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Complete convergence for negatively orthant dependent random variables

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Abstract
In this paper, necessary and sufficient conditions of the complete convergence are obtained for the maximum partial sums of negatively orthant dependent (NOD) random variables. The results extend and improve those in Kuczmaszewska (Acta Math. Hung. 128(1-2):116-130, 2010) for negatively associated (NA) random variables.

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1 Introduction

The concept of complete convergence for a sequence of random variables was introduced by Hsu and Robbins [1] as follows. A sequence $\{U_n, n \ge 1\}$ of random variables *converges completely* to the constant θ if

$$\sum_{n=1}^{\infty} P\bigl(|U_n - \theta| > \varepsilon\bigr) < \infty \quad \text{for all } \varepsilon > 0.$$

Moreover, they proved that the sequence of arithmetic means of independent identically distribution (i.i.d.) random variables converges completely to the expected value if the variance of the summands is finite. This result has been generalized and extended in several directions by many authors. One can refer to [2–16], and so forth. Kuczmaszewska [8] proved the following result.

Theorem A *Let $\{X_n, n \ge 1\}$ be a sequence of negatively associated (NA) random variables and X be a random variables possibly defined on a different space satisfying the condition*

$$\frac{1}{n}\sum_{i=1}^{n} P\bigl(|X_i| > x\bigr) = DP\bigl(|X| > x\bigr)$$

for all $x > 0$, all $n \ge 1$ and some positive constant D. Let $\alpha p > 1$ and $\alpha > 1/2$. Moreover, additionally assume that $EX_n = 0$ for all $n \ge 1$ if $p \ge 1$. Then the following statements are equivalent:

(i) $E|X|^p < \infty$,
(ii) $\sum_{n=1}^{\infty} n^{\alpha p-2} P(\max_{1\le j\le n} |\sum_{i=1}^{j} X_i| \ge \varepsilon n^{\alpha}) < \infty$, $\forall \varepsilon > 0$.

The aim of this paper is to extend and improve Theorem A to negatively orthant dependent (NOD) random variables. The tool in the proof of Theorem A is the Rosenthal maximal inequality for NA sequence (*cf.* [17]), but no one established the kind of maximal inequality for NOD sequence. So the truncated method is different and the proofs of our main results are more complicated and difficult.

The concept of negatively associated (NA) and negatively orthant dependent (NOD) was introduced by Joag-Dev and Proschan [18] in the following way.

Definition 1.1 A finite family of random variables $\{X_i, 1 \le i \le n\}$ is said to be negatively associated (NA) if for every pair of disjoint nonempty subset A_1, A_2 of $\{1, 2, \ldots, n\}$,

$$\left|\operatorname{Cov}\big(f_1(X_i, i \in A_1), f_2(X_j, j \in A_2)\big)\right| \le 0,$$

where f_1 and f_2 are coordinatewise nondecreasing such that the covariance exists. An infinite sequence of $\{X_n, n \ge 1\}$ is NA if every finite subfamily is NA.

Definition 1.2 A finite family of random variables $\{X_i, 1 \le i \le n\}$ is said to be

(a) negatively upper orthant dependent (NUOD) if

$$P(X_i > x_i, i = 1, 2, \ldots, n) \le \prod_{i=1}^{n} P(X_i > x_i)$$

for $\forall x_1, x_2, \ldots, x_n \in R$,

(b) negatively lower orthant dependent (NLOD) if

$$P(X_i \le x_i, i = 1, 2, \ldots, n) \le \prod_{i=1}^{n} P(X_i \le x_i)$$

for $\forall x_1, x_2, \ldots, x_n \in R$,

(c) negatively orthant dependent (NOD) if they are both NUOD and NLOD.

A sequence of random variables $\{X_n, n \ge 1\}$ is said to be NOD if for each n, $X_1, X_2, \ldots, X_n$ are NOD.

Obviously, every sequence of independent random variables is NOD. Joag-Dev and Proschan [18] pointed out that NA implies NOD, neither being NUOD nor being NLOD implies being NA. They gave an example that possesses NOD, but does not possess NA, which shows that NOD is strictly wider than NA. For more details of NOD random variables, one can refer to [3, 6, 11, 14, 19–21], and so forth.

In order to prove our main results, we need the following lemmas.

Lemma 1.1 (Bozorgnia *et al.* [19]) *Let $X_1, X_2, \ldots, X_n$ be NOD random variables.*

(i) *If $f_1, f_2, \ldots, f_n$ are Borel functions all of which are monotone increasing (or all monotone decreasing), then $f_1(X_1), f_2(X_2), \ldots, f_n(X_n)$ are NOD random variables.*

(ii) $E\prod_{i=1}^{n} X_i^+ \le \prod_{i=1}^{n} EX_i^+$, $\forall n \ge 2$.

Lemma 1.2 (Asadian *et al.* [22]) *For any $q \ge 2$, there is a positive constant $C(q)$ depending only on q such that if $\{X_n, n \ge 1\}$ is a sequence of NOD random variables with $EX_n = 0$ for* *every $n \geq 1$, then for all $n \geq 1$,*

$$E\left|\sum_{i=1}^{n} X_i\right|^q \leq C(q)\left\{\sum_{i=1}^{n} E|X_i|^q + \left(\sum_{i=1}^{n} EX_i^2\right)^{q/2}\right\}.$$

Lemma 1.3 *For any $q \geq 2$, there is a positive constant $C(q)$ depending only on q such that if $\{X_n, n \geq 1\}$ is a sequence of NOD random variables with $EX_n = 0$ for every $n \geq 1$, then for all $n \geq 1$,*

$$E\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} X_i\right|^q \leq C(q)\big(\log(4n)\big)^q\left\{\sum_{i=1}^{n} E|X_i|^q + \left(\sum_{i=1}^{n} EX_i^2\right)^{q/2}\right\}.$$

Proof By Lemma 1.2, the proof is similar to that of Theorem 2.3.1 in Stout [23], so it is omitted here. □

Lemma 1.4 (Kuczmaszewska [8]) *Let β, γ be positive constants. Suppose that $\{X_n, n \geq 1\}$ is a sequence of random variables and X is a random variable. There exists constant $D > 0$ such that*

$$\sum_{i=1}^{n} P\big(|X_i| > x\big) \leq DnP\big(|X| > x\big), \quad \forall x > 0, \forall n \geq 1; \tag{1.1}$$

(i) *if $E|X|^\beta < \infty$, then $\frac{1}{n}\sum_{j=1}^{n} E|X_j|^\beta \leq CE|X|^\beta$;*
(ii) $\frac{1}{n}\sum_{j=1}^{n} E|X_j|^\beta I(|X_j| \leq \gamma) \leq C\{E|X|^\beta I(|X| \leq \gamma) + \gamma^\beta P(|X| > \gamma)\}$;
(iii) $\frac{1}{n}\sum_{j=1}^{n} E|X_j|^\beta I(|X_j| > \gamma) \leq CE|X|^\beta I(|X| > \gamma)$.

Recall that a function $h(x)$ is said to be slowly varying at infinity if it is real valued, positive, and measurable on $[0, \infty)$, and if for each $\lambda > 0$

$$\lim_{x\to\infty} \frac{h(\lambda x)}{h(x)} = 1.$$

We refer to Seneta [24] for other equivalent definitions and for a detailed and comprehensive study of properties of slowly varying functions.

We frequently use the following properties of slowly varying functions (*cf.* Seneta [24]).

Lemma 1.5 *If $h(x)$ is a function slowly varying at infinity, then for any $s > 0$*

$$C_1 n^{-s} h(n) \leq \sum_{i=n}^{\infty} i^{-1-s} h(i) \leq C_2 n^{-s} h(n)$$

and

$$C_3 n^{s} h(n) \leq \sum_{i=1}^{n} i^{-1+s} h(i) \leq C_4 n^{s} h(n),$$

where $C_1, C_2, C_3, C_4 > 0$ depend only on s.

Throughout this paper, C will represent positive constants of which the value may change from one place to another.

2 Main results and proofs

Theorem 2.1 *Let $\alpha > 1/2$, $p > 0$, $\alpha p > 1$ and $h(x)$ be a slowly varying function at infinity. Let $\{X_n, n \geq 1\}$ be a sequence of NOD random variables and X be a random variables possibly defined on a different space satisfying the condition* (1.1). *Moreover, additionally assume that for $\alpha \leq 1$, $EX_n = 0$ for all $n \geq 1$. If*

$$E|X|^p h\big(|X|^{1/\alpha}\big) < \infty, \tag{2.1}$$

then the following statements hold:

$$\text{(i)} \quad \sum_{n=1}^{\infty} n^{\alpha p-2} h(n) P\Big(\max_{1\leq j\leq n} |S_j| \geq \varepsilon n^{\alpha}\Big) < \infty, \quad \forall \varepsilon > 0; \tag{2.2}$$

$$\text{(ii)} \quad \sum_{n=2}^{\infty} n^{\alpha p-2} h(n) P\Big(\max_{1\leq k\leq n} \big|S_n^{(k)}\big| \geq \varepsilon n^{\alpha}\Big) < \infty, \quad \forall \varepsilon > 0; \tag{2.3}$$

$$\text{(iii)} \quad \sum_{n=1}^{\infty} n^{\alpha p-2} h(n) P\Big(\max_{1\leq j\leq n} |X_j| \geq \varepsilon n^{\alpha}\Big) < \infty, \quad \forall \varepsilon > 0; \tag{2.4}$$

$$\text{(iv)} \quad \sum_{n=1}^{\infty} n^{\alpha p-2} h(n) P\Big(\sup_{j\geq n} j^{-\alpha} |S_j| \geq \varepsilon\Big) < \infty, \quad \forall \varepsilon > 0; \tag{2.5}$$

$$\text{(v)} \quad \sum_{n=1}^{\infty} n^{\alpha p-2} h(n) P\Big(\sup_{j\geq n} j^{-\alpha} |X_j| \geq \varepsilon\Big) < \infty, \quad \forall \varepsilon > 0. \tag{2.6}$$

Here $S_n = \sum_{i=1}^{n} X_i$, $S_n^{(k)} = S_n - X_k$, $k = 1, 2, \ldots, n$.

Proof First, we prove (2.2). Choose q such that $1/\alpha p < q < 1$. Let $X_i^{(n,1)} = -n^{\alpha q} I(X_i < -n^{\alpha q}) + X_i I(|X_i| \leq n^{\alpha q}) + n^{\alpha q} I(X_i > n^{\alpha q})$, $X_i^{(n,2)} = (X_i - n^{\alpha q}) I(X_i > n^{\alpha q})$, $X_i^{(n,3)} = -(X_i + n^{\alpha q}) I(X_i < -n^{\alpha q})$, $\forall n \geq 1$, $1 \leq i \leq n$. Note that

$$X_i = X_i^{(n,1)} + X_i^{(n,2)} - X_i^{(n,3)}$$

and

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha p-2} h(n) P\Big(\max_{1\leq j\leq n} |S_j| > \varepsilon n^{\alpha}\Big) \\
&\quad\leq \sum_{n=1}^{\infty} n^{\alpha p-2} h(n) P\left(\max_{1\leq j\leq n} \left|\sum_{i=1}^{j} X_i^{(n,1)}\right| > \varepsilon n^{\alpha}/3\right) \\
&\qquad + \sum_{n=1}^{\infty} n^{\alpha p-2} h(n) P\left(\sum_{i=1}^{n} X_i^{(n,2)} > \varepsilon n^{\alpha}/3\right) + \sum_{n=1}^{\infty} n^{\alpha p-2} h(n) P\left(\sum_{i=1}^{n} X_i^{(n,3)} > \varepsilon n^{\alpha}/3\right) \\
&\quad\stackrel{\text{def}}{=} I_1 + I_2 + I_3.
\end{aligned} \tag{2.7}$$

In order to prove (2.2), it suffices to show that $I_l < \infty$ for $l = 1, 2, 3$. Obviously, for $0 < \eta < p$, the condition (2.1) implies $E|X|^{p-\eta} < \infty$. Therefore, we choose $0 < \eta < p$, $\alpha(p - \eta) > \alpha(p -$

$\eta)q > 1$ and $p - \eta - 1 > 0$ if $p > 1$. In order to prove $I_1 < \infty$, we first prove that

$$\lim_{n\to\infty} n^{-\alpha} \max_{1\le j\le n} \left| \sum_{i=1}^{j} EX_i^{(n,1)} \right| = 0. \tag{2.8}$$

This holds when $\alpha \le 1$. Since $\alpha p > 1$, $p > 1$. By $EX_i = 0$, $i \ge 1$, and Lemma 1.4, we have

$$\begin{aligned} n^{-\alpha} \max_{1\le j\le n} \left| \sum_{i=1}^{j} EX_i^{(n,1)} \right| &\le n^{-\alpha} \max_{1\le j\le n} \sum_{i=1}^{j} \bigl\{E|X_i|I\bigl(|X_i| > n^{\alpha q}\bigr) + n^{\alpha q}P\bigl(|X_i| > n^{\alpha q}\bigr)\bigr\} \\ &\le 2n^{-\alpha} \sum_{i=1}^{n} E|X_i|I\bigl(|X_i| > n^{\alpha q}\bigr) \le Cn^{1-\alpha}E|X|I\bigl(|X| > n^{\alpha q}\bigr) \\ &\le Cn^{-\{\alpha(p-\eta)q-1\}-\alpha(1-q)}E|X|^{p-\eta} \\ &\to 0, \quad n \to \infty. \end{aligned}$$

When $\alpha > 1$, $p > 1$,

$$\begin{aligned} n^{-\alpha} \max_{1\le j\le n} \left| \sum_{i=1}^{j} EX_i^{(n,1)} \right| &\le n^{-\alpha} \max_{1\le j\le n} \sum_{i=1}^{j} \bigl\{E|X_i|I\bigl(|X_i| \le n^{\alpha q}\bigr) + n^{\alpha q}P\bigl(|X_i| > n^{\alpha q}\bigr)\bigr\} \\ &\le n^{-\alpha} \sum_{i=1}^{n} E|X_i| \le Cn^{1-\alpha}E|X| \\ &\to 0, \quad n \to \infty. \end{aligned}$$

When $\alpha > 1$, $p \le 1$,

$$\begin{aligned} n^{-\alpha} \max_{1\le j\le n} \left| \sum_{i=1}^{j} EX_i^{(n,1)} \right| &\le n^{-\alpha} \max_{1\le j\le n} \sum_{i=1}^{j} \bigl\{E|X_i|I\bigl(|X_i| \le n^{\alpha q}\bigr) + n^{\alpha q}P\bigl(|X_i| > n^{\alpha q}\bigr)\bigr\} \\ &\le n^{-\alpha} \sum_{i=1}^{n} \bigl\{E|X_i|I\bigl(|X_i| \le n^{\alpha q}\bigr) + n^{\alpha q}P\bigl(|X_i| > n^{\alpha q}\bigr)\bigr\} \\ &\le n^{-\alpha} \sum_{i=1}^{n} \bigl(n^{\alpha(1-p+\eta)q}E|X_i|^{p-\eta}\bigr) \\ &\le Cn^{-\{\alpha(p-\eta)q-1\}-\alpha(1-q)}E|X|^{p-\eta} \\ &\to 0, \quad n \to \infty. \end{aligned}$$

Therefore, (2.8) holds. So, in order to prove $I_1 < \infty$, it is enough to prove that

$$I_1^* := \sum_{n=1}^{\infty} n^{\alpha p-2}h(n)P\left(\max_{1\le j\le n} \left| \sum_{i=1}^{j} \bigl(X_i^{(n,1)} - EX_i^{(n,1)}\bigr) \right| > \varepsilon n^{\alpha}/6 \right) < \infty. \tag{2.9}$$

By Lemma 1.1 for $\forall n \ge 1$, $\{X_i^{(n,1)} - EX_i^{(n,1)}, 1 \le i \le n\}$ is a sequence of NOD random variables. When $0 < p \le 2$, by $\alpha(p-\eta) > 1$ and $0 < q < 1$, we have $\alpha - \frac{1}{2} - \alpha(1 - \frac{p-\eta}{2})q > \alpha - \frac{1}{2} - \alpha(1 - \frac{p-\eta}{2}) > 0$. Taking v such that $v > \max\{2, p, (\alpha p - 1)/(\alpha - 1/2), (\alpha p - 1)/(\alpha -$

$\frac{1}{2} - \alpha(1 - \frac{p-\eta}{2})q), \frac{p-(p-\eta)q}{1-q}\}$, we get by the Markov inequality, the C_r inequality, the Hölder inequality, and Lemma 1.3,

$$I_1^* \le C\sum_{n=1}^{\infty} n^{\alpha p-\alpha\nu-2}h(n)\big(\log(4n)\big)^{\nu}\sum_{i=1}^{n}E\big|X_i^{(n,1)}\big|^{\nu} + C\sum_{n=1}^{\infty} n^{\alpha p-\alpha\nu-2}h(n)\big(\log(4n)\big)^{\nu}\left(\sum_{i=1}^{n}E\big|X_i^{(n,1)}\big|^2\right)^{\nu/2} \stackrel{\text{def}}{=} I_{11}^* + I_{12}^*.$$

By the C_r inequality, Lemma 1.4, and Lemma 1.5, we have

$$\begin{aligned} I_{11}^* &\le C\sum_{n=1}^{\infty} n^{\alpha p-\alpha\nu-2}h(n)\big(\log(4n)\big)^{\nu}\sum_{i=1}^{n}E\big\{|X_i|^{\nu}I\big(|X_i|\le n^{\alpha q}\big) + n^{\alpha q\nu}P\big(|X_i|>n^{\alpha q}\big)\big\} \\ &\le C\sum_{n=1}^{\infty} n^{\alpha p-\alpha\nu-1}h(n)\big(\log(4n)\big)^{\nu}E\big\{|X|^{\nu}I\big(|X|\le n^{\alpha q}\big) + n^{\alpha q\nu}P\big(|X|>n^{\alpha q}\big)\big\} \\ &\le C\sum_{n=1}^{\infty} n^{\alpha\{-(1-q)\nu+p-q(p-\eta)\}-1}h(n)\big(\log(4n)\big)^{\nu}E|X|^{p-\eta} < \infty. \end{aligned}$$

By the C_r inequality and Lemma 1.4,

$$\begin{aligned} I_{12}^* &\le C\sum_{n=1}^{\infty} n^{\alpha p-\alpha\nu-2}h(n)\big(\log(4n)\big)^{\nu}\left\{\sum_{i=1}^{n}\big(E|X_i|^2I\big(|X_i|\le n^{\alpha q}\big) + n^{2\alpha q}P\big(|X_i|>n^{\alpha q}\big)\big)\right\}^{\nu/2} \\ &\le C\sum_{n=1}^{\infty} n^{\alpha p-2-(\alpha-1/2)\nu}h(n)\big(\log(4n)\big)^{\nu}\big\{E|X|^2I\big(|X|\le n^{\alpha q}\big) + n^{2\alpha q}P\big(|X|>n^{\alpha q}\big)\big\}^{\nu/2}. \end{aligned}$$

When $p > 2$,

$$I_{12}^* \le C\sum_{n=1}^{\infty} n^{\alpha p-2-(\alpha-1/2)\nu}h(n)\big(\log(4n)\big)^{\nu}\big(EX^2\big)^{\nu/2} < \infty.$$

When $0 < p \le 2$,

$$\begin{aligned} I_{12}^* &\le C\sum_{n=1}^{\infty} n^{\alpha p-2-(\alpha-1/2)\nu}h(n)\big(\log(4n)\big)^{\nu}\big(E|X|^{p-\eta}\big)^{\nu/2}n^{\alpha q\{2-(p-\eta)\}\nu/2} \\ &\le C\sum_{n=1}^{\infty} n^{\alpha p-2-\{\alpha-\frac{1}{2}-\alpha(1-\frac{p-\eta}{2})q\}\nu}h(n)\big(\log(4n)\big)^{\nu} < \infty. \end{aligned}$$

Therefore, (2.9) holds for I_2. Define $Y_i^{(n,2)} = (X_i - n^{\alpha q})I(n^{\alpha q} < X_i \le n^{\alpha} + n^{\alpha q}) + n^{\alpha}I(X_i > n^{\alpha} + n^{\alpha q})$, $1 \le i \le n$, $n \ge 1$, since $X_i^{(n,2)} = Y_i^{(n,2)} + (X_i - n^{\alpha q} - n^{\alpha})I(X_i > n^{\alpha} + n^{\alpha q})$, we have

$$\begin{aligned} I_2 &\le \sum_{n=1}^{\infty} n^{\alpha p-2}h(n)P\left(\sum_{i=1}^{n}Y_i^{(n,2)} > \varepsilon n^{\alpha}/6\right) \\ &\quad + \sum_{n=1}^{\infty} n^{\alpha p-2}h(n)P\left(\sum_{i=1}^{n}\big(X_i - n^{\alpha q} - n^{\alpha}\big)I\big(X_i > n^{\alpha} + n^{\alpha q}\big) > \varepsilon n^{\alpha}/6\right) \\ &\stackrel{\text{def}}{=} I_{21} + I_{22}. \end{aligned} \tag{2.10}$$

By Lemma 1.5, (2.1), and a standard computation, we have

$$\begin{aligned} I_{22} &\le \sum_{n=1}^{\infty} n^{\alpha p-2} h(n) \sum_{i=1}^{n} P\big(X_i > n^{\alpha} + n^{\alpha q}\big) \le \sum_{n=1}^{\infty} n^{\alpha p-2} h(n) \sum_{i=1}^{n} P\big(|X_i| > n^{\alpha}\big) \\ &\le C \sum_{n=1}^{\infty} n^{\alpha p-1} h(n) P\big(|X| > n^{\alpha}\big) \le C + CE|X|^p h\big(|X|^{1/\alpha}\big) < \infty. \end{aligned} \tag{2.11}$$

Now we prove $I_{21} < \infty$. By (2.1) and Lemma 1.4, we have

$$\begin{aligned} 0 &\le n^{-\alpha} \sum_{i=1}^{n} EY_i^{(n,2)} \\ &\le \begin{cases} n^{-\alpha} \sum_{i=1}^{n} EX_i I(X_i > n^{\alpha q}), & \text{if } p > 1, \\ n^{-\alpha} \sum_{i=1}^{n} \{E|X_i| I(|X_i| \le 2n^{\alpha}) + n^{\alpha} P(|X_i| > 2n^{\alpha q})\}, & \text{if } 0 < p \le 1 \end{cases} \\ &\le \begin{cases} Cn^{-\{\alpha(p-\eta)q-1\}-\alpha(1-q)} E|X|^{p-\eta}, & \text{if } p > 1, \\ Cn^{1-\alpha(p-\eta)q} E|X|^{p-\eta}, & \text{if } 0 < p \le 1 \end{cases} \to 0, \quad n \to \infty. \end{aligned}$$

Therefore, in order to prove $I_{21} < \infty$, it is enough to prove that

$$I_{21}^{*} \le \sum_{n=1}^{\infty} n^{\alpha p-2} h(n) P\left(\sum_{i=1}^{n} \big(Y_i^{(n,2)} - EY_i^{(n,2)}\big) > \varepsilon n^{\alpha}/12 \right) < \infty. \tag{2.12}$$

Taking ν such that $\nu > \max\{2, \frac{\alpha p-1}{\alpha-1/2}, \frac{2(\alpha p-1)}{\alpha(p-\eta)-1}\}$, we get by Lemma 1.1, the Markov inequality, the C_r inequality, the Hölder inequality, and Lemma 1.2,

$$\begin{aligned} I_{21}^{*} &\le C \sum_{n=1}^{\infty} n^{\alpha p-\alpha\nu-2} h(n) E\left| \sum_{i=1}^{n} \big(Y_i^{(n,2)} - EY_i^{(n,2)}\big) \right|^{\nu} \\ &\le C \sum_{n=1}^{\infty} n^{\alpha p-\alpha\nu-2} h(n) \sum_{i=1}^{n} E\big|Y_i^{(n,2)}\big|^{\nu} + C \sum_{n=1}^{\infty} n^{\alpha p-\alpha\nu-2} h(n) \left(\sum_{i=1}^{n} E\big(Y_i^{(n,2)}\big)^2 \right)^{\nu/2} \\ &\stackrel{\text{def}}{=} I_{211}^{*} + I_{212}^{*}. \end{aligned}$$

By the C_r inequality, Lemma 1.4, Lemma 1.5, (2.1), and a standard computation, we have

$$\begin{aligned} I_{211}^{*} &= C \sum_{n=1}^{\infty} n^{\alpha p-\alpha\nu-2} h(n) \sum_{i=1}^{n} E\big|Y_i^{(n,2)}\big|^{\nu} \\ &\le C \sum_{n=1}^{\infty} n^{\alpha p-\alpha\nu-2} h(n) \sum_{i=1}^{n} \big\{EX_i^{\nu} I\big(n^{\alpha q} < X_i \le n^{\alpha q} + n^{\alpha}\big) + n^{\alpha\nu} P\big(X_i > n^{\alpha q} + n^{\alpha}\big)\big\} \\ &\le C \sum_{n=1}^{\infty} n^{\alpha p-\alpha\nu-2} h(n) \sum_{i=1}^{n} \big\{E|X_i|^{\nu} I\big(|X_i| \le 2n^{\alpha}\big) + n^{\alpha\nu} P\big(|X_i| > n^{\alpha}\big)\big\} \\ &\le C \sum_{n=1}^{\infty} n^{\alpha p-\alpha\nu-1} h(n) \big\{E|X|^{\nu} I\big(|X| \le 2n^{\alpha}\big) + n^{\alpha\nu} P\big(|X| > n^{\alpha}\big)\big\} \\ &\le C + CE|X|^p h\big(|X|^{1/\alpha}\big) < \infty \end{aligned}$$

and

$$\begin{aligned}
I_{212}^* &\le C\sum_{n=1}^{\infty} n^{\alpha p-\alpha\nu-2}h(n)\left\{\sum_{i=1}^{n}\left(EX_i^2 I\left(n^{\alpha q} < X_i \le n^{\alpha q}+n^{\alpha}\right)+n^{2\alpha}P\left(X_i > n^{\alpha q}+n^{\alpha}\right)\right)\right\}^{\nu/2} \\
&\le C\sum_{n=1}^{\infty} n^{\alpha p-\alpha\nu+\nu/2-2}h(n)\left\{EX^2 I\left(|X|\le 2n^{\alpha}\right)+n^{2\alpha}P\left(|X|>n^{\alpha}\right)\right\}^{\nu/2} \\
&\le \begin{cases} C\sum_{n=1}^{\infty} n^{\alpha p-(\alpha-1/2)\nu-2}h(n)(EX^2)^{\nu/2}, & \text{if } p>2, \\ C\sum_{n=1}^{\infty} n^{\alpha p-2-\{\alpha(p-\eta)-1\}\nu/2}h(n)(E|X|^{p-\eta})^{\nu/2}, & \text{if } p\le 2 \end{cases} \\
&\le \begin{cases} C\sum_{n=1}^{\infty} n^{\alpha p-(\alpha-1/2)\nu-2}h(n), & \text{if } p>2, \\ C\sum_{n=1}^{\infty} n^{\alpha p-2-\{\alpha(p-\eta)-1\}\nu/2}h(n), & \text{if } p\le 2 \end{cases} \\
&< \infty.
\end{aligned}$$

Therefore, (2.12) holds. By (2.10)-(2.12) we get $I_2 < \infty$. In a similar way of $I_2 < \infty$ we can obtain $I_3 < \infty$. Thus, (2.2) holds.

(2.2) ⇒ (2.3). Note that $|S_n^{(k)}| = |S_n - X_k| \le |S_n| + |X_k| = |S_n| + |S_k - S_{k-1}| \le |S_n| + |S_k| + |S_{k-1}| \le 3\max_{1\le j\le n}|S_j|$, we have $(\max_{1\le k\le n}|S_n^{(k)}| \ge \varepsilon n^{\alpha}) \subseteq (\max_{1\le j\le n}|S_j| \ge \varepsilon n^{\alpha}/3)$, hence, from (2.2), (2.3) holds.

(2.3) ⇒ (2.4). Since $\frac{1}{2}|S_n| \le \frac{n-1}{n}|S_n| = |\frac{1}{n}\sum_{k=1}^{n} S_n^{(k)}| \le \max_{1\le k\le n}|S_n^{(k)}|$, $\forall n\ge 2$, and $|X_k| = |S_n - S_n^{(k)}| \le |S_n| + |S_n^{(k)}|$, we have $(\max_{1\le k\le n}|X_k| \ge \varepsilon n^{\alpha}) \subseteq (|S_n| \ge \varepsilon n^{\alpha}/2) \cup (\max_{1\le k\le n}|S_n^{(k)}| \ge \varepsilon n^{\alpha}/2) \subseteq (\max_{1\le k\le n}|S_n^{(k)}| \ge \varepsilon n^{\alpha}/4)$, $\forall n \ge 1$, hence, from (2.3), (2.4) holds.

(2.2) ⇒ (2.5). By Lemma 1.5 and (2.3), we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha p-2}h(n)P\left(\sup_{j\ge n} j^{-\alpha}|S_j| \ge \varepsilon\right) \\
&\quad = \sum_{i=1}^{\infty}\sum_{2^{i-1}\le n<2^i} n^{\alpha p-2}h(n)P\left(\sup_{j\ge n} j^{-\alpha}|S_j| \ge \varepsilon\right) \\
&\quad \le C\sum_{i=1}^{\infty} 2^{i(\alpha p-1)}h\left(2^i\right)P\left(\sup_{j\ge 2^{i-1}} j^{-\alpha}|S_j| \ge \varepsilon\right) \\
&\quad \le C\sum_{i=1}^{\infty} 2^{i(\alpha p-1)}h\left(2^i\right)\sum_{k=i}^{\infty} P\left(\max_{2^{k-1}\le j<2^k}|S_j| \ge \varepsilon 2^{\alpha(k-1)}\right) \\
&\quad \le C\sum_{k=1}^{\infty} P\left(\max_{2^{k-1}\le j<2^k}|S_j| \ge \varepsilon 2^{\alpha(k-1)}\right)\sum_{i=1}^{k} 2^{i(\alpha p-1)}h\left(2^i\right) \\
&\quad \le C\sum_{k=1}^{\infty} 2^{k(\alpha p-1)}h\left(2^k\right)P\left(\max_{1\le j<2^k}|S_j| \ge \varepsilon 2^{\alpha(k-1)}\right) < \infty.
\end{aligned}$$

(2.5) ⇒ (2.6). The proof of (2.5) ⇒ (2.6) is similar to that of (2.2) ⇒ (2.4), so it is omitted. □

Theorem 2.2 *Let $\alpha > 1/2$, $p > 0$, $\alpha p > 1$ and $h(x)$ be a slowly varying function at infinity. Let $\{X_n, n\ge 1\}$ be a sequence of NOD random variables and X be a random variables possibly* *defined on a different space. Moreover, additionally assume that for* $\alpha \le 1$, $EX_n = 0$ *for all* $n \ge 1$. *If there exist constant* $D_1 > 0$ *and* $D_2 > 0$ *such that*

$$\frac{D_1}{n}\sum_{i=n}^{2n-1} P\big(|X_i| > x\big) \le P\big(|X| > x\big) \le \frac{D_2}{n}\sum_{i=n}^{2n-1} P\big(|X_i| > x\big), \quad \forall x > 0, n \ge 1,$$

then (2.1)-(2.6) *are equivalent.*

Proof From the proof of Theorem 2.1, in order to prove Theorem 2.2, it is enough to show that (2.4) ⇒ (2.6) and (2.6) ⇒ (2.1). The proof of (2.4) ⇒ (2.6) is similar to that of (2.2) ⇒ (2.5). Now, we prove (2.6) ⇒ (2.1). Firstly we prove that

$$\lim_{n\to\infty} P\Big(\sup_{j\ge n} j^{-\alpha}|X_j| \ge \varepsilon\Big) = 0, \quad \forall\varepsilon > 0. \tag{2.13}$$

Otherwise, there are $\varepsilon_0 > 0$, $\delta > 0$, and a sequence of positive integers $\{n_k, k \ge 1\}$, $n_k \uparrow \infty$ such that $P(\sup_{j\ge n_k} j^{-\alpha}|X_j| \ge \varepsilon_0) \ge \delta$, $\forall k \ge 1$. Without loss of generality, we can assume that $n_{k+1} \ge 2n_k$, $\forall k \ge 1$. Therefore, we have

$$P\Big(\sup_{j\ge 2n_k} j^{-\alpha}|X_j| \ge \varepsilon_0\Big) \ge \delta, \quad \forall k \ge 1.$$

By $\alpha p > 1$ we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha p-2}h(n)P\Big(\sup_{j\ge n} j^{-\alpha}|X_j| \ge \varepsilon_0\Big)\\
&\quad\ge \sum_{k=1}^{\infty}\sum_{n=n_k+1}^{2n_k} n^{\alpha p-2}h(n)P\Big(\sup_{j\ge n} j^{-\alpha}|X_j| \ge \varepsilon_0\Big)\\
&\quad\ge C\sum_{k=1}^{\infty} n_k^{\alpha p-1}h(n_k)P\Big(\sup_{j\ge 2n_k} j^{-\alpha}|X_j| \ge \varepsilon_0\Big) = \infty,
\end{aligned}$$

which is in contradiction with (2.6), thus, (2.13) holds. By Lemma 1.1, we get

$$\begin{aligned}
P\Big(\sup_{j\ge n} j^{-\alpha}|X_j| \ge \varepsilon\Big) &\ge P\Big(\max_{n\le j<2n} j^{-\alpha}|X_j| \ge \varepsilon\Big)\\
&\ge P\Big(\max_{n\le j<2n} |X_j| \ge (2n)^{\alpha}\varepsilon\Big)\\
&\ge 1 - P\Big(\max_{n\le j<2n} X_j < (2n)^{\alpha}\varepsilon\Big) = 1 - E\Bigg(\prod_{j=n}^{2n-1} I\big(X_j < (2n)^{\alpha}\varepsilon\big)\Bigg)\\
&\ge 1 - \prod_{j=n}^{2n-1} P\big(X_j < (2n)^{\alpha}\varepsilon\big) = 1 - \prod_{j=n}^{2n-1}\big(1 - P\big(X_j \ge (2n)^{\alpha}\varepsilon\big)\big)\\
&\ge 1 - \exp\Bigg(-\sum_{j=n}^{2n-1} P\big(X_j \ge (2n)^{\alpha}\varepsilon\big)\Bigg).
\end{aligned}$$

By (2.13), we have $\lim_{n\to\infty}\sum_{j=n}^{2n-1} P(X_j \geq (2n)^\alpha\varepsilon) = 0$, $\forall\varepsilon > 0$. Therefore, when n is large enough, we have

$$P\Bigl(\max_{n\leq j<2n} j^{-\alpha}|X_j| \geq \varepsilon\Bigr) \geq 1 - \left\{1 - \sum_{j=n}^{2n-1} P\bigl(X_j \geq (2n)^\alpha\varepsilon\bigr) + \frac{1}{2}\left(\sum_{j=n}^{2n-1} P\bigl(X_j \geq (2n)^\alpha\varepsilon\bigr)\right)^2\right\}$$
$$\geq C\sum_{j=n}^{2n-1} P\bigl(X_j \geq (2n)^\alpha\varepsilon\bigr), \quad \forall\varepsilon > 0.$$

In a similar way, when n is large enough,

$$P\Bigl(\max_{n\leq j<2n} j^{-\alpha}|X_j| \geq \varepsilon\Bigr) \geq C\sum_{j=n}^{2n-1} P\bigl(-X_j \geq (2n)^\alpha\varepsilon\bigr), \quad \forall\varepsilon > 0.$$

Thus, when n is large enough, we have

$$P\Bigl(\max_{n\leq j<2n} j^{-\alpha}|X_j| \geq \varepsilon\Bigr) \geq C\sum_{j=n}^{2n-1} P\bigl(|X_j| \geq (2n)^\alpha\varepsilon\bigr) \geq CnP\bigl(|X| \geq (2n)^\alpha\varepsilon\bigr), \quad \forall\varepsilon > 0. \tag{2.14}$$

Taking $\varepsilon = 2^{-\alpha}$, by (2.6), (2.14), Lemma 1.5, and a standard computation, we have

$$\infty > \sum_{n=1}^{\infty} n^{\alpha p-2}h(n)P\Bigl(\sup_{j\geq n} j^{-\alpha}|X_j| \geq 2^{-\alpha}\Bigr) \geq \sum_{n=1}^{\infty} n^{\alpha p-2}h(n)P\Bigl(\max_{n\leq j<2n} j^{-\alpha}|X_j| \geq 2^{-\alpha}\Bigr)$$
$$\geq C\sum_{n=1}^{\infty} n^{\alpha p-1}h(n)P\bigl(|X| \geq n^\alpha\bigr)$$
$$\geq CE|X|^p h\bigl(|X|^{1\alpha}\bigr).$$

Thus, (2.1) holds. □

In the following, let $\{\tau_n, n \geq 1\}$ be a sequence of non-negative, integer valued random variables and τ a positive random variable. All random variables are defined on the same probability space.

Theorem 2.3 *Let $\alpha > 1/2$, $p > 0$, $\alpha p > 1$ and $h(x) > 0$ be a slowly varying function as $x \to +\infty$. Let $\{X_n, n \geq 1\}$ be a sequence of NOD random variables and X be a random variables possibly defined on a different space satisfying the condition* (1.1) *and* (2.1). *Moreover, additionally assume that for $\alpha \leq 1$, $EX_n = 0$ for all $n \geq 1$. If there exists $\lambda > 0$ such that $\sum_{n=1}^{\infty} n^{\alpha p-2}h(n)P(\frac{\tau_n}{n} < \lambda) < \infty$, then*

$$\sum_{n=1}^{\infty} n^{\alpha p-2}h(n)P\bigl(|S_{\tau_n}| \geq \varepsilon\tau_n^\alpha\bigr) < \infty, \quad \forall\varepsilon > 0. \tag{2.15}$$

Proof Note that

$$\bigl(|S_{\tau_n}| \geq \varepsilon\tau_n^\alpha\bigr) \subseteq (\tau_n/n < \lambda) \cup \bigl(|S_{\tau_n}| \geq \varepsilon\tau_n^\alpha, \tau_n \geq \lambda n\bigr) \subseteq (\tau_n/n < \lambda) \cup \Bigl(\sup_{j\geq\lambda n} j^{-\alpha}|S_j| \geq \varepsilon\Bigr).$$

Thus, by (2.5) of Theorem 2.1, we have (2.15). □

Theorem 2.4 *Let $\alpha > 1/2$, $p > 0$, $\alpha p > 1$ and $h(x)$ be a slowly varying function at infinity. Let $\{X_n, n \geq 1\}$ be a sequence of NOD random variables and X be a random variables possibly defined on a different space satisfying the condition* (1.1) *and* (2.1). *Moreover, additionally assume that for $\alpha \leq 1$, $EX_n = 0$ for all $n \geq 1$. If there exists $\theta > 0$ such that $\sum_{n=1}^{\infty} n^{\alpha p-2} h(n) P(|\frac{\tau_n}{n} - \tau| > \theta) < \infty$ with $P(\tau \leq B) = 1$ for some $B > 0$, then*

$$\sum_{n=1}^{\infty} n^{\alpha p-2} h(n) P\bigl(|S_{\tau_n}| \geq \varepsilon n^{\alpha}\bigr) < \infty, \quad \forall \varepsilon > 0. \tag{2.16}$$

Proof Note that

$$\begin{aligned}
\bigl(|S_{\tau_n}| \geq \varepsilon n^{\alpha}\bigr) &\subseteq \left(\left|\frac{\tau_n}{n} - \tau\right| > \theta\right) \cup \left(|S_{\tau_n}| \geq \varepsilon n^{\alpha}, \left|\frac{\tau_n}{n} - \tau\right| \leq \theta\right) \\
&\subseteq \left(\left|\frac{\tau_n}{n} - \tau\right| > \theta\right) \cup \bigl(|S_{\tau_n}| \geq \varepsilon n^{\alpha}, \tau_n \leq (\tau + \theta)n\bigr) \\
&\subseteq \left(\left|\frac{\tau_n}{n} - \tau\right| > \theta\right) \cup \bigl(|S_{\tau_n}| \geq \varepsilon n^{\alpha}, \tau_n \leq (B + \theta)n\bigr) \\
&\subseteq \left(\left|\frac{\tau_n}{n} - \tau\right| > \theta\right) \cup \Bigl(\max_{1 \leq j \leq (B+\theta)n} |S_j| \geq \varepsilon n^{\alpha}\Bigr).
\end{aligned}$$

Thus, by (2.2) of Theorem 2.1, we have (2.16). □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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