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Certain Grüss type inequalities involving the generalized fractional integral operator

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Abstract

A remarkably large number of Grüss type fractional integral inequalities involving the special function have been investigated by many authors. Very recently, Kalla and Rao (Matematiche LXVI(1):57-64, 2011) gave two Grüss type inequalities involving the Saigo fractional integral operator. Using the same technique, in this paper, we establish certain new Grüss type fractional integral inequalities involving the Gauss hypergeometric function. Moreover, we also consider their relevances for other related known results.

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1 Introduction and preliminaries

In recent years, a number of inequalities involving the fractional operators (like Erdélyi-Kober, Riemann-Liouville, Saigo fractional integral operators *etc.*) have been considered by many authors (see, *e.g.*, [1–10]; for very recent work, see also [11] and [12]). The above-mentioned works largely have motivated us to perform the present study.

We begin by recalling some known functions and earlier works.

Let f and g be two functions which are defined and integrable on $[a, b]$. Then the following inequalities hold (see also [7], [13, p.296]):

$$l \leq f(x) \leq L, \quad m \leq g(x) \leq M. \quad (1.1)$$

Let, for each $x \in [a, b]$, l, L, m and M be real constants satisfying the inequalities (1.1). Then the following Grüss type inequalities hold:

$$\left| \frac{1}{(b-a)} \int_a^b f(x)g(x) dx - \frac{1}{(b-a)} \int_a^b f(x) dx \cdot \frac{1}{(b-a)} \int_a^b g(x) dx \right| \leq \frac{1}{4}(L-l)(M-m), \quad (1.2)$$

where $\frac{1}{4}$ is a best possible constant.

The inequality (1.2) has various generalizations that have appeared in the literature, for example, *e.g.* [5, 6, 13–16] and the references cited therein.

Very recently, Kalla and Rao [12] gave two Grüss type inequalities involving the Saigo fractional integral operator. Using the same technique, in this paper, we establish certain

new Grüss type fractional integral inequalities involving the generalized fractional integral operator due to Curiel and Galué (see [17]). Moreover, we also consider their relevant connections with other known results.

Throughout the present paper, we shall investigate a fractional integral over the space C_λ introduced in [18] and defined as follows.

Definition 1.1 The space of functions C_λ , $\lambda \in \mathbb{R}$, the set of real numbers, consists of all functions $f(x)$, $x > 0$, that can be represented in the form $f(x) = x^p f_1(x)$ with $p > \lambda$ and $f_1 \in C[0, \infty)$, where $C[0, \infty)$ is the set of continuous functions in the interval $[0, \infty)$.

We define a fractional integral operator $K_t^{\alpha, \beta, \eta, \delta}$ associated with the Gauss hypergeometric function as follows.

Definition 1.2 Let $f \in C_\lambda$. For $\alpha > \max\{0, -(\delta + \eta + 1)\}$, $\beta - 1 < \eta < 0$, $\beta < 1$ and $\delta > -1$ we define a fractional integral $K_t^{\alpha, \beta, \eta, \delta} f$ as follows:

$$(K_t^{\alpha, \beta, \eta, \delta} f)(x) = \frac{\Gamma(1 - \beta)\Gamma(\alpha + \delta + \eta + 1)}{\Gamma(\eta - \beta + 1)\Gamma(\delta + 1)} x^{\beta + \delta} (I_t^{\alpha, \beta, \eta, \delta} f)(x), \tag{1.3}$$

where $I_t^{\alpha, \beta, \eta, \delta}$ is the Gauss hypergeometric fractional integral of order α and is defined in the following.

Definition 1.3 Let $\alpha > 0$, $\delta > -1$, $\beta, \eta \in \mathbb{R}$. Then the generalized fractional integral $I_t^{\alpha, \beta, \eta, \delta}$ (in terms of the Gauss hypergeometric function) of order α for real-valued continuous function $f(t)$ is defined by [17] (see also [19])

$$I_t^{\alpha, \beta, \eta, \delta} \{f(x)\} = \frac{x^{-\alpha - \beta - 2\delta}}{\Gamma(\alpha)} \int_0^x t^\delta (x - t)^{\alpha - 1} {}_2F_1\left(\alpha + \beta + \delta, -\eta; \alpha; 1 - \frac{t}{x}\right) f(t) dt, \tag{1.4}$$

where the function ${}_2F_1(\cdot)$ appearing as a kernel for the operator (1.4) is the Gaussian hypergeometric function defined by

$${}_2F_1(a, b; c; t) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{t^n}{n!}, \tag{1.5}$$

and $(a)_n$ is the Pochhammer symbol defined by $(n \in \mathbb{N})$, and

$$(a)_n = a(a + 1) \cdots (a + n - 1); \quad (a)_0 = 1, \tag{1.6}$$

where \mathbb{N} denotes the set of positive integers.

The above integral (1.4) has the following commutative property:

$$I_t^{\alpha, \beta, \eta, \delta} I_t^{a, b, c, d} f(x) = I_t^{a, b, c, d} I_t^{\alpha, \beta, \eta, \delta} f(x). \tag{1.7}$$

In the sequel, we use the following well-known result to establish our main results in the present paper:

$${}_2F_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \quad (\Re(c-a-b) > 0; c \in \Xi/\mathbb{Z}_0^-), \tag{1.8}$$

where Ξ and \mathbb{Z}_0^- denotes the sets of complex numbers and nonpositive integers, respectively.

Definition 1.4 Two functions f and g are said to be synchronous functions on $[0, \infty)$ if

$$A(u, v) = (f(u) - f(v))(g(u) - g(v)) \geq 0; \quad u, v \in [0, \infty). \tag{1.9}$$

Next, we discuss some results regarding the fractional integral operator $K_t^{\alpha, \beta, \eta, \delta}$ which have been used in the present work.

Lemma 1.1 For $\mu > \max\{0, -(\eta - \beta)\} - 1$, $\alpha > \max\{0, -(\delta + \eta + 1)\}$; and $\eta - \beta > -1$, $\beta < 1$, $\delta + \mu > -1$, we have

$$K_t^{\alpha, \beta, \eta, \delta}(x^\mu) = \frac{\Gamma(1 - \beta)\Gamma(\alpha + \delta + \eta + 1)\Gamma(\delta + \mu + 1)\Gamma(\mu - \beta + \eta + 1)}{\Gamma(\eta - \beta + 1)\Gamma(\delta + 1)\Gamma(\mu - \beta + 1)\Gamma(\mu + \delta + \alpha + \eta + 1)} x^\mu \tag{1.10}$$

and

$$K_t^{\alpha, \beta, \eta, \delta}(C) = C, \tag{1.11}$$

where C is constant.

Proof Using the result (1.4), (1.3) reduces to

$$\begin{aligned} K_t^{\alpha, \beta, \eta, \delta}(x^\mu) &= \frac{\Gamma(1 - \beta)\Gamma(\alpha + \delta + \eta + 1)x^{\beta+\delta}}{\Gamma(\eta - \beta + 1)\Gamma(\delta + 1)} \frac{x^{-\alpha-\beta-2\delta}}{\Gamma(\alpha)} \\ &\quad \times \int_0^x t^{\delta+\mu}(x-t)^{\alpha-1} {}_2F_1\left(\alpha + \beta + \delta, -\eta; \alpha; 1 - \frac{t}{x}\right) dt. \end{aligned} \tag{1.12}$$

Using (1.5), (1.12) reduces to the following form:

$$\begin{aligned} K_t^{\alpha, \beta, \eta, \delta}(x^\mu) &= \frac{\Gamma(1 - \beta)\Gamma(\alpha + \delta + \eta + 1)x^{-\alpha-\delta}}{\Gamma(\eta - \beta + 1)\Gamma(\delta + 1)} \frac{1}{\Gamma(\alpha)} \\ &\quad \times \sum_{n=0}^{\infty} \frac{(\alpha + \beta + \delta)_n (-\eta)_n}{(\alpha)_n n!} x^{-n} \int_0^x t^{\delta+\mu+1-1}(x-t)^{\alpha+n-1} dt \\ &= \frac{\Gamma(1 - \beta)\Gamma(\alpha + \delta + \eta + 1)x^{-\alpha-\delta}}{\Gamma(\eta - \beta + 1)\Gamma(\delta + 1)} \frac{1}{\Gamma(\alpha)} \sum_{n=0}^{\infty} \frac{(\alpha + \beta + \delta)_n (-\eta)_n}{(\alpha)_n n!} x^{-n} \\ &\quad \times x^{\delta+\mu+\alpha+n+1-1} B(\delta + \mu + 1, \alpha + n) \\ &= \frac{\Gamma(1 - \beta)\Gamma(\alpha + \delta + \eta + 1)\Gamma(\delta + \mu + 1)}{\Gamma(\eta - \beta + 1)\Gamma(\delta + 1)\Gamma(\alpha + \delta + \mu + 1)} \sum_{n=0}^{\infty} \frac{(\alpha + \beta + \delta)_n (-\eta)_n}{(\alpha + \delta + \mu + 1)_n n!} x^\mu \end{aligned}$$

$$\begin{aligned}
 &= \frac{\Gamma(1-\beta)\Gamma(\alpha+\delta+\eta+1)\Gamma(\delta+\mu+1)}{\Gamma(\eta-\beta+1)\Gamma(\delta+1)\Gamma(\alpha+\delta+\mu+1)} \\
 &\quad \times {}_2F_1(\alpha+\beta+\delta, -\eta; \alpha+\delta+\mu+1; 1)x^\mu. \tag{1.13}
 \end{aligned}$$

Applying the result (1.8) to (1.13), after a little simplification, we easily arrive at the required result (1.10).

To prove (1.11), we again use the result (1.4), and (1.3) reduces to

$$\begin{aligned}
 K_t^{\alpha,\beta,\eta,\delta}(C) &= \frac{\Gamma(1-\beta)\Gamma(\alpha+\delta+\eta+1)x^{\beta+\delta}}{\Gamma(\eta-\beta+1)\Gamma(\delta+1)} \frac{x^{-\alpha-\beta-2\delta}}{\Gamma(\alpha)} \\
 &\quad \times \int_0^x t^\delta (x-t)^{\alpha-1} {}_2F_1\left(\alpha+\beta+\delta, -\eta; \alpha; 1-\frac{t}{x}\right) C dt. \tag{1.14}
 \end{aligned}$$

Using (1.5), (1.14) gets the following form:

$$\begin{aligned}
 K_t^{\alpha,\beta,\eta,\delta}(C) &= C \frac{\Gamma(1-\beta)\Gamma(\alpha+\delta+\eta+1)x^{-\alpha-\delta}}{\Gamma(\eta-\beta+1)\Gamma(\delta+1)} \frac{1}{\Gamma(\alpha)} \\
 &\quad \times \sum_{n=0}^{\infty} \frac{(\alpha+\beta+\delta)_n(-\eta)_n}{(\alpha)_n n!} x^{-n} \int_0^x t^{\delta+1-1} (x-t)^{\alpha+n-1} dt \\
 &= C \frac{\Gamma(1-\beta)\Gamma(\alpha+\delta+\eta+1)x^{-\alpha-\delta}}{\Gamma(\eta-\beta+1)\Gamma(\delta+1)} \frac{1}{\Gamma(\alpha)} \sum_{n=0}^{\infty} \frac{(\alpha+\beta+\delta)_n(-\eta)_n}{(\alpha)_n n!} x^{-n} \\
 &\quad \times x^{\delta+\alpha+n+1-1} B(\delta+1, \alpha+n) \\
 &= C \frac{\Gamma(1-\beta)\Gamma(\alpha+\delta+\eta+1)}{\Gamma(\eta-\beta+1)\Gamma(\alpha+\delta+1)} \sum_{n=0}^{\infty} \frac{(\alpha+\beta+\delta)_n(-\eta)_n}{(\alpha+\delta+1)_n n!} \\
 &= C \frac{\Gamma(1-\beta)\Gamma(\alpha+\delta+\eta+1)}{\Gamma(\eta-\beta+1)\Gamma(\alpha+\delta+1)} {}_2F_1(\alpha+\beta+\delta, -\eta; \alpha+\delta+1; 1). \tag{1.15}
 \end{aligned}$$

Using the result (1.8) in (1.15), after a little simplification, we easily arrive at the required result (1.11).

This completes the proof of the Lemma 1.1. □

Lemma 1.2 Let $h \in C_\lambda$ and $m, M \in \mathbb{R}$ with $m \leq h(x) \leq M$. Then we have

$$\begin{aligned}
 &K_t^{\alpha,\beta,\eta,\delta} h^2(x) - (K_x^{\alpha,\beta,\eta,\delta} t h(x))^2 \\
 &= (M - K_t^{\alpha,\beta,\eta,\delta} h(x))(K_t^{\alpha,\beta,\eta,\delta} h(x) - m) - K_t^{\alpha,\beta,\eta,\delta} (M - h(x))(h(x) - m), \tag{1.16}
 \end{aligned}$$

for all $x \in [0, \infty)$; $\alpha > 0$, $\delta > -1$, and $\beta, \eta \in \mathbb{R}$ with $\alpha + \beta + \delta \geq 0$ and $\eta \leq 0$.

Proof Let $h \in C_\lambda$ and $m, M \in \mathbb{R}$; $m \leq h(x) \leq M$, for all $x \in [0, \infty)$. Then, for any $u, v \in [0, \infty)$, we have

$$\begin{aligned}
 &(M - h(u))(h(v) - m) + (M - h(v))(h(u) - m) - (M - h(u))(h(u) - m) \\
 &\quad - (M - h(v))(h(v) - m) = h^2(u) + h^2(v) - 2h(u)h(v). \tag{1.17}
 \end{aligned}$$

If $h \in C_\lambda$, then h is integrable on $[0, x]$, $x > 0$. Thus multiplying (1.17) by $\frac{u^\delta(x-u)^{\alpha-1}}{\Gamma(\alpha)} {}_2F_1(\alpha + \delta + \beta, -\eta; \alpha; 1 - \frac{u}{x})$; using $u \in (0, x)$; $x > 0$ and integrating with respect to u from 0 to x , and then applying Definition 1.2 and Lemma 1.1, we obtain

$$\begin{aligned} & (M - K_t^{\alpha, \beta, \eta, \delta} h(x))(h(v) - m) + (M - h(v))(K_t^{\alpha, \beta, \eta, \delta} h(x) - m) \\ & \quad - K_t^{\alpha, \beta, \eta, \delta} (M - h(x))(h(x) - m) - (M - h(v))(h(v) - m) \\ & = K_t^{\alpha, \beta, \eta, \delta} h^2(x) + h^2(v) - 2K_t^{\alpha, \beta, \eta, \delta} h(x)h(v). \end{aligned} \tag{1.18}$$

Again multiplying (1.18) by

$$\frac{v^\delta(x-v)^{\alpha-1}}{\Gamma(\alpha)} {}_2F_1\left(\alpha + \delta + \beta, -\eta; \alpha; 1 - \frac{v}{x}\right); \quad v \in (0, x); x > 0,$$

then integrating with respect to v from 0 to x , we obtain the required result (1.16). This completes the proof of Lemma 1.2. \square

2 Main results

In this section, we establish two inequalities involving the composition formula of the fractional integral (1.3) with a power function.

Theorem 2.1 *Let f and g be two functions defined and integrable on $[a, b]$ with $f, g \in C_\lambda$ and satisfying the condition (1.1) on $[0, \infty)$. Thus we have*

$$\left| K_t^{\alpha, \beta, \eta, \delta} f g(x) - K_t^{\alpha, \beta, \eta, \delta} f(x) K_t^{\alpha, \beta, \eta, \delta} g(x) \right| \leq \frac{1}{4} (L - l)(M - m), \tag{2.1}$$

for all $x \in [0, \infty)$; $\alpha > 0$, $\delta > -1$, and $\beta, \eta \in \mathbb{R}$ with $\alpha + \beta + \delta \geq 0$ and $\eta \leq 0$.

Proof Let us define a function

$$A(u, v) = (f(u) - f(v))(g(u) - g(v)) \quad (u, v \in [0, x]). \tag{2.2}$$

First multiplying (2.2) by

$$\begin{aligned} & \frac{(uv)^\delta(x-u)^{\alpha-1}(x-v)^{\alpha-1}}{(\Gamma(\alpha))^2} {}_2F_1\left(\alpha + \delta + \beta, -\eta; \alpha; 1 - \frac{u}{x}\right) \\ & \quad \times {}_2F_1\left(\alpha + \delta + \beta, -\eta; \alpha; 1 - \frac{v}{x}\right), \end{aligned}$$

and then integrating twice with respect to u and v from 0 to x , we obtain the following result with the aid of (1.3), (1.4), and property (1.5):

$$\begin{aligned} & \frac{1}{(\Gamma(\alpha))^2} \int_0^x \int_0^x (uv)^\delta(x-u)^{\alpha-1}(x-v)^{\alpha-1} {}_2F_1\left(\alpha + \delta + \beta, -\eta; \alpha; 1 - \frac{u}{x}\right) \\ & \quad \times {}_2F_1\left(\alpha + \delta + \beta, -\eta; \alpha; 1 - \frac{v}{x}\right) A(u, v) \, du \, dv \\ & = 2K_t^{\alpha, \beta, \eta, \delta} f g(x) - 2K_t^{\alpha, \beta, \eta, \delta} f(x) K_t^{\alpha, \beta, \eta, \delta} g(x). \end{aligned} \tag{2.3}$$

Making use of the well-known Cauchy-Schwarz inequality for a linear operator [13, Eq. (1.3), p.296], we find that

$$\begin{aligned} & (K_t^{\alpha,\beta,\eta,\delta}fg(x) - K_t^{\alpha,\beta,\eta,\delta}f(x)K_t^{\alpha,\beta,\eta,\delta}g(x))^2 \\ & \leq (K_t^{\alpha,\beta,\eta,\delta}f^2(x) - (K_t^{\alpha,\beta,\eta,\delta}f(x))^2)(K_t^{\alpha,\beta,\eta,\delta}g^2(x) - (K_t^{\alpha,\beta,\eta,\delta}g(x))^2). \end{aligned} \tag{2.4}$$

Since

$$(L - f(x))(f(x) - l) \geq 0 \quad \text{and} \quad (M - g(x))(g(x) - m) \geq 0,$$

we therefore have

$$K_t^{\alpha,\beta,\eta,\delta}(L - f(x))(f(x) - l) \geq 0 \quad \text{and} \quad K_t^{\alpha,\beta,\eta,\delta}(M - g(x))(g(x) - m) \geq 0. \tag{2.5}$$

Thus by using Lemma 1.2, we have

$$K_t^{\alpha,\beta,\eta,\delta}f^2(x) - (K_t^{\alpha,\beta,\eta,\delta}f(x))^2 \leq (L - K_t^{\alpha,\beta,\eta,\delta}f(x))(K_t^{\alpha,\beta,\eta,\delta}f(x) - l) \tag{2.6}$$

and

$$K_t^{\alpha,\beta,\eta,\delta}g^2(x) - (K_t^{\alpha,\beta,\eta,\delta}g(x))^2 \leq (M - K_t^{\alpha,\beta,\eta,\delta}g(x))(K_t^{\alpha,\beta,\eta,\delta}g(x) - m). \tag{2.7}$$

Using the inequalities (2.6) and (2.7), (2.4) reduces to the following form:

$$\begin{aligned} & (K_t^{\alpha,\beta,\eta,\delta}fg(x) - K_t^{\alpha,\beta,\eta,\delta}f(x)K_t^{\alpha,\beta,\eta,\delta}g(x))^2 \\ & \leq (L - K_t^{\alpha,\beta,\eta,\delta}f(x))(K_t^{\alpha,\beta,\eta,\delta}f(x) - l)(M - K_t^{\alpha,\beta,\eta,\delta}g(x)) \\ & \quad \times (K_t^{\alpha,\beta,\eta,\delta}g(x) - m). \end{aligned} \tag{2.8}$$

Applying the well-known inequality $4ab \leq (a + b)^2$; and using $a, b \in \mathbb{R}$ in the right-hand side of the inequality (2.8) and simplifying it, we obtain the required result (2.1). This completes the proof of Theorem 2.1. \square

Theorem 2.2 *Let f and g be two synchronous functions on $[0, \infty)$. Then the following inequality holds:*

$$K_t^{\alpha,\beta,\eta,\delta}fg(x) \geq K_t^{\alpha,\beta,\eta,\delta}f(x)K_t^{\alpha,\beta,\eta,\delta}g(x), \tag{2.9}$$

for all $x \in [0, \infty)$; $\alpha > 0$, $\delta > -1$, and $\beta, \eta \in \mathbb{R}$ with $\alpha + \beta + \delta \geq 0$ and $\eta \leq 0$.

Proof For the synchronous function f and g , the inequality (1.9) holds for all $u, v \in [0, \infty)$.

This implies that

$$f(u)g(u) - f(v)g(v) \geq f(u)g(v) + f(v)g(u). \tag{2.10}$$

Following the procedure of the Lemma 1.2 for applying the fractional integral $K_t^{\alpha,\beta,\eta,\delta}$, after a little simplification, we arrive at the required result (2.9). This completes the proof of Theorem 2.2. \square

3 Concluding remarks

We consider some consequences of the results derived in the previous section. Following Curiel and Galué [17], the operator (1.4) would reduce immediately to the extensively investigated Saigo, Erdélyi-Kober, and Riemann-Liouville type fractional integral operators, respectively, given by the following relationships (see also [20] and [19]):

$$I_{0,t}^{\alpha,\beta,\eta} \{f(t)\} = I_t^{\alpha,\beta,\eta,0} \{f(t)\} = \frac{t^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} {}_2F_1\left(\alpha+\beta, -\eta; \alpha; 1-\frac{\tau}{t}\right) f(\tau) d\tau$$

$$(\alpha > 0; \beta, \eta \in \mathbb{R}), \tag{3.1}$$

$$I^{\alpha,\eta} \{f(t)\} = I_t^{\alpha,0,\eta,0} \{f(t)\} = \frac{t^{-\alpha-\eta}}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \tau^\eta f(\tau) d\tau \quad (\alpha > 0; \eta \in \mathbb{R}) \tag{3.2}$$

and

$$R^\alpha \{f(t)\} = I_t^{\alpha,-\alpha,\eta,0} \{f(t)\} = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau \quad (\alpha > 0). \tag{3.3}$$

Remarks We obtain the special cases of the operator $K_t^{\alpha,\beta,\eta,\delta}$ as follows by setting $\delta = 0$, $\delta = \beta = 0$ and $\delta = 0$ and $\beta = -\alpha$ in (1.2). Immediately Definition 1.2 would reduce to the Saigo, Erdélyi-Kober, and Riemann-Liouville type fractional integral operators, respectively, given as follows:

$$(K_t^{\alpha,\beta,\eta} f)(x) = \frac{\Gamma(1-\beta)\Gamma(\alpha+\eta+1)}{\Gamma(\eta-\beta+1)} x^\beta (I_0^{\alpha,\beta,\eta} t f)(x), \tag{3.4}$$

$$(K_t^{\alpha,\eta} f)(x) = \frac{\Gamma(\eta+\alpha+1)}{\Gamma(1+\eta)} (I^{\alpha,\eta} f)(x) \tag{3.5}$$

and

$$(K_t^\alpha f)(x) = \frac{\Gamma(\alpha+1)}{x^\alpha} (R^\alpha f)(x), \tag{3.6}$$

where $(I_0^{\alpha,\beta,\eta} t)$, $(I^{\alpha,\eta})$ and (R^α) are given by (3.1), (3.2), and (3.3), respectively.

We conclude our present investigation by remarking further that the results obtained here are useful in deriving various fractional integral inequalities involving such relatively more familiar fractional integral operators. For example, if we consider $\delta = 0$ and make use of (3.1), Theorems 2.1 and 2.2 provide, respectively, the known fractional integral inequalities due to Kalla and Rao [12, pp.60-62, Eqs. (14) and (22)].

Again, for $\delta = 0$ and $\beta = 0$ in Theorems 2.1 and 2.2 and making use of the relation (3.2), Theorems 2.1 and 2.2 provide, respectively, the known fractional integral inequalities due to Kalla and Rao [12, p.62, Eqs. (24) and (25)].

Finally, taking $\delta = 0$ and $\beta = -\alpha$ in Theorems 2.1 and 2.2 yields the known result due to Dahmani *et al.* [5, Theorem 3.1].

It is noted that the results derived in this paper are general in character and give some contributions to the theory of integral inequalities and fractional calculus. Moreover, they are expected to lead to some applications for establishing uniqueness of solutions in fractional boundary value problems, and in fractional partial differential equations.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have equal contributions. All authors read and approved the final manuscript.

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