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On fixed point theorems involving altering distances in Menger probabilistic metric spaces

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Abstract
In this paper, we show by means of an example that the results of Babačev (Appl. Anal. Discrete Math. 6:257-264, 2012) do not hold for the class of t-norms $T \leq T_p$. Further, we prove a fixed point theorem for quasi-type contraction involving altering distance functions which is weaker than that proposed by Babačev but for any continuous t-norm in a complete Menger space.

MSC: 47H10; 54H25

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1 Introduction

Probabilistic metric spaces (in short PM-spaces) are a probabilistic generalization of metric spaces which are appropriate to carry out the study of those situations wherein distances are measured in terms of distribution functions rather than non-negative real numbers. The study of PM-spaces was initiated by Menger [1]. Schweizer and Sklar [2] further enriched this concept and provided a new impetus by proving some fundamental results on this theme.

The first result on fixed point theory in PM-spaces was given by Sehgal and Bharueha-Reid [3] wherein the notion of probabilistic B-contraction was introduced and a generalization of the classical Banach fixed point principle to complete Menger PM-spaces was given. In [3], it was proved that any B-contraction on a complete Menger space (S, F, T_M), where t-norm T_M is defined by $T_M(x, y) = \min\{x, y\}$, has a unique fixed point. In 1982, Hadžić [4] extended the result contained in [3] for a more general class of t-norms called H-type t-norms (see also [5]).

After that several types of contractions and associated fixed point theorems have been established in PM-spaces by various authors, *e.g.*, [6–8] (see also [9–12]). We also refer to a nice book on this topic by Hadžić and Pap [13]. In this continuation, Choudhury and Das [14] extended the classical metric fixed point result of Khan *et al.* [15] by introducing the idea of altering distance functions in PM-spaces. In [14], it was proved that any probabilistic ϕ-contraction on a complete Menger space (S, F, T_M) has a unique fixed point.

An open problem that remains to be investigated is whether the results are valid in the cases of any arbitrary continuous t-norm. (However, in [16] Miheţ gave an affirmative answer to the question raised in [14] using the idea of probabilistic boundedness and H-type t-norms along with some additional conditions.) Very recently, Babačev in [17] extended and improved the results of Choudhury and Das [14] for a nonlinear generalized contraction wherein she used the associated t-norm as a min norm.

In this paper, we show by means of an example that Babačev's [17] results do not hold for the class of t-norms $T \leq T_p$. Further, we prove a fixed point theorem for quasi-type contraction involving altering distance functions in a complete Menger space for any continuous t-norm T.

2 Preliminaries

Consistent with Choudhury and Das [14], Choudhury *et al.* [18] and Babačev [17], the following definitions and results will be needed in the sequel.

In the standard notation, let $\mathcal{D}^+$ be the set of all distribution functions $F\colon \mathbb{R} \to [0,1]$ such that F is a nondecreasing, left-continuous mapping which satisfies $F(0) = 0$ and $\sup_{x\in\mathbb{R}} F(x) = 1$. The space $\mathcal{D}^+$ is partially ordered by the usual point-wise ordering of functions, *i.e.*, $F \leq G$ if and only if $F(t) \leq G(t)$ for all $t \in \mathbb{R}$. The maximal element for $\mathcal{D}^+$ in this order is the distribution function given by

$$\epsilon_0(t) = \begin{cases} 0, & t \leq 0, \\ 1, & t > 0. \end{cases}$$

Definition 2.1 [2] A binary operation $T\colon [0,1] \times [0,1] \to [0,1]$ is a continuous t-norm if it satisfies the following conditions:

(a) T is commutative and associative,
(b) T is continuous,
(c) $T(a,1) = a$ for all $a \in [0,1]$,
(d) $T(a,b) \leq T(c,d)$ whenever $a \leq c$ and $b \leq d$, and $a,b,c,d \in [0,1]$.

The following are the three basic continuous t-norms:

(i) The minimum t-norm, T_M, is defined by $T_M(a,b) = \min\{a,b\}$.
(ii) The product t-norm, T_p, is defined by $T_p(a,b) = a \cdot b$.
(iii) The Lukasiewicz t-norm, T_L, is defined by $T_L(a,b) = \max\{a+b-1,0\}$.

Regarding the pointwise ordering, the following inequalities hold:

$$T_L < T_p < T_M.$$

Definition 2.2 A Menger probabilistic metric space (briefly, Menger PM-space) is a triple (X,F,T), where X is a nonempty set, T is a continuous t-norm, and F is a mapping from $X \times X$ into $\mathcal{D}^+$ such that if F_{xy} denotes the value of F at the pair (x,y), the following conditions hold:

(PM1) $F_{xy}(t) = \epsilon_0(t)$ if and only if $x = y$,
(PM2) $F_{xy}(t) = F_{yx}(t)$,
(PM3) $F_{xy}(t+s) \geq T(F_{xz}(t), F_{zy}(s))$ for all $x,y,z \in X$ and $s,t \geq 0$.

Remark 2.1 [3] Every metric space is a PM-space. Let (X,d) be a metric space and $T(a,b) = \min\{a,b\}$ be a continuous t-norm. Define $F_{xy}(t) = \epsilon_0(t - d(x,y))$ for all $x,y \in X$ and $t > 0$. The triple (X,F,T) is a PM-space induced by the metric d.

Definition 2.3 Let (X,F,T) be a Menger PM-space.

(1) A sequence $\{x_n\}_n$ in X is said to be convergent to x in X if for every $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer N such that $F_{x_n x}(\epsilon) > 1 - \lambda$ whenever $n \geq N$.
(2) A sequence $\{x_n\}_n$ in X is called a Cauchy sequence if for every $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer N such that $F_{x_n x_m}(\epsilon) > 1 - \lambda$ whenever $n, m \geq N$.
(3) The space X is said to be complete if every Cauchy sequence in X is convergent to a point in X.

The (ϵ, λ)-topology [2] in a Menger space (X, F, T) is introduced by the family of neighborhoods N_x of a point $x \in X$ given by

$$N_x = \{N_x(\epsilon, \lambda) : \epsilon > 0, \lambda \in (0,1)\},$$

where

$$N_x(\epsilon, \lambda) = \{y \in X : F_{xy}(\epsilon) > 1 - \lambda\}.$$

The (ϵ, λ)-topology is a Hausdorff topology. In this topology the function f is continuous in $x_0 \in X$ if and only if for every sequence $x_n \to x_0$ it holds that $f(x_n) \to f(x_0)$.

Definition 2.4 (Altering distance function [15]) The control function $\psi : [0, \infty) \to [0, \infty)$ is called an altering distance function if it has the following properties:
(i) ψ is monotone increasing and continuous,
(ii) $\psi(t) = 0$ if and only if $t = 0$.

The following category of functions was introduced in [14].

Definition 2.5 [14] A function $\phi\colon [0, \infty) \to [0, \infty)$ is said to be a Φ-function if it satisfies the following conditions:
(i) $\phi(t) = 0$ if and only if $t = 0$,
(ii) $\phi(t)$ is strictly monotone increasing and $\phi(t) \to \infty$ as $t \to \infty$,
(iii) ϕ is left continuous in $(0, \infty)$,
(iv) ϕ is continuous at 0.

The class of all Φ-functions will be denoted by Φ.

Lemma 2.1 [17] *Let (X, F, T) be a Menger PM-space. Let $\phi\colon [0, \infty) \to [0, \infty)$ be a Φ-function. Then the following statement holds.*
If for $x, y \in X$, $0 < c < 1$, we have $F_{xy}(\phi(t)) \geq F_{xy}(\phi(t/c))$ for all $t > 0$, then $x = y$.

Theorem 2.1 [17] *Let (X, F, T) be a complete Menger PM-space with a continuous t-norm T which satisfies $T(a, a) \geq a$ for every $a \in [0,1]$. Let $c \in (0,1)$ be fixed. If for a Φ-function ϕ and a self-mapping f on X,*

$$\begin{aligned} F_{fxfy}(\phi(t)) \geq \min\biggl\{ & F_{xy}\left(\phi\left(\frac{t}{c}\right)\right), F_{xfx}\left(\phi\left(\frac{t}{c}\right)\right), F_{yfy}\left(\phi\left(\frac{t}{c}\right)\right), \\ & F_{xfy}\left(2\phi\left(\frac{t}{c}\right)\right), F_{yfx}\left(2\phi\left(\frac{t}{c}\right)\right)\biggr\} \end{aligned} \tag{2.1}$$

holds for every $x, y \in X$ and all $t > 0$, then f has a unique fixed point in X.

3 Main results

We begin with the following example.

Example 3.1 Let $X = [0, \infty)$ and $T = T_p$. For each $t \in (0, \infty)$, define

$$F_{xy}(t) = \begin{cases} \frac{\min\{x,y\}}{\max\{x,y\}}, & \forall t > 0, x \neq y, \\ 1, & \forall t > 0, x = y. \end{cases}$$

It is clear that (X, F, T_p) is a complete Menger PM-space (see [19]) (but here (X, F, T_M) is not a PM-space). Let us consider the function

$$f : X \to X, \qquad fx = x + 1, \quad \forall x \in X.$$

Then it can be easily seen that the above example satisfies Theorem 2.1 for T_p. Indeed,

- Case I. If $x = y$, then inequality (2.1) is obviously true.
- Case II. If $x \neq y$ and $x < y$, then we have $x + 1 < y + 1$, $x < x + 1$, $y < y + 1$, $x < y + 1$.

For any function $\phi \in \Phi$ and $c \in (0, 1)$, inequality (2.1) becomes

$$\frac{x+1}{y+1} \geq \min\left\{\frac{x}{y}, \frac{x}{x+1}, \frac{y}{y+1}, \frac{x}{y+1}, \frac{\min\{y, x+1\}}{\max\{y, x+1\}}\right\}.$$

If $x + 1 < y$, then we have

$$\frac{x+1}{y+1} \geq \min\left\{\frac{x}{y}, \frac{x}{x+1}, \frac{y}{y+1}, \frac{x}{y+1}, \frac{x+1}{y}\right\}.$$

Clearly, the inequality holds with the minimum value $x/(y + 1)$.

If $x + 1 = y$, then we have

$$\frac{x+1}{y+1} \geq \min\left\{\frac{x}{y}, \frac{x}{x+1}, \frac{y}{y+1}, \frac{x}{y+1}, 1\right\}.$$

Then the inequality holds with the minimum value $x/(y + 1)$.

And if $x + 1 > y$, then we have

$$\frac{x+1}{y+1} \geq \min\left\{\frac{x}{y}, \frac{x}{x+1}, \frac{y}{y+1}, \frac{x}{y+1}, \frac{y}{x+1}\right\}.$$

The inequality holds with the minimum value $x/(y + 1)$.

- Case III. If $x \neq y$ and $x > y$, then we have $x + 1 > y + 1$, $x + 1 > y$.

By inequality (2.1), we have

$$\frac{y+1}{x+1} \geq \min\left\{\frac{y}{x}, \frac{x}{x+1}, \frac{y}{y+1}, \frac{\min\{x, y+1\}}{\max\{x, y+1\}}, \frac{y}{x+1}\right\}.$$

If $x > y + 1$, then we have

$$\frac{y+1}{x+1} \geq \min\left\{\frac{y}{x}, \frac{x}{x+1}, \frac{y}{y+1}, \frac{y+1}{x}, \frac{y}{x+1}\right\}.$$

The inequality holds with the minimum value $y/(x+1)$.

If $x = y+1$, then we have

$$\frac{y+1}{x+1} \geq \min\left\{\frac{y}{x}, \frac{x}{x+1}, \frac{y}{y+1}, 1, \frac{y}{x+1}\right\}.$$

The inequality holds with the minimum value $y/(x+1)$.

Finally, if $x < y+1$, then we have

$$\frac{y+1}{x+1} \geq \min\left\{\frac{y}{x}, \frac{x}{x+1}, \frac{y}{y+1}, \frac{x}{y+1}, \frac{y}{x+1}\right\}.$$

Again inequality holds with the minimum value $y/(x+1)$.

Thus the above example satisfies all the conditions of Theorem 2.1 with the t-norm T_p, but here the mapping f has no fixed point. Therefore, Theorem 2.1 cannot be generalized for $T \leq T_p$.

Now, we are motivated to introduce our result.

Theorem 3.1 *Let (X, F, T) be a complete Menger PM-space with a continuous t-norm, and let $c \in (0,1)$ be fixed. If for a Φ-function ϕ and a self-mapping f on X,*

$$\begin{aligned} F_{fxfy}\bigl(\phi(t)\bigr) \geq \min\biggl\{&F_{xy}\biggl(\phi\biggl(\frac{t}{c}\biggr)\biggr), F_{xfx}\biggl(\phi\biggl(\frac{t}{c}\biggr)\biggr), \\ &F_{yfy}\biggl(\phi\biggl(\frac{t}{c}\biggr)\biggr), F_{yfx}\biggl(\phi\biggl(\frac{t}{c}\biggr)\biggr)\biggr\} \end{aligned} \tag{3.1}$$

holds for every $x, y \in X$ and all $t > 0$, then f has a unique fixed point in X.

Proof Let $x_0 \in X$. Now, construct a sequence $\{x_n\}$ in X as follows:

$$x_n = fx_{n-1}, \quad n = 1, 2, \ldots.$$

Applying (3.1) for $x = x_{n-1}$ and $y = x_n$, we have

$$\begin{aligned} F_{x_n x_{n+1}}\bigl(\phi(t)\bigr) &= F_{fx_{n-1}fx_n}\bigl(\phi(t)\bigr) \\ &\geq \min\bigl\{F_{x_{n-1}x_n}\bigl(\phi(t/c)\bigr), F_{x_{n-1}x_n}\bigl(\phi(t/c)\bigr), F_{x_n x_{n+1}}\bigl(\phi(t/c)\bigr), F_{x_n x_n}\bigl(\phi(t/c)\bigr)\bigr\} \\ &= \min\bigl\{F_{x_{n-1}x_n}\bigl(\phi(t/c)\bigr), F_{x_n x_{n+1}}\bigl(\phi(t/c)\bigr)\bigr\} \end{aligned}$$

for all $t > 0$.

We should prove that

$$\min\bigl\{F_{x_{n-1}x_n}\bigl(\phi(t/c)\bigr), F_{x_n x_{n+1}}\bigl(\phi(t/c)\bigr)\bigr\} = F_{x_{n-1}x_n}\bigl(\phi(t/c)\bigr) \tag{3.2}$$

for all $t > 0$.

If it is not, there exists $p > 0$ such that

$$F_{x_{n-1}x_n}\bigl(\phi(p/c)\bigr) > F_{x_n x_{n+1}}\bigl(\phi(p/c)\bigr). \tag{3.3}$$

Then, using (3.1), we have

$$\begin{aligned} F_{x_{n+1}x_n}\bigl(\phi(p)\bigr) &\geq F_{x_nx_{n+1}}\bigl(\phi(p/c)\bigr) \\ &\geq \min\bigl\{F_{x_{n-1}x_n}\bigl(\phi\bigl(p/c^2\bigr)\bigr), F_{x_nx_{n+1}}\bigl(\phi\bigl(p/c^2\bigr)\bigr)\bigr\}. \end{aligned}$$

If

$$\min\bigl\{F_{x_{n-1}x_n}\bigl(\phi\bigl(p/c^2\bigr)\bigr), F_{x_nx_{n+1}}\bigl(\phi\bigl(p/c^2\bigr)\bigr)\bigr\} = F_{x_{n-1}x_n}\bigl(\phi\bigl(p/c^2\bigr)\bigr), \tag{3.4}$$

then by (3.3) and (3.4)

$$F_{x_{n-1}x_n}\bigl(\phi(p/c)\bigr) > F_{x_nx_{n+1}}\bigl(\phi(p/c)\bigr) \geq F_{x_{n-1}x_n}\bigl(\phi\bigl(p/c^2\bigr)\bigr).$$

So, we get a contradiction.

Accordingly, $\min\{F_{x_{n-1}x_n}(\phi(p/c^2)), F_{x_nx_{n+1}}(\phi(p/c^2))\} = F_{x_nx_{n+1}}(\phi(p/c^2))$.

Now, we have

$$\begin{aligned} F_{x_nx_{n+1}}\bigl(\phi(p)\bigr) &\geq F_{x_nx_{n+1}}\bigl(\phi(p/c)\bigr) \\ &\geq F_{x_nx_{n+1}}\bigl(\phi\bigl(p/c^2\bigr)\bigr) \\ &\geq \min\bigl\{F_{x_{n-1}x_n}\bigl(\phi\bigl(p/c^3\bigr)\bigr), F_{x_nx_{n+1}}\bigl(\phi\bigl(p/c^3\bigr)\bigr)\bigr\}. \end{aligned}$$

Repeating the same procedure, we conclude that

$$F_{x_{n-1}x_n}\bigl(\phi(p/c)\bigr) > F_{x_nx_{n+1}}\bigl(\phi(p/c)\bigr) \geq F_{x_nx_{n+1}}\bigl(\phi\bigl(p/c^2\bigr)\bigr) \geq \cdots \geq F_{x_nx_{n+1}}\bigl(\phi\bigl(p/c^k\bigr)\bigr).$$

Since $F_{x_nx_{n+1}}(\phi(p/c^k)) \to 1$, $k \to \infty$, we get a contradiction. Accordingly, (3.2) is true. Therefore,

$$F_{x_nx_{n+1}}\bigl(\phi(t)\bigr) \geq F_{x_{n-1}x_n}\bigl(\phi(t/c)\bigr) \geq F_{x_1x_0}\bigl(\phi\bigl(t/c^n\bigr)\bigr) \to 1$$

as $n \to \infty$.

By the property of ϕ, given $s > 0$, there exists $t > 0$ such that $s > \phi(t)$. Thus,

$$F_{x_nx_{n+1}}(s) \to 1 \quad \text{as } n \to \infty \tag{3.5}$$

for all $s > 0$.

Now, we claim that $\{x_n\}$ is a Cauchy sequence. If not, then $\exists\epsilon > 0$ and $\lambda > 0$, and subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ such that $m(k) < n(k)$ and

$$F_{x_{m(k)}x_{n(k)}}(\epsilon) < 1 - \lambda, \tag{3.6}$$

$$F_{x_{m(k)}x_{n(k)-1}}(\epsilon) \geq 1 - \lambda. \tag{3.7}$$

Since F is non-decreasing, we have

$$\bigl\{x : F_{xp}\bigl(\epsilon''\bigr) \geq 1 - \lambda\bigr\} \subseteq \bigl\{x : F_{xp}(\epsilon) \geq 1 - \lambda\bigr\}$$

for all $p \in X$, $\lambda > 0$, and $0 < \epsilon'' < \epsilon$. It follows that whenever the above construction is possible for $\epsilon > 0$, $\lambda > 0$, it is also possible to construct $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ satisfying (3.6) and (3.7) corresponding to $\epsilon'' > 0$, $\lambda > 0$ whenever $\epsilon'' < \epsilon$.

Since ϕ is continuous at 0 and strictly monotonic increasing with $\phi(0) = 0$, it is possible to obtain $\epsilon_1 > 0$ such that $\phi(\epsilon_1) < \epsilon$. Then, by the above argument, it is possible to obtain increasing sequences of integers $m(k)$ and $n(k)$ with $m(k) < n(k)$ such that

$$F_{x_{m(k)}x_{n(k)}}\bigl(\phi(\epsilon_1)\bigr) < 1 - \lambda, \tag{3.8}$$

$$F_{x_{m(k)}x_{n(k)-1}}\bigl(\phi(\epsilon_1)\bigr) \geq 1 - \lambda. \tag{3.9}$$

Since $0 < c < 1$ and $\phi \in \Phi$, we can choose $\eta > 0$ such that $0 < \eta < \phi(\epsilon_1/c) - \phi(\epsilon_1)$. Since ϕ is strictly increasing, therefore

$$\phi(\epsilon_1/c) - \eta > \phi(\epsilon_1).$$

From (3.9), we get

$$F_{x_{m(k)}x_{n(k)-1}}\bigl(\phi(\epsilon_1/c) - \eta\bigr) > F_{x_{m(k)}x_{n(k)-1}}\phi(\epsilon_1) \geq 1 - \lambda. \tag{3.10}$$

By (3.5), for $\lambda_1 < \lambda < 1$, it is possible to find a positive integer N_1 such that for all $k > N_1$,

$$\left.\begin{array}{l} F_{x_{m(k)}x_{m(k)-1}}\phi(\eta) \geq 1 - \lambda_1, \\ F_{x_{n(k)}x_{n(k)-1}}\phi(\eta) \geq 1 - \lambda_1. \end{array}\right\} \tag{3.11}$$

By (PM3), we have

$$F_{x_{m(k)-1}x_{n(k)-1}}\bigl(\phi(\epsilon_1/c)\bigr) \geq T\bigl(F_{x_{m(k)-1}x_{m(k)}}(\eta), F_{x_{m(k)}x_{n(k)-1}}\bigl(\phi(\epsilon_1/c) - \eta\bigr)\bigr). \tag{3.12}$$

Let $0 < \lambda_2 < \lambda_1 < \lambda < 1$ be arbitrary. Then by (3.5) there exists a positive integer N_2 such that for all $k > N_2$,

$$F_{x_{m(k)-1}x_{m(k)}}(\eta) \geq 1 - \lambda_2. \tag{3.13}$$

Now, using (3.10), (3.12), and (3.13), we have, for all $k > \max\{N_1, N_2\}$,

$$F_{x_{m(k)-1}x_{n(k)-1}}\bigl(\phi(\epsilon_1/c)\bigr) \geq T(1 - \lambda_2, 1 - \lambda).$$

As λ_2 is arbitrary and T is continuous, we have

$$F_{x_{m(k)-1}x_{n(k)-1}}\bigl(\phi(\epsilon_1/c)\bigr) \geq T(1, 1 - \lambda) = 1 - \lambda. \tag{3.14}$$

Now, using (3.1), (3.9), (3.11), and (3.14), we have

$$\begin{aligned} 1 - \lambda &> F_{x_{m(k)}x_{n(k)}}\bigl(\phi(\epsilon_1)\bigr) \\ &= F_{fx_{m(k)-1}fx_{n(k)-1}}\bigl(\phi(\epsilon_1)\bigr) \end{aligned}$$

$$\begin{aligned}
&\geq \min\bigl\{F_{x_{m(k)-1}x_{n(k)-1}}\bigl(\phi(\epsilon_1/c)\bigr), F_{x_{m(k)-1}x_{m(k)}}\bigl(\phi(\epsilon_1/c)\bigr), \\
&\quad F_{x_{n(k)-1}x_{n(k)}}\bigl(\phi(\epsilon_1/c)\bigr), F_{x_{n(k)-1}x_{m(k)}}\bigl(\phi(\epsilon_1/c)\bigr)\bigr\} \\
&\geq \min\{1-\lambda, 1-\lambda, 1-\lambda, 1-\lambda\} \\
&= 1-\lambda,
\end{aligned}$$

which is a contradiction. Therefore $\{x_n\}$ is a Cauchy sequence in a complete Menger PM-space X, thus there exists $z \in X$ such that $z = \lim_{n\to\infty} x_n$.

Now, we will show that z is a fixed point of f. Since $\phi \in \Phi$, we have that for every $x, y \in X$ and all $s > 0$, there exists $r > 0$ such that $s > \phi(r)$ and $n_0 \in N$ such that for all $n \geq n_0$,

$$F_{fzz}(s) \geq T\bigl(F_{fzx_n}\bigl(\phi(r)\bigr), F_{x_nz}\bigl(s-\phi(r)\bigr)\bigr). \tag{3.15}$$

Since $s > \phi(r)$, thus $(s-\phi(r)) > 0$. Also, since $z = \lim_{n\to\infty} x_n$, for arbitrary $\delta \in (0,1)$, we have

$$F_{x_nz}\bigl(s-\phi(r)\bigr) > 1-\delta. \tag{3.16}$$

Hence, from (3.15) and (3.16), we get

$$F_{fzz}(s) \geq T\bigl(F_{fzx_n}\bigl(\phi(r)\bigr), 1-\delta\bigr).$$

Since $\delta > 0$ is arbitrary and the t-norm T is continuous, we get

$$\begin{aligned}
F_{fzz}(s) &\geq F_{fzx_n}\bigl(\phi(r)\bigr) \\
&\geq F_{fzfx_{n-1}}\bigl(\phi(r)\bigr) \\
&\geq \min\bigl\{F_{zx_{n-1}}\bigl(\phi(r/c)\bigr), F_{zfz}\bigl(\phi(r/c)\bigr), F_{x_{n-1}fx_{n-1}}\bigl(\phi(r/c)\bigr), F_{x_{n-1}fz}\bigl(\phi(r/c)\bigr)\bigr\}.
\end{aligned}$$

Letting $n \to \infty$ in the above inequality and using the fact that the t-norm T is continuous, we obtain

$$F_{fzz}\bigl(\phi(r)\bigr) \geq F_{zfz}\bigl(\phi(r/c)\bigr)$$

and applying Lemma 2.1, we get $z = fz$.

Next, we prove the uniqueness of a fixed point. Let $w \in X$ be another fixed point of f, *i.e.*, $fw = w$. Since $\phi \in \Phi$, for all $s > 0$, there exists $r > 0$ such that $s > \phi(r)$. Then we have

$$\begin{aligned}
F_{zw}(s) &\geq F_{zw}\bigl(\phi(r)\bigr) \\
&= F_{fzfw}\bigl(\phi(r)\bigr) \\
&\geq \min\bigl\{F_{zw}\bigl(\phi(r/c)\bigr), F_{zfz}\bigl(\phi(r/c)\bigr), F_{wfw}\bigl(\phi(r/c)\bigr), F_{wfz}\bigl(\phi(r/c)\bigr)\bigr\} \\
&= \min\bigl\{F_{zw}\bigl(\phi(r/c)\bigr), F_{zz}\bigl(\phi(r/c)\bigr), F_{ww}\bigl(\phi(r/c)\bigr), F_{wz}\bigl(\phi(r/c)\bigr)\bigr\} \\
&= F_{zw}\bigl(\phi(r/c)\bigr).
\end{aligned}$$

From Lemma 2.1, it follows that $z = w$, *i.e.*, z is the unique fixed point of f. □

4 Connection with metric spaces

It is well known that every metric space (X,d) is also a Menger space (X,F,T_M) if F is defined in the following way:

$$F_{xy}(t) = \begin{cases} 1, & d(x,y) < t, \\ 0, & d(x,y) \geq t. \end{cases}$$

If ψ is an altering distance function defined in Definition 2.4 with additional property $\psi(t) \to \infty$ as $t \to \infty$, then the function

$$\phi(t) = \begin{cases} \inf\{\alpha : \psi(\alpha) \geq t\}, & t > 0, \\ 0, & t = 0 \end{cases}$$

is a Φ-function (see [18]).

We will present that (3.1) in this case implies

$$\psi\bigl(d(fx,fy)\bigr) \leq c \max\bigl\{\psi\bigl(d(fx,x)\bigr), \psi\bigl(d(fy,y)\bigr), \psi\bigl(d(x,y)\bigr), \psi\bigl(d(fx,y)\bigr)\bigr\},$$

$c \in (0,1)$ and $x,y \in X$ in a metric space.

Suppose the contrary, *i.e.*, there exists $t > 0$ such that $F_{fxfy}(\phi(t)) = 0$ and all of

$$F_{xfx}\bigl(\phi(t/c)\bigr) = 1, \qquad F_{yfy}\bigl(\phi(t/c)\bigr) = 1, \qquad F_{x,y}\bigl(\phi(t/c)\bigr) = 1, \quad \text{and} \quad F_{fxy}\bigl(\phi(t/c)\bigr) = 1.$$

So, $F_{fxfy}(\phi(t)) = 0$ implies that $d(fx,fy) \geq \phi(t)$, and since ψ is continuous, we have

$$\psi\bigl(d(fx,fy)\bigr) \geq t.$$

Similarly, since $F_{xfx}(\phi(t/c)) = 1$, we have that $d(fx,x) < \phi(t/c)$, which implies

$$\psi\bigl(d(fx,x)\bigr) < \frac{t}{c}.$$

Also, we have the following:

$$\psi\bigl(d(fy,y)\bigr) < \frac{t}{c}, \qquad \psi\bigl(d(x,y)\bigr) < \frac{t}{c}, \quad \text{and} \quad \psi\bigl(d(fx,y)\bigr) < \frac{t}{c}.$$

Thus, we have

$$\psi\bigl(d(fx,fy)\bigr) > c \max\bigl\{\psi\bigl(d(fx,x)\bigr), \psi\bigl(d(fy,y)\bigr), \psi\bigl(d(x,y)\bigr), \psi\bigl(d(fx,y)\bigr)\bigr\},$$

and we get a contradiction.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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