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Generalized Gronwall inequalities and their applications to fractional differential equations

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Abstract

In this paper, we provide several generalizations of the Gronwall inequality and present their applications to prove the uniqueness of solutions for fractional differential equations with various derivatives.

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1 Introduction

The Gronwall inequality has an important role in numerous differential and integral equations. The classical form of this inequality is described as follows, *cf.* [1].

Theorem 1.1 *For any* $t \in [t_0, T)$,

$$u(t) \le a(t) + \int_{t_0}^{t} b(s)u(s)\,ds,$$

where $b \ge 0$, *then*

$$u(t) \le a(t) + \int_{t_0}^{t} a(s)b(s)\exp\left[\int_{s}^{t} b(u)\,du\right]ds, \quad t \in [t_0, T).$$

In particular, if $a(t)$ *is not decreasing, then*

$$u(t) \le a(t)\exp\left[\int_{t_0}^{t} b(s)\,ds\right], \quad t \in [t_0, T).$$

In recent years, an increasing number of Gronwall inequality generalizations have been discovered to address difficulties encountered in differential equations, *cf.* [2–7]. Among these generalizations, we focus on the works of Ye, Gao and Qian, Gong, Li, the generalized Gronwall inequality with Riemann-Liouville fractional derivative and the Hadamard derivative which are presented as follows.

Theorem 1.2 ([6]) *For any $t \in [0, T)$,*

$$u(t) \le a(t) + b(t) \int_0^t (t-s)^{\beta-1} u(s)\, ds,$$

where all the functions are not negative and continuous. The constant $\beta > 0$. b is a bounded and monotonic increasing function on $[0, T)$, then

$$u(t) \le a(t) + \int_0^t \left[\sum_{n=1}^{\infty} \frac{(b(t)\Gamma(\beta))^n}{\Gamma(n\beta)} (t-s)^{n\beta-1} a(s) \right] ds, \quad t \in [0, T).$$

Theorem 1.3 ([5]) *For any $t \in [1, T)$,*

$$u(t) \le a(t) + b(t) \int_1^t \left(\ln \frac{t}{s} \right)^{\beta-1} \frac{u(s)}{s}\, ds,$$

where all the functions are not negative and continuous. The constant $\beta > 0$. b is a bounded and monotonic increasing function on $[1, T)$, then

$$u(t) \le a(t) + \int_1^t \left[\sum_{n=1}^{\infty} \frac{(b(t)\Gamma(\beta))^n}{\Gamma(n\beta)} \left(\ln \frac{t}{s} \right)^{n\beta-1} a(s) \right] \frac{ds}{s}, \quad t \in [1, T).$$

The aforementioned inequalities are obtained using the estimation method of the composition operators. This method is usually applied in studying qualitative theory of fractional differential equations. However, this method is not suitable for more complex situations. Therefore, we shall use a simpler technique to prove the main results obtained in this work.

Theorem 1.4 *For any $t \in [0, T)$,*

$$u(t) \le a(t) + \sum_{i=1}^{n} b_i(t) \int_0^t (t-s)^{\beta_i-1} u(s)\, ds,$$

where all the functions are not negative and continuous. The constants $\beta_i > 0$. b_i ($i = 1, 2, \ldots, n$) are the bounded and monotonic increasing functions on $[0, T)$, then

$$u(t) \le a(t) + \sum_{k=1}^{\infty} \left(\sum_{1', 2', \ldots, k'=1}^{n} \frac{\prod_{i=1}^{k} [b_{i'}(t)\Gamma(\beta_{i'})]}{\Gamma(\sum_{i=1}^{k} \beta_{i'})} \int_0^t (t-s)^{\sum_{i=1}^{k} \beta_{i'}-1} a(s)\, ds \right).$$

Theorem 1.5 *For any $t \in [1, T)$,*

$$u(t) \le a(t) + \sum_{i=1}^{n} b_i(t) \int_1^t \left(\ln \frac{t}{s} \right)^{\beta_i-1} \frac{u(s)}{s}\, ds,$$

where all the functions are not negative and continuous. The constants $\beta_i > 0$. b_i ($i = 1, 2, \ldots, n$) are the bounded and monotonic increasing functions on $[1, T)$, then

$$u(t) \le a(t) + \sum_{k=1}^{\infty} \left(\sum_{1', 2', \ldots, k'=1}^{n} \frac{\prod_{i=1}^{k} [b_{i'}(t)\Gamma(\beta_{i'})]}{\Gamma(\sum_{i=1}^{k} \beta_{i'})} \int_1^t \left[\left(\ln \frac{t}{s} \right)^{\sum_{i=1}^{k} \beta_{i'}-1} a(s) \right] \frac{ds}{s} \right).$$

2 The proof of the main results

In this section, we use the following critical lemmas to prove our main results.

Lemma 2.1 *For any* $t \in [0, T)$,

$$H(t) \geq \sum_{i=1}^{n} b_i(t) \int_0^t (t-s)^{\beta_i - 1} H(s)\, ds,$$

where all the functions are continuous. The constants $\beta_i > 0$. b_i $(i = 1, 2, \ldots, n)$ *are the bounded, not negative, and monotonic increasing functions on* $[0, T)$, *then* $H(t) \geq 0$, $t \in [0, T)$.

Proof Obviously, $H(0) \geq 0$. If the proposition is false, that is,

$$\{t \mid t \in [0, T), H(t) < 0\} \neq \Phi,$$

where Φ is an empty set, then a number t_0 exists on $[0, T)$ which satisfies $H_{|[0,t_0]} \geq 0$, $H(t_0) = 0$. H is a strictly monotonic decreasing function on $(t_0, t_0 + \xi_0) \subset [0, T)$. Here, $\xi_0 > 0$. Therefore, for any $t \in (t_0, t_0 + \xi_0)$, $H(t) < 0$ and

$$\begin{aligned} H(t) &\geq \sum_{i=1}^{n} b_i(t) \int_0^t (t-s)^{\beta_i - 1} H(s)\, ds \\ &\geq \sum_{i=1}^{n} b_i(t) \int_{t_0}^t (t-s)^{\beta_i - 1} H(s)\, ds \\ &\geq \sum_{i=1}^{n} b_i(t) H(t) \int_{t_0}^t (t-s)^{\beta_i - 1}\, ds \\ &= H(t) \cdot \sum_{i=1}^{n} b_i(t) \frac{(t-t_0)^{\beta_i}}{\beta_i}, \end{aligned}$$

which implies that

$$\sum_{i=1}^{n} b_i(t) \frac{(t-t_0)^{\beta_i}}{\beta_i} \geq 1.$$

Let $t \to t_0$, then we have a contradiction, that is, $0 \geq 1$. This process completes the proof of Lemma 2.1. □

The next lemma is given following the same method as for the previous lemma. The proving process is relatively similar, thus we do not include it in this paper.

Lemma 2.2 *For any* $t \in [1, T)$,

$$H(t) \geq \sum_{i=1}^{n} b_i(t) \int_1^t \left(\ln \frac{t}{s}\right)^{\beta_i - 1} \frac{H(s)}{s}\, ds,$$

where all the functions are continuous. The constants $\beta_i > 0$. b_i ($i = 1, 2, \dots, n$) are the bounded, not negative, and monotonic increasing functions on $[1, T)$, then $H(t) \geq 0$, $t \in [1, T)$.

Our next task is proving our main results. To prove Theorem 1.4, we initially suppose that

$$g(t) = a(t) + \sum_{k=1}^{\infty}\left(\sum_{1',2',\dots,k'=1}^{n} \frac{\prod_{i=1}^{k}[b_{i'}(t)\Gamma(\beta_{i'})]}{\Gamma(\sum_{i=1}^{k}\beta_{i'})}\int_0^t (t-s)^{\sum_{i=1}^{k}\beta_{i'}-1}a(s)\,ds\right).$$

Then, the following equality is given:

$$\begin{aligned}
&\int_0^t \int_0^s (t-s)^{\beta_j-1}(s-u)^{\sum_{i=1}^{k}\beta_{i'}-1}a(u)\,du\,ds \\
&\quad = \int_0^t \int_u^t (t-s)^{\beta_j-1}(s-u)^{\sum_{i=1}^{k}\beta_{i'}-1}a(u)\,ds\,du \\
&\quad = \frac{\Gamma(\beta_j)\Gamma(\sum_{i=1}^{k}\beta_{i'})}{\Gamma(\beta_j+\sum_{i=1}^{k}\beta_{i'})}\int_0^t (t-s)^{\beta_j+\sum_{i=1}^{k}\beta_{i'}-1}a(s)\,ds.
\end{aligned}$$

This equality, combined with the fact that b_i ($i = 1, 2, \dots, n$) are the monotonic increasing functions on $[0, T)$, yields

$$\begin{aligned}
&\sum_{i=1}^{n} b_i(t)\int_0^t (t-s)^{\beta_i-1}g(s)\,ds \\
&\quad \leq \sum_{k=1}^{\infty}\sum_{j=1}^{n}\sum_{1',2',\dots,k'=1}^{n} b_j(t)\int_0^t\int_0^s \frac{\prod_{i=1}^{k}[b_{i'}(s)\Gamma(\beta_{i'})]}{\Gamma(\sum_{i=1}^{k}\beta_{i'})}(t-s)^{\beta_j-1}(s-u)^{\sum_{i=1}^{k}\beta_{i'}-1}\cdot a(u)\,du\,ds \\
&\qquad + \sum_{i=1}^{n} b_i(t)\int_0^t (t-s)^{\beta_i-1}a(s)\,ds \\
&\quad \leq \sum_{k=1}^{\infty}\sum_{j=1}^{n}\sum_{1',2',\dots,k'=1}^{n} b_j(t)\int_0^t\int_0^s \frac{\prod_{i=1}^{k}[b_{i'}(t)\Gamma(\beta_{i'})]}{\Gamma(\sum_{i=1}^{k}\beta_{i'})}(t-s)^{\beta_j-1}(s-u)^{\sum_{i=1}^{k}\beta_{i'}-1}\cdot a(u)\,du\,ds \\
&\qquad + \sum_{i=1}^{n} b_i(t)\int_0^t (t-s)^{\beta_i-1}a(s)\,ds \\
&\quad \leq \sum_{k=1}^{\infty}\left(\sum_{1',2',\dots,k'=1}^{n} \frac{\prod_{i=1}^{k}[b_{i'}(t)\Gamma(\beta_{i'})]}{\Gamma(\sum_{i=1}^{k}\beta_{i'})}\int_0^t (t-s)^{\sum_{i=1}^{k}\beta_{i'}-1}a(s)\,ds\right) \\
&\quad = g(t) - a(t),
\end{aligned}$$

which indicates that

$$u(t) - \sum_{i=1}^{n} b_i(t)\int_0^t (t-s)^{\beta_i-1}u(s)\,ds \leq a(t) \leq g(t) - \sum_{i=1}^{n} b_i(t)\int_0^t (t-s)^{\beta_i-1}g(s)\,ds.$$

Let $H(t) = g(t) - u(t)$, then we obtain

$$H(t) \geq \sum_{i=1}^{n} b_i(t)\int_0^t (t-s)^{\beta_i-1}H(s)\,ds.$$

According to Lemma 2.1, $H(t) \geq 0$. That is, $u(t) \leq g(t)$ and $t \in [0, T)$. This process completes the proof of Theorem 1.4.

We can prove Theorem 1.5 by applying Lemma 2.2 in the same manner as in the previous theorem. We conclude the main results of this work.

3 Applications

In this section, we apply the main results of this work to demonstrate the uniqueness of the solution for fractional differential equations. First, the following initial value problems with the Riemann-Liouville fractional derivative are considered:

$$\begin{cases} \sum_{i=1}^{n} D_R^{\beta_i} u(t) = f(t, u(t)), \\ \sum_{i=1}^{n} I_R^{1-\beta_i} u(t)|_{t=0} = \delta, \end{cases} \tag{3.1}$$

where $0 < \beta_1 < \beta_2 < \cdots < \beta_n < 1$, $t \in [0, T)$, D_R^{β} and I_R^{β} denote the Riemann-Liouville fractional derivative and fractional integral operators, respectively.

Definition 3.1 ([8–13]) The βth Riemann-Liouville-type fractional order integral of a function $u : [0, +\infty) \mapsto \mathbb{R}$ is defined by

$$I_R^{\beta} u(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} u(s)\, ds,$$

where $\beta > 0$ and Γ is the gamma function.

Definition 3.2 ([9, 10, 12–14]) For any $0 < \beta < 1$, the βth Riemann-Liouville-type fractional order derivative of a function $u : [0, +\infty) \mapsto \mathbb{R}$ is defined by

$$D_R^{\beta} u(t) = \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \int_0^t (t-s)^{-\beta} u(s)\, ds.$$

Lemma 3.1 ([8, 10, 14]) *For any* $0 < \beta < 1$,

$$I_R^{\beta} D_R^{\beta} u(t) = u(t) + c \cdot t^{\beta-1},$$

where c is a constant in $\mathbb{R}$.

Lemma 3.2 ([15]) *For any* $\alpha, \beta > 0$,

$$I_R^{\alpha} I_R^{\beta} u(t) = I_R^{\alpha+\beta} u(t).$$

We can then state the next theorem.

Theorem 3.1 *For any $t \in [0, T)$, suppose that $|f(t, y) - f(t, z)| \leq \gamma(t)|y - z|$, and $\gamma(t) \geq 0$ is a bounded and monotonic increasing function. If initial value problem* (3.1) *has a solution, then the solution is unique.*

Proof Since $0 < \beta_1 < \beta_2 < \cdots < \beta_n < 1$, then according to Lemma 3.1, we can suppose that

$$I_R^{\beta_i} D_R^{\beta_i} u(t) = u(t) + c_i t^{\beta-1},$$

where c_i, $i = 1, 2, \ldots, n$, are some constants. This equality, combined with Lemma 3.2, enables us to change problem (3.1) into the following:

$$\begin{aligned} I_R^{\beta_n} f\bigl(t, u(t)\bigr) &= \sum_{i=1}^{n} I_R^{\beta_n} D_R^{\beta_i} u(t) = \sum_{i=1}^{n} I_R^{\beta_n-\beta_i}\bigl(I_R^{\beta_i} D_R^{\beta_i} u(t)\bigr) \\ &= \sum_{i=1}^{n} I_R^{\beta_n-\beta_i}\bigl(u(t) + c_i t^{\beta_i-1}\bigr) \\ &= \sum_{i=1}^{n} I_R^{\beta_n-\beta_i} u(t) + \frac{t^{\beta_n-1}}{\Gamma(\beta_n)} \cdot \sum_{i=1}^{n}\bigl(c_i \Gamma(\beta_i)\bigr). \end{aligned}$$

Therefore,

$$\begin{aligned} 0 &= I_R^{1-\beta_n} I_R^{\beta_n} f|_{t=0} \\ &= \sum_{i=1}^{n} I_R^{1-\beta_n} I_R^{\beta_n-\beta_i} u|_{t=0} + \frac{I_R^{1-\beta_n}(t^{\beta_n-1})|_{t=0}}{\Gamma(\beta_n)} \cdot \sum_{i=1}^{n}\bigl(c_i\Gamma(\beta_i)\bigr) \\ &= \sum_{i=1}^{n} I_R^{1-\beta_i} u|_{t=0} + \sum_{i=1}^{n}\bigl(c_i\Gamma(\beta_i)\bigr) \\ &= \delta + \sum_{i=1}^{n}\bigl(c_i\Gamma(\beta_i)\bigr). \end{aligned}$$

That is,

$$\sum_{i=1}^{n}\bigl(c_i\Gamma(\beta_i)\bigr) = -\delta,$$

which implies that

$$u(t) = I_R^{\beta_n} f\bigl(t, u(t)\bigr) - \sum_{i=1}^{n-1} I_R^{\beta_n-\beta_i} u(t) + \delta \cdot \frac{t^{\beta_n-1}}{\Gamma(\beta_n)}.$$

If $u_1(t)$ and $u_2(t)$ are two solutions to problem (3.1), then they also satisfy the above equality. Thus we have

$$\begin{aligned} \bigl|u_1(t) - u_2(t)\bigr| &= \left| I_R^{\beta_n}\bigl[f\bigl(t, u_1(t)\bigr) - f\bigl(t, u_2(t)\bigr)\bigr] - \sum_{i=1}^{n-1} I_R^{\beta_n-\beta_i}\bigl[u_1(t) - u_2(t)\bigr] \right| \\ &\le \frac{1}{\Gamma(\beta_n)} \int_0^t (t-s)^{\beta_n-1} \bigl|f\bigl(s, u_1(s)\bigr) - f\bigl(s, u_2(s)\bigr)\bigr|\, ds \\ &\quad + \sum_{i=1}^{n-1} \frac{1}{\Gamma(\beta_n-\beta_i)} \int_0^t (t-s)^{\beta_n-\beta_i-1} \bigl|u_1(s) - u_2(s)\bigr|\, ds \\ &\le \frac{\gamma(t)}{\Gamma(\beta_n)} \int_0^t (t-s)^{\beta_n-1} \bigl|u_1(s) - u_2(s)\bigr|\, ds \\ &\quad + \sum_{i=1}^{n-1} \frac{1}{\Gamma(\beta_n-\beta_i)} \int_0^t (t-s)^{\beta_n-\beta_i-1} \bigl|u_1(s) - u_2(s)\bigr|\, ds. \end{aligned}$$

According to Theorem 1.4, we can conclude that

$$\left|u_1(t) - u_2(t)\right| \le 0,$$

which indicates that

$$u_1(t) = u_2(t), \quad t \in [0, T).$$

This process completes the proof of Theorem 3.1. □

Next, we study the uniqueness of the solution for the following initial value problems with the Hadamard-type fractional derivative:

$$\begin{cases} \sum_{i=1}^{n} D_H^{\alpha_i} v(t) = g(t, v(t)), \\ \sum_{i=1}^{n} I_H^{1-\alpha_i} v(t)|_{t=1} = \eta, \end{cases} \tag{3.2}$$

where $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_n < 1$, $t \in [1, T)$. For any $\alpha \in (0,1)$, D_H^{α} and I_H^{α} are defined as follows:

$$\begin{cases} D_H^{\alpha} v(t) = \frac{1}{\Gamma(1-\alpha)} (t \frac{d}{dt}) \int_1^t (\ln \frac{t}{s})^{-\alpha} \frac{v(s)}{s}\, ds, \\ I_H^{\alpha} v(t) = \frac{1}{\Gamma(\alpha)} \int_1^t (\ln \frac{t}{s})^{\alpha-1} \frac{v(s)}{s}\, ds. \end{cases} \tag{3.3}$$

From [16], we obtain

$$\begin{aligned} I_H^{\alpha_n} g\bigl(t, v(t)\bigr) &= \sum_{i=1}^{n} I_H^{\alpha_n} D_H^{\alpha_i} v(t) \\ &= \sum_{i=1}^{n} I_H^{\alpha_n-\alpha_i} \bigl(I_H^{\alpha_i} D_H^{\alpha_i} v(t)\bigr) \\ &= \sum_{i=1}^{n} I_H^{\alpha_n-\alpha_i} \bigl[v(t) + d_i \cdot (\log t)^{\alpha_i-1}\bigr] \\ &= \sum_{i=1}^{n} I_H^{\alpha_n-\alpha_i} v(t) + \frac{(\log t)^{\alpha_n-1}}{\Gamma(\alpha_n)} \cdot \sum_{i=1}^{n} \bigl(d_i \Gamma(\alpha_i)\bigr), \end{aligned} \tag{3.4}$$

where d_i, $i = 1, 2, \ldots, n$, are some constants. Therefore,

$$\begin{aligned} 0 &= \frac{1}{\Gamma(1)} \int_1^t \left(\ln \frac{t}{s}\right)^{1-1} \frac{g(s, v(s))}{s}\, ds \bigg|_{t=1} \\ &= I_H^{1-\alpha_n} I_H^{\alpha_n} g|_{t=1} \\ &= \sum_{i=1}^{n} I_H^{1-\alpha_n} I_H^{\alpha_n-\alpha_i} v|_{t=1} + \frac{I_H^{1-\alpha_n}((\log t)^{\alpha_n-1})|_{t=1}}{\Gamma(\alpha_n)} \cdot \sum_{i=1}^{n} \bigl(d_i \Gamma(\alpha_i)\bigr) \\ &= \sum_{i=1}^{n} I_H^{1-\alpha_i} v|_{t=0} + \sum_{i=1}^{n} \bigl(d_i \Gamma(\alpha_i)\bigr) \\ &= \eta + \sum_{i=1}^{n} \bigl(d_i \Gamma(\alpha_i)\bigr). \end{aligned}$$

When the aforementioned equality is plugged into problem (3.4), we obtain

$$\nu(t) = I_H^{\alpha_n} g\big(t, \nu(t)\big) - \sum_{i=1}^{n-1} I_H^{\alpha_n - \alpha_i} \nu(t) + \frac{(\log t)^{\alpha_n - 1}}{\Gamma(\alpha_n)} \cdot \eta. \tag{3.5}$$

This equality, combined with Theorem 1.5, can derive the next theorem. The procedure is relatively similar to the proof of Theorem 3.1, thus we do not include it in this paper.

Theorem 3.2 *For any $t \in [0, T)$, suppose that $|g(t,y) - g(t,z)| \le \zeta(t)|y - z|$, and $\zeta(t) \ge 0$ is a bounded and monotonic increasing function. If initial value problem* (3.2) *has a solution, then the solution is unique.*

4 Concluding remarks

In this work, we have obtained generalizations of the Gronwall inequality using several mathematical techniques. In addition, we have listed the initial value problems, namely (3.1) and (3.2), and proved the uniqueness of solutions to these problems by applying the generalized Gronwall inequalities.

Competing interests
The author declares that they have no competing interests.

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