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Some exact constants for the approximation of the quantity in the Wallis' formula

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Abstract
In this article, a sharp two-sided bounding inequality and some best constants for the approximation of the quantity associated with the Wallis' formula are presented.
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1 Introduction and main result

Throughout the paper, $\mathbb{Z}$ denotes the set of all integers, $\mathbb{N}$ denotes the set of all positive integers,

$$\begin{aligned} &\mathbb{N}_0 := \mathbb{N} \cup \{0\}, \\ &n!! := \prod_{i=0}^{[(n-1)/2]} (n-2i), \end{aligned} \tag{1}$$

and

$$W_n := \frac{(2n-1)!!}{(2n)!!}. \tag{2}$$

Here in (1), the floor function $[t]$ denotes the integer which is less than or equal to the number t.

The Euler gamma function is defined and denoted for $\operatorname{Re} z > 0$ by

$$\Gamma(z) := \int_0^\infty t^{z-1} e^{-t}\, dt. \tag{3}$$

One of the elementary properties of the gamma function is that

$$\Gamma(x+1) = x\Gamma(x). \tag{4}$$

In particular,

$$\Gamma(n+1) = n!, \quad n \in \mathbb{N}_0. \tag{5}$$

Also, note that

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}. \tag{6}$$

For the approximation of $n!$, a well-known result is the following Stirling's formula:

$$n! \sim \sqrt{2\pi n}n^n e^{-n}, \quad n \to \infty, \tag{7}$$

which is an important tool in analytical probability theory, statistical physics and physical chemistry.

Consider the quantity W_n, defined by (2). This quantity is important in the probability theory - for example, the three events, (a) a return to the origin takes place at time $2n$, (b) no return occurs up to and including time $2n$, and (c) the path is non-negative between 0 and $2n$, have the common probability W_n. Also, the probability that in the time interval from 0 to $2n$ the particle spends $2k$ time units on the positive side and $2n - 2k$ time units on the negative side is $W_k W_{n-k}$. For details of these interesting results, one may see [1, Chapter III].

W_n is closely related to the Wallis' formula.

The Wallis' formula

$$\frac{2}{\pi} = \prod_{n=1}^{\infty} \frac{(2n-1)(2n+1)}{(2n)^2} \tag{8}$$

can be obtained by taking

$$x = \frac{\pi}{2}$$

in the infinite product representation of $\sin x$ (see [2, p.10], [3, p.211])

$$\sin x = x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2\pi^2}\right), \quad x \in \mathbb{R}. \tag{9}$$

Since

$$\prod_{n=1}^{\infty} \frac{(2n-1)(2n+1)}{(2n)^2} = \lim_{n\to\infty} (2n+1) W_n^2, \tag{10}$$

another important form of Wallis' formula is (see [4, pp.181-184])

$$\lim_{n\to\infty} (2n+1) W_n^2 = \frac{2}{\pi}. \tag{11}$$

The following generalization of Wallis' formula was given in [5].

$$\frac{\pi}{t\sin(\pi/t)} = \frac{1}{t-1} \prod_{i=1}^{\infty} \frac{(it)^2}{(it+t-1)(it-t+1)}, \quad t > 1. \tag{12}$$

In fact, by letting

$$x = (1 - 1/t)\pi, \quad t \neq 0$$

in (9), we have

$$\sin\frac{\pi}{t} = \frac{\pi}{t}(t-1)\prod_{i=1}^{\infty}\frac{(it+t-1)(it-t+1)}{(it)^2}, \quad t \neq 0. \tag{13}$$

From (13), we get

$$\frac{\pi}{t\sin(\pi/t)} = \frac{1}{t-1}\prod_{i=1}^{\infty}\frac{(it)^2}{(it+t-1)(it-t+1)} \tag{14}$$

for

$$t \neq 0, \qquad t \neq \frac{1}{k}, \quad k \in \mathbb{Z}.$$

(12) is a special case of (14). The proof of (12) in [5] involves integrating powers of a generalized sine function.

There is a close relationship between Stirling's formula and Wallis' formula. The determination of the constant $\sqrt{2\pi}$ in the usual proof of Stirling's formula (7) or Stirling's asymptotic formula

$$\Gamma(x) \sim \sqrt{2\pi}x^{x-1/2}e^{-x}, \quad x \to \infty, \tag{15}$$

relies on Wallis' formula (see [2, pp.18-20], [3, pp.213-215], [4, pp.181-184]).

Also, note that

$$W_n = \left[(2n+1)\int_0^{\pi/2}\sin^{2n+1}x\,dx\right]^{-1} \tag{16}$$

$$= \left[(2n+1)\int_0^{\pi/2}\cos^{2n+1}x\,dx\right]^{-1} \tag{17}$$

and Wallis' sine (cosine) formula (see [6, p.258])

$$W_n = \frac{2}{\pi}\int_0^{\pi/2}\sin^{2n}x\,dx \tag{18}$$

$$= \frac{2}{\pi}\int_0^{\pi/2}\cos^{2n}x\,dx. \tag{19}$$

Some inequalities involving W_n were given in [7–12].

In this article, we give a sharp two-sided bounding inequality and some exact constants for the approximation of W_n, defined by (2). The main result of the paper is as follows.

Theorem 1 *For all $n \in \mathbb{N}$, $n \geq 2$,*

$$\sqrt{\frac{e}{\pi}}\left(1-\frac{1}{2n}\right)^n\frac{\sqrt{n-1}}{n} < W_n \leq \frac{4}{3}\left(1-\frac{1}{2n}\right)^n\frac{\sqrt{n-1}}{n}. \tag{20}$$

The constants $\sqrt{e/\pi}$ and $4/3$ in (20) *are best possible.*

Moreover,

$$W_n \sim \sqrt{\frac{e}{\pi}}\left(1-\frac{1}{2n}\right)^n \frac{\sqrt{n-1}}{n}, \quad n \to \infty. \tag{21}$$

Remark 1 By saying that the constants $\sqrt{e/\pi}$ and 4/3 in (20) are best possible, we mean that the constant $\sqrt{e/\pi}$ in (20) cannot be replaced by a number which is greater than $\sqrt{e/\pi}$ and the constant 4/3 in (20) cannot be replaced by a number which is less than 4/3.

2 Lemmas

We need the following lemmas to prove our result.

Lemma 1 ([13, Theorem 1.1]) *The function*

$$f(x) := \frac{x^{x+\frac{1}{2}}}{e^x \Gamma(x+1)} \tag{22}$$

is strictly logarithmically concave and strictly increasing from $(0,\infty)$ *onto* $(0, \frac{1}{\sqrt{2\pi}})$.

Lemma 2 ([13, Theorem 1.3]) *The function*

$$h(x) := \frac{e^x \sqrt{x-1}\,\Gamma(x+1)}{x^{x+1}} \tag{23}$$

is strictly logarithmically concave and strictly increasing from $(1,\infty)$ *onto* $(0, \sqrt{2\pi})$.

Lemma 3 ([6, p.258]) *For all* $n \in \mathbb{N}$,

$$\Gamma\left(n+\frac{1}{2}\right) = \sqrt{\pi}\, n!\, W_n, \tag{24}$$

where W_n *is defined by* (2).

Remark 2 Some functions associated with the functions $f(x)$ and $h(x)$, defined by (22) and (23) respectively, were proved to be logarithmically completely monotonic in [14–16]. For more recent work on (logarithmically) completely monotonic functions, please see, for example, [17–43].

3 Proof of the main result

Proof of Theorem 1 By Lemma 1, we have

$$\frac{3}{e\sqrt{e\pi}} = f\left(\frac{3}{2}\right) \le f\left(n-\frac{1}{2}\right) = \frac{(n-\frac{1}{2})^n}{e^{n-1/2}\Gamma(n+1/2)} < \frac{1}{\sqrt{2\pi}}, \quad n \ge 2, \tag{25}$$

and

$$\lim_{n\to\infty} \frac{(n-\frac{1}{2})^n}{e^{n-1/2}\Gamma(n+1/2)} = \frac{1}{\sqrt{2\pi}}. \tag{26}$$

The lower and upper bounds in (25) are best possible.

By Lemma 3, (25) and (26) can be rewritten respectively as

$$\frac{3}{e^2} \leq \frac{(n-\frac{1}{2})^n}{W_n e^n n!} < \frac{1}{\sqrt{2e}}, \quad n \geq 2, \tag{27}$$

and

$$\lim_{n\to\infty} \frac{(n-\frac{1}{2})^n}{W_n e^n n!} = \frac{1}{\sqrt{2e}}. \tag{28}$$

The constants $3/e^2$ and $1/\sqrt{2e}$ in (27) are best possible.

By Lemma 2, we get

$$\left(\frac{e}{2}\right)^2 = h(2) \leq h(n) = \frac{e^n n! \sqrt{n-1}}{n^{n+1}} < \sqrt{2\pi}, \quad n \geq 2, \tag{29}$$

and

$$\lim_{n\to\infty} \frac{e^n n! \sqrt{n-1}}{n^{n+1}} = \sqrt{2\pi}. \tag{30}$$

The lower bound $(e/2)^2$ and the upper bound $\sqrt{2\pi}$ in (29) are best possible.

From (27) and (29), we obtain that for all $n \geq 2$,

$$\frac{3}{4} \leq \frac{\sqrt{n-1}(n-\frac{1}{2})^n}{W_n n^{n+1}} < \sqrt{\frac{\pi}{e}}. \tag{31}$$

The constants $3/4$ and $\sqrt{\pi/e}$ in (31) are best possible. From (31) we get that for all $n \geq 2$,

$$\sqrt{\frac{e}{\pi}}\left(1-\frac{1}{2n}\right)^n \frac{\sqrt{n-1}}{n} < W_n \leq \frac{4}{3}\left(1-\frac{1}{2n}\right)^n \frac{\sqrt{n-1}}{n}. \tag{32}$$

The constants $\sqrt{e/\pi}$ and $4/3$ in (32) are best possible.

From (28) and (30), we see that

$$\lim_{n\to\infty} \frac{\sqrt{n-1}(n-\frac{1}{2})^n}{W_n n^{n+1}} = \sqrt{\frac{\pi}{e}}, \tag{33}$$

which is equivalent to (21).

The proof is thus completed. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All the authors contributed to the writing of the present article. They also read and approved the final manuscript.

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