (0,\infty)^2 | x_1^{\alpha_1}x_2^{\alpha_2}u_1^{\alpha_1}u_2^{\alpha_2} + x_1^{\alpha_2}x_2^{\alpha_1}u_1^{\alpha_2}u_2^{\alpha_1} = c^{\alpha_1+\alpha_2}\right\}.$$

We arbitrarily fixed x_1, x_2, which satisfies condition (8), then inequality (6) can be rewritten as

$$F(u_1,u_2) \geq (c+1)^{\alpha_1+\alpha_2}, \quad \forall (u_1,u_2) \in D. \tag{10}$$

We consider the following Lagrange function:

$$L = F(u_1,u_2) + \lambda\left(x_1^{\alpha_1}x_2^{\alpha_2}u_1^{\alpha_1}u_2^{\alpha_2} + x_1^{\alpha_2}x_2^{\alpha_1}u_1^{\alpha_2}u_2^{\alpha_1} - c^{\alpha_1+\alpha_2}\right).$$

Set

$$\begin{aligned}\frac{\partial L}{\partial u_1} &= \alpha_1 x_1^{\alpha_1}x_2^{\alpha_2}(1+u_1)^{\alpha_1-1}(1+u_2)^{\alpha_2} + \alpha_2 x_1^{\alpha_2}x_2^{\alpha_1}(1+u_1)^{\alpha_2-1}(1+u_2)^{\alpha_1}\\ &\quad + \lambda\left(\alpha_1 x_1^{\alpha_1}x_2^{\alpha_2}u_1^{\alpha_1-1}u_2^{\alpha_2} + \alpha_2 x_1^{\alpha_2}x_2^{\alpha_1}u_1^{\alpha_2-1}u_2^{\alpha_1}\right) = 0,\end{aligned} \tag{11}$$

and

$$\begin{aligned}\frac{\partial L}{\partial u_2} &= \alpha_2 x_1^{\alpha_1}x_2^{\alpha_2}(1+u_1)^{\alpha_1}(1+u_2)^{\alpha_2-1} + \alpha_1 x_1^{\alpha_2}x_2^{\alpha_1}(1+u_1)^{\alpha_2}(1+u_2)^{\alpha_1-1}\\ &\quad + \lambda\left(\alpha_2 x_1^{\alpha_1}x_2^{\alpha_2}u_1^{\alpha_1}u_2^{\alpha_2-1} + \alpha_1 x_1^{\alpha_2}x_2^{\alpha_1}u_1^{\alpha_2}u_2^{\alpha_1-1}\right) = 0.\end{aligned} \tag{12}$$

From (11) and (12), we get

$$\begin{aligned}&\frac{\alpha_1 x_1^{\alpha_1}x_2^{\alpha_2}(1+u_1)^{\alpha_1-1}(1+u_2)^{\alpha_2} + \alpha_2 x_1^{\alpha_2}x_2^{\alpha_1}(1+u_1)^{\alpha_2-1}(1+u_2)^{\alpha_1}}{\alpha_2 x_1^{\alpha_1}x_2^{\alpha_2}(1+u_1)^{\alpha_1}(1+u_2)^{\alpha_2-1} + \alpha_1 x_1^{\alpha_2}x_2^{\alpha_1}(1+u_1)^{\alpha_2}(1+u_2)^{\alpha_1-1}}\\ &\quad = \frac{\alpha_1 x_1^{\alpha_1}x_2^{\alpha_2}u_1^{\alpha_1-1}u_2^{\alpha_2} + \alpha_2 x_1^{\alpha_2}x_2^{\alpha_1}u_1^{\alpha_2-1}u_2^{\alpha_1}}{\alpha_2 x_1^{\alpha_1}x_2^{\alpha_2}u_1^{\alpha_1}u_2^{\alpha_2-1} + \alpha_1 x_1^{\alpha_2}x_2^{\alpha_1}u_1^{\alpha_2}u_2^{\alpha_1-1}}.\end{aligned} \tag{13}$$

Write

$$\mu \triangleq \left(\frac{x_2}{x_1}\right)^{\alpha_1-\alpha_2} > 0, \qquad g(t) \triangleq t\frac{\alpha_1 + \mu\alpha_2 t^{\alpha_1-\alpha_2}}{\alpha_2 + \mu\alpha_1 t^{\alpha_1-\alpha_2}}, \quad t > 0. \tag{14}$$

Since

$$\frac{\alpha_1 x_1^{\alpha_1} x_2^{\alpha_2} u_1^{\alpha_1-1} u_2^{\alpha_2} + \alpha_2 x_1^{\alpha_2} x_2^{\alpha_1} u_1^{\alpha_2-1} u_2^{\alpha_1}}{\alpha_2 x_1^{\alpha_1} x_2^{\alpha_2} u_1^{\alpha_1} u_2^{\alpha_2-1} + \alpha_1 x_1^{\alpha_2} x_2^{\alpha_1} u_1^{\alpha_2} u_2^{\alpha_1-1}} = \frac{u_2}{u_1} \frac{\alpha_1 + \alpha_2 (\frac{x_2}{x_1})^{\alpha_1-\alpha_2} (\frac{u_2}{u_1})^{\alpha_1-\alpha_2}}{\alpha_2 + \alpha_1 (\frac{x_2}{x_1})^{\alpha_1-\alpha_2} (\frac{u_2}{u_1})^{\alpha_1-\alpha_2}},$$

equation (13) can be rewritten as

$$g\left(\frac{1+u_2}{1+u_1}\right) = g\left(\frac{u_2}{u_1}\right). \tag{15}$$

By

$$\log\bigl[g(t)\bigr] = \log(t) + \log\bigl(\alpha_1 + \mu\alpha_2 t^{\alpha_1-\alpha_2}\bigr) - \log\bigl(\alpha_2 + \mu\alpha_1 t^{\alpha_1-\alpha_2}\bigr),$$

we get

$$\begin{aligned}
\frac{g'(t)}{g(t)} &= \frac{1}{t} + \frac{(\alpha_1-\alpha_2)\mu\alpha_2 t^{\alpha_1-\alpha_2-1}}{\alpha_1 + \mu\alpha_2 t^{\alpha_1-\alpha_2}} - \frac{(\alpha_1-\alpha_2)\mu\alpha_1 t^{\alpha_1-\alpha_2-1}}{\alpha_2 + \mu\alpha_1 t^{\alpha_1-\alpha_2}} \\
&= \frac{\alpha_1\alpha_2[1 + (\mu t^{\alpha_1-\alpha_2})^2] + [\alpha_1^2 + \alpha_2^2 + (\alpha_1-\alpha_2)(\alpha_2^2 - \alpha_1^2)]\mu t^{\alpha_1-\alpha_2}}{t(\alpha_1 + \mu\alpha_2 t^{\alpha_1-\alpha_2})(\alpha_2 + \mu\alpha_1 t^{\alpha_1-\alpha_2})} \\
&\geq \frac{2\alpha_1\alpha_2\mu t^{\alpha_1-\alpha_2} + [\alpha_1^2 + \alpha_2^2 + (\alpha_1-\alpha_2)(\alpha_2^2 - \alpha_1^2)]\mu t^{\alpha_1-\alpha_2}}{t(\alpha_1 + \mu\alpha_2 t^{\alpha_1-\alpha_2})(\alpha_2 + \mu\alpha_1 t^{\alpha_1-\alpha_2})} \\
&= \frac{(\alpha_1+\alpha_2)[\alpha_1 + \alpha_2 - (\alpha_1-\alpha_2)^2]}{t(\alpha_1 + \mu\alpha_2 t^{\alpha_1-\alpha_2})(\alpha_2 + \mu\alpha_1 t^{\alpha_1-\alpha_2})}\mu t^{\alpha_1-\alpha_2} \\
&> 0,
\end{aligned}$$

hence

$$g'(t) > 0, \quad \forall t > 0. \tag{16}$$

By (15) and (16), we get

$$\frac{1+u_2}{1+u_1} = \frac{u_2}{u_1}. \tag{17}$$

By (16), (17), (8) and (9), we get

$$u_1 = u_2 = c. \tag{18}$$

According to the theory of mathematical analysis, we just need to prove that inequality (10) holds for a stationary point (c, c) of $F(u_1, u_2)$ and boundary points of D.

If $(u_1, u_2) = (c, c) \in D$ is a stationary point of $F(u_1, u_2)$, then equality in (10) holds. Here we assume that (u_1, u_2) is a boundary point of D. Then we have $(u_1, u_2) = (0, \infty)$ or $(u_1, u_2) = (\infty, 0)$. Since

$$F(u_1, u_2) = \infty > (c+1)^{\alpha_1+\alpha_2},$$

inequality (10) also holds. So we have proved inequalities (7) and (6).

Next, note the continuity of both sides of (6) for the variable $\boldsymbol{\alpha}$, hence inequality (6) also holds if

$$\boldsymbol{\alpha} \in [0,\infty)^2, \qquad |\boldsymbol{\alpha}| - (\alpha_1 - \alpha_2)^2 = 0, \qquad |\boldsymbol{\alpha}| > 0.$$

From the above analysis we know that equality in (6) holds if $u_1 = u_2$, *i.e.*, $\mathbf{x}$, $\mathbf{y}$ are linearly dependent. This completes the proof of Lemma 1. □

Lemma 2 *If* $\boldsymbol{\alpha} \in (0,1)^2$ *and* $0 < |\boldsymbol{\alpha}| < 1$, *then for any* $\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w} \in (0,\infty)^n$, $n \geq 1$, *we have the inequality*

$$\left[\sum_{i=1}^{n} (x_i + y_i)^{\alpha_1} (z_i + w_i)^{\alpha_2}\right]^{\frac{1}{|\boldsymbol{\alpha}|}} \geq \left(\sum_{i=1}^{n} x_i^{\alpha_1} z_i^{\alpha_2}\right)^{\frac{1}{|\boldsymbol{\alpha}|}} + \left(\sum_{i=1}^{n} y_i^{\alpha_1} w_i^{\alpha_2}\right)^{\frac{1}{|\boldsymbol{\alpha}|}}. \tag{19}$$

Equation in (19) *holds if and only if*

$$\frac{x_1}{y_1} = \frac{x_2}{y_2} = \cdots = \frac{x_n}{y_n} = \frac{z_1}{w_1} = \frac{z_2}{w_2} = \cdots = \frac{z_n}{w_n}. \tag{20}$$

Proof Write

$$\frac{y_i}{x_i} = u_i, \qquad \frac{w_i}{z_i} = v_i, \quad i = 1, 2.$$

Then inequality (19) can be rewritten as

$$\left[\sum_{i=1}^{n} x_i^{\alpha_1} z_i^{\alpha_2} (1 + u_i)^{\alpha_1} (1 + v_i)^{\alpha_2}\right]^{\frac{1}{|\boldsymbol{\alpha}|}} \geq \left(\sum_{i=1}^{n} x_i^{\alpha_1} z_i^{\alpha_2} u_i^{\alpha_1} v_i^{\alpha_2}\right)^{\frac{1}{|\boldsymbol{\alpha}|}} + \left(\sum_{i=1}^{n} x_i^{\alpha_1} z_i^{\alpha_2}\right)^{\frac{1}{|\boldsymbol{\alpha}|}}. \tag{21}$$

Without loss of generality, we can assume that

$$\sum_{i=1}^{n} x_i^{\alpha_1} z_i^{\alpha_2} = 1, \quad \mathbf{x}, \mathbf{z} \in (0,\infty)^n, \tag{22}$$

and

$$\sum_{i=1}^{n} x_i^{\alpha_1} z_i^{\alpha_2} u_i^{\alpha_1} v_i^{\alpha_2} = c^{\alpha_1 + \alpha_2}, \quad c > 0. \tag{23}$$

Write

$$G(u,v) \triangleq \sum_{i=1}^{n} x_i^{\alpha_1} z_i^{\alpha_2} (1 + u_i)^{\alpha_1} (1 + v_i)^{\alpha_2}, \quad u, v \in (0,\infty)^n,$$

and

$$D_* \triangleq \left\{ (u,v) \in (0,\infty)^{2n} \,\middle|\, \sum_{i=1}^{n} x_i^{\alpha_1} z_i^{\alpha_2} u_i^{\alpha_1} v_i^{\alpha_2} = c^{\alpha_1 + \alpha_2} \right\}.$$

Then inequality (21) can be rewritten as

$$G(u,v) \geq (c+1)^{\alpha_1+\alpha_2}, \quad \forall (u,v) \in D_*. \tag{24}$$

We define the following Lagrange function:

$$L = G(u,v) + \lambda\left(\sum_{i=1}^{n} x_i^{\alpha_1} z_i^{\alpha_2} u_i^{\alpha_1} v_i^{\alpha_2} - c^{\alpha_1+\alpha_2}\right).$$

Set

$$\frac{\partial L}{\partial u_k} = \alpha_1 x_k^{\alpha_1} z_k^{\alpha_2} (1+u_k)^{\alpha_1-1}(1+v_k)^{\alpha_2} + \lambda \alpha_1 x_k^{\alpha_1} z_k^{\alpha_2} u_k^{\alpha_1-1} v_k^{\alpha_2} = 0, \quad k = 1,2,\ldots,n, \tag{25}$$

and

$$\frac{\partial L}{\partial v_k} = \alpha_2 x_k^{\alpha_1} z_k^{\alpha_2} (1+u_k)^{\alpha_1}(1+v_k)^{\alpha_2-1} + \lambda \alpha_2 x_k^{\alpha_1} z_k^{\alpha_2} u_k^{\alpha_1} v_k^{\alpha_2-1} = 0, \quad k = 1,2,\ldots,n. \tag{26}$$

Then equations (25) and (26) can be rewritten as

$$\alpha_1 x_k^{\alpha_1} z_k^{\alpha_2} (1+u_k)^{\alpha_1-1}(1+v_k)^{\alpha_2} = -\lambda \alpha_1 x_k^{\alpha_1} z_k^{\alpha_2} u_k^{\alpha_1-1} v_k^{\alpha_2}, \quad k = 1,2,\ldots,n, \tag{27}$$

and

$$\alpha_2 x_k^{\alpha_1} z_k^{\alpha_2} (1+u_k)^{\alpha_1}(1+v_k)^{\alpha_2-1} = -\lambda \alpha_2 x_k^{\alpha_1} z_k^{\alpha_2} u_k^{\alpha_1} v_k^{\alpha_2-1}, \quad k = 1,2,\ldots,n, \tag{28}$$

respectively. From (27) divided by (28), we get

$$\frac{1+v_k}{1+u_k} = \frac{v_k}{u_k} \quad \Leftrightarrow \quad u_k = v_k, \quad k = 1,2,\ldots,n. \tag{29}$$

From (29) and (27), we get

$$\left(1+u_k^{-1}\right)^{\alpha_1+\alpha_2-1} = -\lambda, \quad k = 1,2,\ldots,n. \tag{30}$$

By (30) and $\alpha_1 + \alpha_2 - 1 = |\boldsymbol{\alpha}| - 1 < 0$, we get

$$u_1 = u_2 = \cdots = u_n. \tag{31}$$

From (31), (29), (22) and (23), we get

$$u_1 = u_2 = \cdots = u_n = v_1 = v_2 = \cdots = v_n = c. \tag{32}$$

That is to say, the function $G(u,v)$ has a unique stationary point $(c,\ldots,c,c,\ldots,c)$ in D_*.

Next, we use the mathematical induction to prove that inequality (24) holds as follows.

According to the theory of mathematical analysis, we only need to prove that inequality (24) holds for a stationary point $(c,\ldots,c,c,\ldots,c)$ of $G(u,v)$ and boundary points of D_*. To complete our proof, we need to divide it into two steps (A) and (B).

(A) Let $n = 1$. If (c, c) is a stationary point of $G(u, v)$ in D_*, then equality in (24) holds. Here we assume that (u, v) is a boundary point of D_*. From (23) we know that $(u_1, v_1) = (0, \infty)$ or $(u_1, v_1) = (\infty, 0)$. Hence

$$G(u,v) = x_1^{\alpha_1} z_1^{\alpha_2} (1 + u_1)^{\alpha_1} (1 + v_1)^{\alpha_2} = \infty > c^{\alpha_1+\alpha_2}.$$

That is to say, inequality (24) also holds. According to the theory of mathematical analysis, inequality (24) is proved.

Let $n = 2$. If (c, c, c, c) is a stationary point of $G(u, v)$ in D_*, then equality in (24) holds. Here we assume that (u, v) is a boundary point of D_*, then there is a 0 among u_1, u_2, v_1, v_2. Without loss of generality, we can assume that $u_2 = 0$. From (23) we have

$$x_1^{\alpha_1} z_1^{\alpha_2} u_1^{\alpha_1} v_1^{\alpha_2} = c^{\alpha_1+\alpha_2}, \quad c > 0. \tag{33}$$

By (33), we get

$$\begin{aligned} G(u,v) &= x_1^{\alpha_1} z_1^{\alpha_2} (1 + u_1)^{\alpha_1} (1 + v_1)^{\alpha_2} + x_2^{\alpha_1} z_2^{\alpha_2} (1 + u_2)^{\alpha_1} (1 + v_2)^{\alpha_2} \\ &= x_1^{\alpha_1} z_1^{\alpha_2} (1 + u_1)^{\alpha_1} (1 + v_1)^{\alpha_2} + x_2^{\alpha_1} z_2^{\alpha_2} (1 + v_2)^{\alpha_2} \\ &> x_1^{\alpha_1} z_1^{\alpha_2} (1 + u_1)^{\alpha_1} (1 + v_1)^{\alpha_2} \\ &> x_1^{\alpha_1} z_1^{\alpha_2} u_1^{\alpha_1} v_1^{\alpha_2} \\ &= c^{\alpha_1+\alpha_2}. \end{aligned}$$

That is to say, inequality (24) still holds for the case when (u, v) is a boundary point of D_*. According to the theory of mathematical analysis, we know that inequality (24) is proved.

(B) Suppose that inequality (24) holds if we use $n - 1$ ($n \geq 3$) instead of n, we prove that inequality (24) holds as follows.

For a stationary point $(c, \ldots, c, c, \ldots, c)$ of $G(u, v)$ in D_*, equation in (24) holds. Here we assume that (u, v) is a boundary point of D_*, then there is a 0 among $u_1, u_2, \ldots, u_n$, $v_1, v_2, \ldots, v_n$. Without loss of generality, we can assume that $u_n = 0$. From (23), we get

$$\sum_{i=1}^{n-1} x_i^{\alpha_1} z_i^{\alpha_2} u_i^{\alpha_1} v_i^{\alpha_2} = c^{\alpha_1+\alpha_2}, \quad c > 0. \tag{34}$$

Set

$$x_i^* = x_i \left(1 - x_n^{\alpha_1} z_n^{\alpha_2}\right)^{-\frac{1}{\alpha_1+\alpha_2}}, \qquad z_i^* = z_i \left(1 - x_n^{\alpha_1} z_n^{\alpha_2}\right)^{-\frac{1}{\alpha_1+\alpha_2}}, \quad i = 1, 2, \ldots, n-1,$$

then equation (22) can be rewritten as

$$\sum_{i=1}^{n-1} x_i^{*\alpha_1} z_i^{*\alpha_2} = 1, \quad x^*, z^* \in (0, \infty)^{n-1}, \tag{35}$$

and equation (23) can be rewritten as

$$\sum_{i=1}^{n-1} x_i^{*\alpha_1} z_i^{*\alpha_2} u_i^{\alpha_1} v_i^{\alpha_2} = c_*^{\alpha_1+\alpha_2}, \quad c_* = c\left(1 - x_n^{\alpha_1} z_n^{\alpha_2}\right)^{-\frac{1}{\alpha_1+\alpha_2}} > 0, \tag{36}$$

as well as the function $G(u,v)$ can be rewritten as

$$G(u,v) = \left(1 - x_n^{\alpha_1} z_n^{\alpha_2}\right) \sum_{i=1}^{n-1} x_i^{*\alpha_1} z_i^{*\alpha_2} (1+u_i)^{\alpha_1} (1+v_i)^{\alpha_2} + x_n^{\alpha_1} z_n^{\alpha_2} (1+v_n)^{\alpha_2}. \tag{37}$$

By the induction hypothesis we have

$$\sum_{i=1}^{n-1} x_i^{*\alpha_1} z_i^{*\alpha_2} (1+u_i)^{\alpha_1} (1+v_i)^{\alpha_2} \geq (c_* + 1)^{\alpha_1+\alpha_2}. \tag{38}$$

By (37) and (38), we get

$$\begin{aligned}
G(u,v) &= \left(1 - x_n^{\alpha_1} z_n^{\alpha_2}\right) \sum_{i=1}^{n-1} x_i^{*\alpha_1} z_i^{*\alpha_2} (1+u_i)^{\alpha_1} (1+v_i)^{\alpha_2} + x_n^{\alpha_1} z_n^{\alpha_2} (1+v_n)^{\alpha_2} \\
&\geq \left(1 - x_n^{\alpha_1} z_n^{\alpha_2}\right) \sum_{i=1}^{n-1} x_i^{*\alpha_1} z_i^{*\alpha_2} (1+u_i)^{\alpha_1} (1+v_i)^{\alpha_2} + x_n^{\alpha_1} z_n^{\alpha_2} \\
&\geq \left(1 - x_n^{\alpha_1} z_n^{\alpha_2}\right) (c_* + 1)^{\alpha_1+\alpha_2} + x_n^{\alpha_1} z_n^{\alpha_2} \\
&= \left(1 - x_n^{\alpha_1} z_n^{\alpha_2}\right) \left[c\left(1 - x_n^{\alpha_1} z_n^{\alpha_2}\right)^{-\frac{1}{\alpha_1+\alpha_2}} + 1\right]^{\alpha_1+\alpha_2} + x_n^{\alpha_1} z_n^{\alpha_2} \\
&= \left[c + \left(1 - x_n^{\alpha_1} z_n^{\alpha_2}\right)^{\frac{1}{\alpha_1+\alpha_2}}\right]^{\alpha_1+\alpha_2} + x_n^{\alpha_1} z_n^{\alpha_2},
\end{aligned}$$

i.e.,

$$G(u,v) \geq \varphi(\delta), \tag{39}$$

where

$$\delta \triangleq \left(1 - x_n^{\alpha_1} z_n^{\alpha_2}\right)^{\frac{1}{\alpha_1+\alpha_2}} \in (0,1), \quad x_n^{\alpha_1} z_n^{\alpha_2} = 1 - \delta^{\alpha_1+\alpha_2},$$

and

$$\varphi(\delta) \triangleq (c+\delta)^{\alpha_1+\alpha_2} + 1 - \delta^{\alpha_1+\alpha_2}, \quad \delta \in (0,1). \tag{40}$$

Since

$$\alpha_1 + \alpha_2 > 0, \qquad \alpha_1 + \alpha_2 - 1 = |\alpha| - 1 < 0, \qquad c > 0,$$

we have

$$\frac{d\varphi(\delta)}{d\delta} \triangleq (\alpha_1+\alpha_2)\left[(c+\delta)^{\alpha_1+\alpha_2-1} - \delta^{\alpha_1+\alpha_2-1}\right] < 0, \quad \delta \in (0,1). \tag{41}$$

From $\delta \in (0,1)$ and (41), we get

$$\varphi(\delta) \triangleq (c+\delta)^{\alpha_1+\alpha_2} + 1 - \delta^{\alpha_1+\alpha_2} > \varphi(1) = (c+1)^{\alpha_1+\alpha_2}. \tag{42}$$

Combining with inequalities (39) and (42), we get

$$G(u,v) \geq \varphi(\delta) > (c+1)^{\alpha_1+\alpha_2}. \tag{43}$$

By inequality (43) we know that inequality (24) holds.

According to the theory of mathematical analysis, we know that inequality (24) is proved, hence inequality (19) is also proved by the above analysis. Inequality (19) is an equation if and only if equations (20) hold. This completes the proof of Lemma 2. □

Next we turn to the proof of Theorem 1.

Proof First of all, we prove that inequality (5) holds if $\boldsymbol{\alpha} \in (0,1)^n$ and $0 < |\boldsymbol{\alpha}| < 1$ by induction for n. To complete our proof, we need to divide it into two steps (A) and (B).

(A) When $n = 1$, then inequality (5) is an equation. Let $n = 2$. According to the hypothesis of Theorem 1, we know that

$$|\alpha_1 - \alpha_2| < |\alpha_1 + \alpha_2| < 1, \qquad \alpha_1 + \alpha_2 - (\alpha_1 - \alpha_2)^2 > \alpha_1 + \alpha_2 - (\alpha_1 + \alpha_2)^2 > 0.$$

By Lemma 1, inequality (5) holds.

(B) Suppose that inequality (5) holds if we use $n-1$ ($n \geq 3$) instead of n, we prove that inequality (5) holds as follows.

For convenience, we use the following notations:

$$\mathbf{x}(j) = (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n), \qquad \boldsymbol{\alpha}(n) = (\alpha_1, \alpha_2, \ldots, \alpha_{n-1}).$$

According to the Laplace theorem (see [2]), we obtain that

$$\operatorname{per} H_n(\mathbf{x}, \boldsymbol{\alpha}) = \sum_{j=1}^{n} x_j^{\alpha_n} \operatorname{per} H_{n-1}\bigl(\mathbf{x}(j), \boldsymbol{\alpha}(n)\bigr). \tag{44}$$

Write

$$z_j \triangleq \bigl(\operatorname{per} H_{n-1}\bigl(\mathbf{x}(j), \boldsymbol{\alpha}(n)\bigr)\bigr)^{\frac{1}{|\boldsymbol{\alpha}(n)|}} \quad \text{and} \quad w_j \triangleq \bigl(\operatorname{per} H_{n-1}\bigl(\mathbf{y}(j), \boldsymbol{\alpha}(n)\bigr)\bigr)^{\frac{1}{|\boldsymbol{\alpha}(n)|}}.$$

By

$$0 < \bigl|\boldsymbol{\alpha}(n)\bigr| < \alpha_n + \bigl|\boldsymbol{\alpha}(n)\bigr| = |\boldsymbol{\alpha}| < 1,$$

(44), the induction hypothesis and Lemma 2, we get

$$\begin{aligned}
&\bigl[\operatorname{per} H_n(\mathbf{x}+\mathbf{y}, \boldsymbol{\alpha})\bigr]^{\frac{1}{|\boldsymbol{\alpha}|}} \\
&\quad = \left\{ \sum_{j=1}^{n} (x_j + y_j)^{\alpha_n} \bigl[\operatorname{per} H_{n-1}\bigl(\mathbf{x}(j) + \mathbf{y}(j), \boldsymbol{\alpha}(n)\bigr)\bigr]^{\frac{|\boldsymbol{\alpha}(n)|}{|\boldsymbol{\alpha}(n)|}} \right\}^{\frac{1}{|\boldsymbol{\alpha}|}} \\
&\quad \geq \left[\sum_{j=1}^{n} (x_j + y_j)^{\alpha_n} (z_j + w_j)^{|\boldsymbol{\alpha}(n)|} \right]^{\frac{1}{|\boldsymbol{\alpha}|}}
\end{aligned}$$

$$\geq \left[\sum_{i=1}^{n} x_i^{\alpha_n} \operatorname{per} H_{n-1}\big(\mathbf{x}(j), \boldsymbol{\alpha}(n)\big)\right]^{\frac{1}{|\boldsymbol{\alpha}|}} + \left[\sum_{i=1}^{n} y_i^{\alpha_n} \operatorname{per} H_{n-1}\big(\mathbf{y}(j), \boldsymbol{\alpha}(n)\big)\right]^{\frac{1}{|\boldsymbol{\alpha}|}}$$
$$= \big[H_n(\mathbf{x}, \boldsymbol{\alpha})\big]^{\frac{1}{|\boldsymbol{\alpha}|}} + \big[\operatorname{per} H_n(\mathbf{y}, \boldsymbol{\alpha})\big]^{\frac{1}{|\boldsymbol{\alpha}|}}.$$

That is to say, inequality (5) holds.

According to the theory of mathematical induction, inequality (5) is proved.

Next, note the continuity of both sides of (5) for the variable $\boldsymbol{\alpha}$. We know that inequality (5) also holds for the case

$$\boldsymbol{\alpha} \in [0,1]^n, \qquad 0 < |\boldsymbol{\alpha}| \leq 1.$$

From the above analysis we know that equality in (5) holds if $\mathbf{x}$, $\mathbf{y}$ are linearly dependent.

This completes the proof of Theorem 1. □

3 Applications in the theory of symmetric function

We use the following notations in this section (see [2, 6, 12, 13]):

$$\mathbb{N} \triangleq \{0,1,2,\ldots\}, \qquad \mathscr{B}_k^+ \triangleq \big\{\boldsymbol{\alpha} \in \mathbb{N}^n \,\big|\, |\boldsymbol{\alpha}| = k, k \in \mathbb{N}\big\},$$
$$\overline{P}_{k,n}[\mathbf{x}] = \left\{\sum_{\boldsymbol{\alpha}\in\mathscr{B}_k^+} \frac{\lambda(\boldsymbol{\alpha})}{n!} \operatorname{per} H_n(\mathbf{x}, \boldsymbol{\alpha}) \,\middle|\, \lambda : \mathscr{B}_k^+ \to (-\infty,\infty)\right\} \Big\backslash \{0\},$$
$$\overline{P}^+_{k,n}[\mathbf{x}] = \left\{\sum_{\boldsymbol{\alpha}\in\mathscr{B}_k^+} \frac{\lambda(\boldsymbol{\alpha})}{n!} \operatorname{per} H_n(\mathbf{x}, \boldsymbol{\alpha}) \,\middle|\, \lambda : \mathscr{B}_k^+ \to [0,\infty)\right\} \Big\backslash \{0\},$$
$$\sqrt[k]{\mathbf{x}} \triangleq \big(\sqrt[k]{x_1}, \ldots, \sqrt[k]{x_n}\big), \qquad a - \mathbf{x} = (a - x_1, \ldots, a - x_n),$$
$$G_n(\mathbf{x}) \triangleq \sqrt[n]{x_1 \cdots x_n}, \qquad \mathbf{I}_n \triangleq (1,\ldots,1) \in (-\infty,\infty)^n,$$
$$\mathbf{O}_n \triangleq (0,\ldots,0) \in (-\infty,\infty)^n, \qquad f_1'(\mathbf{I}_n) \triangleq \frac{\partial f(\mathbf{x})}{\partial x_1}\Big|_{\mathbf{x}=\mathbf{I}_n}.$$

If $f(\mathbf{x}) \in \overline{P}_{k,n}[\mathbf{x}]$, then we call $f(\mathbf{x})$ a k-degree *homogeneous and symmetric polynomial* (see [6]). Obviously, we have that

$$\overline{P}^+_{k,n}[\mathbf{x}] \subset \overline{P}_{k,n}[\mathbf{x}].$$

Theorem 1 implies the following result.

Theorem 2 *Let* $f(\mathbf{x}) \in \overline{P}^+_{k,n}[\mathbf{x}]$, $k \geq 2$. *Then, for any* $\mathbf{x}, \mathbf{y} \in (0,\infty)^n$, *we have the following Minkowski-type inequality*:

$$f\big(\sqrt[k]{\mathbf{x}+\mathbf{y}}\big) \geq f\big(\sqrt[k]{\mathbf{x}}\big) + f\big(\sqrt[k]{\mathbf{y}}\big). \tag{45}$$

Equality in (45) *holds if* $\mathbf{x}$, $\mathbf{y}$ *are linearly dependent*.

Proof If $n = 1$, then inequality (45) is an equation. We suppose that $n \geq 2$ below.

Note that

$$\frac{\boldsymbol{\alpha}}{k} \in [0,1]^n, \qquad 0 < \left|\frac{\boldsymbol{\alpha}}{k}\right| = 1 \le 1, \quad \forall \boldsymbol{\alpha} \in \mathscr{B}_k^+.$$

According to Theorem 1, we get

$$\begin{aligned} f\left(\sqrt[k]{\mathbf{x}+\mathbf{y}}\right) &= \sum_{\boldsymbol{\alpha}\in\mathscr{B}_k^+} \frac{\lambda(\boldsymbol{\alpha})}{n!} \operatorname{per} H_n\left(\sqrt[k]{\mathbf{x}+\mathbf{y}}, \boldsymbol{\alpha}\right) \\ &= \sum_{\boldsymbol{\alpha}\in\mathscr{B}_k^+} \frac{\lambda(\boldsymbol{\alpha})}{n!} \left[\operatorname{per} H_n\left(\mathbf{x}+\mathbf{y}, \frac{\boldsymbol{\alpha}}{k}\right)\right]^{\frac{1}{|\frac{\boldsymbol{\alpha}}{k}|}} \\ &\ge \sum_{\boldsymbol{\alpha}\in\mathscr{B}_k^+} \frac{\lambda(\boldsymbol{\alpha})}{n!} \left\{\left[\operatorname{per} H_n\left(\mathbf{x}, \frac{\boldsymbol{\alpha}}{k}\right)\right]^{\frac{1}{|\frac{\boldsymbol{\alpha}}{k}|}} + \left[\operatorname{per} H_n\left(\mathbf{y}, \frac{\boldsymbol{\alpha}}{k}\right)\right]^{\frac{1}{|\frac{\boldsymbol{\alpha}}{k}|}}\right\} \\ &= \sum_{\boldsymbol{\alpha}\in\mathscr{B}_k^+} \frac{\lambda(\boldsymbol{\alpha})}{n!} \left[\operatorname{per} H_n\left(\mathbf{x}, \frac{\boldsymbol{\alpha}}{k}\right) + \operatorname{per} H_n\left(\mathbf{y}, \frac{\boldsymbol{\alpha}}{k}\right)\right] \\ &= \sum_{\boldsymbol{\alpha}\in\mathscr{B}_k^+} \frac{\lambda(\boldsymbol{\alpha})}{n!} \operatorname{per} H_n\left(\sqrt[k]{\mathbf{x}}, \boldsymbol{\alpha}\right) + \sum_{\boldsymbol{\alpha}\in\mathscr{B}_k^+} \frac{\lambda(\boldsymbol{\alpha})}{n!} \operatorname{per} H_n\left(\sqrt[k]{\mathbf{y}}, \boldsymbol{\alpha}\right) \\ &= f\left(\sqrt[k]{\mathbf{x}}\right) + f\left(\sqrt[k]{\mathbf{y}}\right), \end{aligned}$$

that is to say, inequality (45) is proved. Equality in (45) holds if $\mathbf{x}$, $\mathbf{y}$ are linearly dependent by Theorem 1.

The proof of Theorem 2 is completed. □

Theorem 1 also contains the following result.

Theorem 3 *Let* $f : [0,a)^n \to [0,\infty)$ *be a symmetric function, and* $f(\mathbf{x})$ *can be expressed as a convergent Taylor series*:

$$f(\mathbf{x}) = \sum_{k=0}^{\infty} \sum_{\boldsymbol{\alpha}\in\mathscr{B}_k^+} \frac{\lambda(\boldsymbol{\alpha})}{n!} \operatorname{per} H_n(\mathbf{x}, \boldsymbol{\alpha}), \quad \forall \mathbf{x} \in [0,a)^n, \tag{46}$$

where

$$a > 1, \quad \lambda(\boldsymbol{\alpha}) \triangleq \frac{1}{\alpha_1!\alpha_2!\cdots\alpha_n!} \frac{\partial^k f(\mathbf{x})}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\cdots\partial x_n^{\alpha_n}}\bigg|_{x=\mathbf{O}_n} \ge 0, \quad \forall k \in \mathbb{N}, \forall \boldsymbol{\alpha} \in \mathscr{B}_k.$$

Then, for any $\mathbf{x}, \mathbf{y}, \mathbf{x}+\mathbf{y} \in (0,a)^n$, *we have the following inequality*:

$$\left[\frac{f(\mathbf{x}+\mathbf{y})}{f(\mathbf{I}_n)}\right]^{\frac{f(\mathbf{I}_n)}{nf_1'(\mathbf{I}_n)}} \ge G_n(\mathbf{x}) + G_n(\mathbf{y}). \tag{47}$$

Equality in (47) *holds if there is a real* $\theta \in (0,1)$ *such that*

$$\mathbf{x} = \theta\mathbf{I}_n \quad \textit{and} \quad \mathbf{y} = (1-\theta)\mathbf{I}_n,$$

or $f(\mathbf{x}) = x_1x_2\cdots x_n$ *and* $\mathbf{x}$, $\mathbf{y}$ *are linearly dependent.*

Proof Obviously, we have that

$$\sum_{k=0}^{\infty}\sum_{\boldsymbol{\alpha}\in\mathcal{B}_k^+}\lambda(\boldsymbol{\alpha}) = f(\mathbf{I}_n). \tag{48}$$

Here we show that

$$\sum_{k=0}^{\infty}\sum_{\boldsymbol{\alpha}\in\mathcal{B}_k^+}|\boldsymbol{\alpha}|\lambda(\boldsymbol{\alpha}) = nf_1'(\mathbf{I}_n). \tag{49}$$

Note the following identities:

$$\begin{aligned} f_1'(\mathbf{x}) &= \frac{\partial f(\mathbf{x})}{\partial x_1} \\ &= \frac{\partial}{\partial x_1}\sum_{k=0}^{\infty}\sum_{\boldsymbol{\alpha}\in\mathcal{B}_k^+}\lambda(\boldsymbol{\alpha})\frac{\operatorname{per} H_n(\mathbf{x},\boldsymbol{\alpha})}{n!} \\ &= \sum_{k=0}^{\infty}\sum_{\boldsymbol{\alpha}\in\mathcal{B}_k^+}\frac{\lambda(\boldsymbol{\alpha})}{n!}\frac{\partial}{\partial x_1}\sum_{\sigma\in S_n} x_1^{\alpha_{\sigma(1)}}x_2^{\alpha_{\sigma(2)}}\cdots x_n^{\alpha_{\sigma(n)}} \\ &= \sum_{k=0}^{\infty}\sum_{\boldsymbol{\alpha}\in\mathcal{B}_k^+}\frac{\lambda(\boldsymbol{\alpha})}{n!}\sum_{\sigma\in S_n}\alpha_{\sigma(1)}x_1^{\alpha_{\sigma(1)}-1}x_2^{\alpha_{\sigma(2)}}\cdots x_n^{\alpha_{\sigma(n)}}. \end{aligned}$$

Hence

$$\begin{aligned} f_1'(\mathbf{I}_n) &= \sum_{k=0}^{\infty}\sum_{\boldsymbol{\alpha}\in\mathcal{B}_k^+}\frac{\lambda(\boldsymbol{\alpha})}{n!}\sum_{\sigma\in S_n}\alpha_{\sigma(1)} \\ &= \sum_{k=0}^{\infty}\sum_{\boldsymbol{\alpha}\in\mathcal{B}_k^+}\frac{\lambda(\boldsymbol{\alpha})}{n!}(n-1)!\sum_{\sigma(1)=1}^{n}\alpha_{\sigma(1)} \\ &= \sum_{k=0}^{\infty}\sum_{\boldsymbol{\alpha}\in\mathcal{B}_k^+}\frac{|\boldsymbol{\alpha}|\lambda(\boldsymbol{\alpha})}{n} \\ &= \frac{1}{n}\sum_{k=0}^{\infty}\sum_{\boldsymbol{\alpha}\in\mathcal{B}_k^+}|\boldsymbol{\alpha}|\lambda(\boldsymbol{\alpha}). \end{aligned}$$

That is to say, equation (49) holds.

Set $\boldsymbol{\alpha} = (n^{-1}, n^{-1}, \dots, n^{-1})$ in Theorem 1, we get

$$G_n(\mathbf{x}+\mathbf{y}) \geq G_n(\mathbf{x}) + G_n(\mathbf{y}), \quad \forall \mathbf{x},\mathbf{y}\in(0,\infty)^n, \forall n\geq 1. \tag{50}$$

According to the A-G inequality (see [12]) or Hardy's inequality (1), we have

$$\frac{\operatorname{per} H_n(\mathbf{x},\boldsymbol{\alpha})}{n!} \geq \left[G_n(\mathbf{x})\right]^{|\boldsymbol{\alpha}|}, \quad \forall \mathbf{x}\in(0,\infty)^n. \tag{51}$$

Note the A-G inequality with weights:

$$\sum_{i=1}^{\infty} \lambda_i x_i \geq |\boldsymbol{\lambda}| \left(\prod_{i=1}^{\infty} x_i^{\lambda_i} \right)^{\frac{1}{|\boldsymbol{\lambda}|}}, \quad \forall \mathbf{x}, \boldsymbol{\lambda} \in (0, \infty)^{\infty}. \tag{52}$$

By (48)-(52), we get

$$\begin{aligned} f(\mathbf{x}+\mathbf{y}) &\geq \sum_{k=0}^{\infty} \sum_{\boldsymbol{\alpha} \in \mathscr{B}_k^+} \lambda(\boldsymbol{\alpha}) \big[G_n(\mathbf{x}+\mathbf{y}) \big]^{|\boldsymbol{\alpha}|} \\ &\geq \sum_{k=0}^{\infty} \sum_{\boldsymbol{\alpha} \in \mathscr{B}_k^+} \lambda(\boldsymbol{\alpha}) \big[G_n(\mathbf{x}) + G_n(\mathbf{y}) \big]^{|\boldsymbol{\alpha}|} \\ &\geq f(\mathbf{I}_n) \left\{ \prod_{k=0}^{\infty} \prod_{\boldsymbol{\alpha} \in \mathscr{B}_k^+} \big[G_n(\mathbf{x}) + G_n(\mathbf{y}) \big]^{|\boldsymbol{\alpha}| \lambda(\boldsymbol{\alpha})} \right\}^{\frac{1}{f(\mathbf{I}_n)}} \\ &= f(\mathbf{I}_n) \big[G_n(\mathbf{x}) + G_n(\mathbf{y}) \big]^{\frac{1}{f(\mathbf{I}_n)} \sum_{k=0}^{\infty} \sum_{\boldsymbol{\alpha} \in \mathscr{B}_k^+} |\boldsymbol{\alpha}| \lambda(\boldsymbol{\alpha})} \\ &= f(\mathbf{I}_n) \big[G_n(\mathbf{x}) + G_n(\mathbf{y}) \big]^{\frac{n f_1'(\mathbf{I}_n)}{f(\mathbf{I}_n)}}. \end{aligned}$$

That is to say, inequality (47) holds.

According to the above analysis, we know that a sufficient condition of inequality (47) to be an equality is as follows: there is a real $\theta \in (0,1)$ such that

$$\mathbf{x} = \theta \mathbf{I}_n \quad \text{and} \quad \mathbf{y} = (1-\theta)\mathbf{I}_n,$$

or $f(\mathbf{x}) = x_1 x_2 \cdots x_n$ and $\mathbf{x}$, $\mathbf{y}$ are linearly dependent.

This completes the proof of Theorem 3. □

Theorem 3 implies the following result.

Corollary 1 *Let* $a \in (0, \infty)$, $\mathbf{x}, \mathbf{y}, \mathbf{x}+\mathbf{y} \in (0, 1+a)^n$. *Then we have the following inequality*:

$$\big[G_n(1+a-\mathbf{x}-\mathbf{y}) \big]^a \big[G_n(\mathbf{x}) + G_n(\mathbf{y}) \big] \leq a^a. \tag{53}$$

Equality in (53) *holds if there exists a real* $\theta \in (0,1)$ *such that*

$$\mathbf{x} = \theta \mathbf{I}_n \quad \textit{and} \quad \mathbf{y} = (1-\theta)\mathbf{I}_n.$$

Proof We construct an auxiliary function $f : [0, 1+a)^n \to (0, \infty)$ as follows:

$$\begin{aligned} f(\mathbf{x}) &\triangleq \prod_{i=1}^{n} \frac{1}{1+a-x_i} = (1+a)^{-n} \prod_{i=1}^{n} \left(1 - \frac{x_i}{1+a} \right)^{-1} \\ &= (1+a)^{-n} \prod_{i=1}^{n} \sum_{j=0}^{\infty} \left(\frac{x_i}{1+a} \right)^j \end{aligned}$$

$$= (1+a)^{-n}\sum_{k=0}^{\infty}\sum_{\boldsymbol{\alpha}\in\mathscr{B}_k^+}\frac{\mathrm{per}H_n(\frac{\mathbf{x}}{1+a},\boldsymbol{\alpha})}{n!}$$

$$= \sum_{k=0}^{\infty}\sum_{\boldsymbol{\alpha}\in\mathscr{B}_k^+}(1+a)^{-k-n}\frac{\mathrm{per}H_n(\mathbf{x},\boldsymbol{\alpha})}{n!},$$

i.e.,

$$f(\mathbf{x}) = \sum_{k=0}^{\infty}\sum_{\boldsymbol{\alpha}\in\mathscr{B}_k^+}\frac{\lambda(\boldsymbol{\alpha})}{n!}\mathrm{per}H_n(\mathbf{x},\boldsymbol{\alpha}), \quad \forall\mathbf{x}\in[0,1+a)^n, \tag{54}$$

where

$$\lambda(\boldsymbol{\alpha}) \equiv (1+a)^{-k-n} > 0.$$

Then

$$f(\mathbf{I}_n) = a^{-n}, \qquad f_1'(\mathbf{I}_n) = a^{-n-1}.$$

According to Theorem 3, for any $\mathbf{x},\mathbf{y},\mathbf{x}+\mathbf{y}\in(0,1+a)^n$, inequality (47) holds, *i.e.*,

$$\left[aG_n\left(\frac{1}{1+a-\mathbf{x}-\mathbf{y}}\right)\right]^a \geq G_n(\mathbf{x}) + G_n(\mathbf{y}),$$

that is to say, inequality (53) holds. Equality in (53) holds if there exists a real $\theta\in(0,1)$ such that

$$\mathbf{x} = \theta\mathbf{I}_n \quad \text{and} \quad \mathbf{y} = (1-\theta)\mathbf{I}_n$$

by Theorem 3. This ends the proof. □

Corollary 1 implies the following result.

Corollary 2 *Let the functions* $\varphi:[b,c]\to(0,1+a)$ *and* $\psi:[b,c]\to(0,1+a)$ *be continuous, and let them satisfy the following conditions*:

$$a>0, \qquad b<c, \qquad \varphi(t)+\psi(t)\in(0,1+a), \qquad \forall t\in[b,c].$$

Then we have the following inequality:

$$\exp\left[a\frac{\int_b^c \log(1+a-\varphi-\psi)}{c-b}\right]\left[\exp\left(\frac{\int_b^c\log\varphi}{c-b}\right)+\exp\left(\frac{\int_b^c\log\psi}{c-b}\right)\right] \leq a^a. \tag{55}$$

Set $\mathbf{x}\in(0,\infty)^{|\boldsymbol{\varepsilon}|}$, $f(\mathbf{x}) = x_1x_2\cdots x_{|\boldsymbol{\varepsilon}|}$ in Theorem 3. By

$$\mathrm{det}H_n(\mathbf{x},\boldsymbol{\varepsilon}) = \prod_{1\leq i<j\leq n}(x_j-x_i)$$

(see [6]) and Theorem 3, we have the following Corollary 3.

Corollary 3 *Let* $\mathbf{x}, \mathbf{y} \in (-\infty, \infty)^n$, *and let*

$$x_1 < x_2 < \cdots < x_n, \qquad y_1 < y_2 < \cdots < y_n, \quad n \geq 2.$$

Then we have the following Minkowski-type inequality:

$$\sqrt[|\boldsymbol{\varepsilon}|]{\det H_n(\mathbf{x}+\mathbf{y}, \boldsymbol{\varepsilon})} \geq \sqrt[|\boldsymbol{\varepsilon}|]{\det H_n(\mathbf{x}, \boldsymbol{\varepsilon})} + \sqrt[|\boldsymbol{\varepsilon}|]{\det H_n(\mathbf{y}, \boldsymbol{\varepsilon})}. \tag{56}$$

Equality in (56) *holds if* $\mathbf{x}$, $\mathbf{y}$ *are linearly dependent.*

Remark 1 If there exists a function $M(\mathbf{x}) > 0$ such that for any $\mathbf{x} \in [0, a)^n$, any non-negative integer k and any $\boldsymbol{\alpha} \in \mathscr{B}_k$ we have

$$\left| \frac{\partial^k f(\mathbf{x})}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}} \right| \leq M(\mathbf{x}),$$

then (46) holds and the Taylor series (46) converges (see [14]) by the theory of mathematical analysis.

Remark 2 The significance of Theorem 2 and Theorem 3 is to estimate the lower bounds of the increment of the symmetric functions

$$f\bigl(\sqrt[k]{\mathbf{x}}\bigr) \quad \text{and} \quad \left[\frac{f(\mathbf{x})}{f(\mathbf{I}_n)}\right]^{\frac{f(\mathbf{I}_n)}{nf_1'(\mathbf{I}_n)}},$$

respectively.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in this paper. All authors read and approved the final manuscript.

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