Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH

Open Access

Singular potential Hamiltonian system

Tacksun Jung[1] and Q-Heung Choi[2*]

*Correspondence:
qheung@inha.ac.kr
[2]Department of Mathematics
Education, Inha University, Incheon,
402-751, Korea
Full list of author information is
available at the end of the article

Abstract

We investigate multiple solutions for the Hamiltonian system with singular potential nonlinearity and periodic condition. We get a theorem which shows the existence of the nontrivial weak periodic solution for the Hamiltonian system with singular potential nonlinearity. We obtain this result by using the variational method, critical point theory for indefinite functional.

MSC: 35Q70

Keywords: Hamiltonian system; singular potential nonlinearity; variational method; critical point theory for indefinite functional; $(P.S.)_c$ condition

1 Introduction

Let D be an open subset in R^{2n} with a compact complement $C = R^{2n}\backslash D$, $n \geq 2$. Let $p \in R^n$, $q \in R^n$, $(p(t), q(t)) \in C^2([0, 2\pi], D)$ and $G(t, (p(t), q(t))) \in C^2([0, 2\pi] \times D, R^1)$. In this paper we investigate the number of solutions $(p(t), q(t)) \in C^2([0, 2\pi], D)$ with singular potential nonlinearity and periodic condition

$$
\begin{aligned}
\dot{p}(t) &= -G_q(t, p(t), q(t)), \\
\dot{q}(t) &= G_p(t, p(t), q(t)), \\
p(0) &= p(2\pi), \qquad q(0) = q(2\pi).
\end{aligned}
\tag{1.1}
$$

Let us set $z = (p, q)$ and

$$
J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}.
$$

Then (1.1) can be rewritten as

$$
\begin{aligned}
-J\dot{z} &= G_z(t, z(t)), \\
z(0) &= z(2\pi),
\end{aligned}
\tag{1.2}
$$

where $\dot{z} = \frac{dz}{dt}$. We assume that $G(t, z(t)) \in C^2([0, 2\pi] \times D, R^1)$ satisfies the following conditions:

(G1) There exists $R_0 > 0$ such that

$$
\sup\{|G(t, z(t))| + \|\mathrm{grad}_z G(t, z(t))\|_{R^{2n}} \mid (t, z) \in [0, 2\pi] \times (R^{2n}\backslash B_{R_0})\} < +\infty.
$$

Springer

(G2) There is a neighborhood Z of C in R^{2n} such that

$$G(t, z) \geq \frac{A}{d^2(z, C)} \quad \text{for } (t, z) \in [0, 2\pi] \times Z,$$

where $d(z, C)$ is the distance function to C and $A > 0$ is a constant.

Several authors ([1–4], *etc.*) studied the Hamiltonian system with nonsingular potential nonlinearity. Jung and Choi [2, 3] considered (1.1) with nonsingular potential nonlinearity or jumping nonlinearity crossing one eigenvalue, or two eigenvalues, or several eigenvalues. Chang [1] proved that (1.1) has at least two nontrivial 2π-periodic weak solutions under some asymptotic nonlinearity. Jung and Choi [2] proved that (1.1) has at least m weak solutions, which are geometrically distinct and nonconstant under some jumping nonlinearity.

In this paper we are trying to find the weak solutions $z \in C^2([0, 2\pi], D)$ of system (1.2) such that

$$\int_0^{2\pi} \dot{z} \cdot w - JG_z(t, z(t)) \cdot Jw \, dt = 0 \quad \text{for all } w \in E,$$

i.e.,

$$\int_0^{2\pi} \left[\left(\dot{p} + G_q(t, z(t)) \right) \cdot \psi - \left(\dot{q} - G_p(t, z(t)) \right) \cdot \phi \right] dt = 0 \quad \text{for all } \zeta = (\phi, \psi) \in E,$$

where E is introduced in Section 2.

Our main result is as follows.

Theorem 1.1 *Assume that G satisfies conditions* (G1)-(G2). *Then system* (1.1) *has at least one* 2π-*periodic solution.*

For the proof of Theorem 1.1, we introduce the perturbed operator A_ϵ, such that A_ϵ^{-1} is a compact operator and the associated functional $I(z)$ corresponding to the operator A_ϵ, and approach the variational method, the critical point theory. In Section 2, we investigate the Fréchet differentiability of the associated functional $I(z)$ and recall the critical point theorem for indefinite functional. In Section 3, we show that the associated functional $I(z)$ satisfies the geometrical assumptions of the critical point theorem for indefinite functional and prove Theorem 1.1.

2 Variational method

Let $L^2([0, 2\pi], R^{2n})$ denote the set of $2n$-tuples of the square integrable 2π-periodic functions and choose $z \in L^2([0, 2\pi], R^{2n})$. Then it has a Fourier expansion $z(t) = \sum_{k=-\infty}^{k=+\infty} a_k e^{ikt}$, with $a_k = \frac{1}{2\pi} \int_0^{2\pi} z(t) e^{-ikt} \, dt \in C^{2n}$, $a_{-k} = \bar{a}_k$ and $\sum_{k \in Z} |a_k|^2 < \infty$. Let

$$A : z(t) \mapsto -J\dot{z}(t)$$

with the domain

$$D(A) = \left\{ z(t) \in H^1([0, 2\pi], R^{2n}) \mid z(0) = z(2\pi) \right\}$$

$$= \left\{ z(t) \in L^2([0, 2\pi], R^{2n}) \mid \sum_{k \in Z} (\epsilon + |k|)^2 |a_k|^2 < +\infty \right\},$$

where ϵ is a positive small number. Then A is a self-adjoint operator. Let $\{M_\lambda\}$ be the spectral resolution of A, and let α be a positive number such that $\alpha \notin \sigma(A)$ and $[-\alpha, \alpha]$ contains only one element 0 of σA. Let

$$P_0 = \int_{-\alpha}^{\alpha} dM_\lambda, \qquad P_+ = \int_{\alpha}^{+\infty} dM_\lambda, \qquad P_- = \int_{-\infty}^{-\alpha} dM_\lambda.$$

Let

$$L_0 = P_0 L^2\big([0, 2\pi], R^{2n}\big), \qquad L_+ = P_+ L^2\big([0, 2\pi], R^{2n}\big), \qquad L_- = P_- L^2\big([0, 2\pi], R^{2n}\big).$$

For each $u \in L^2([0, 2\pi], R^{2n})$, we have the composition

$$u = u_0 + u_+ + u_-,$$

where $u_0 \in L_0$, $u_+ \in L_+$, $u_- \in L_-$. According to A, there exists a small number $\epsilon > 0$ such that $-\epsilon \notin \sigma(A)$. Let us define the space E as follows:

$$E = D\big(|A|^{\frac{1}{2}}\big) = \left\{ z \in L^2\big([0, 2\pi], R^{2n}\big) \mid \sum_{k \in Z} \big(\epsilon + |k|\big)|a_k|^2 < \infty \right\}$$

with the scalar product

$$(z, w)_E = \epsilon(z, w)_{L^2} + \big(|A|^{\frac{1}{2}} z, |A|^{\frac{1}{2}} w\big)_{L^2}$$

and the norm

$$\|z\| = (z, z)_E^{\frac{1}{2}} = \left(\sum_{k \in Z} \big(\epsilon + |k|\big)|a_k|^2 \right)^{\frac{1}{2}}.$$

The space E endowed with this norm is a real Hilbert space continuously embedded in $L^2([0, 2\pi], R^{2n})$. The scalar product in L^2 naturally extends as the duality pairing between E and $E' = W^{-\frac{1}{2}, 2}([0, 2\pi], R^{2n})$. We note that the operator $(\epsilon + |A|)^{-1}$ is a compact linear operator from $L^2([0, 2\pi], R^{2n})$ to E such that

$$\big((\epsilon + |A|)^{-1} w, z\big)_E = \int_0^{2\pi} \big(w(t), z(t)\big)\, dt.$$

Let

$$A_\epsilon = \epsilon I + A.$$

Let

$$E_0 = |A_\epsilon|^{-\frac{1}{2}} L_0, \qquad E_+ = |A_\epsilon|^{-\frac{1}{2}} L_+, \qquad E_- = |A_\epsilon|^{-\frac{1}{2}} L_-.$$

Then $E = E_0 \oplus E_+ \oplus E_-$ and for $z \in E$, z has the decomposition $z = z_0 + z_+ + z_- \in E$, where

$$z_0 = |A_\epsilon|^{-\frac{1}{2}} u_0, \qquad z_+ = |A_\epsilon|^{-\frac{1}{2}} u_+, \qquad z_- = |A_\epsilon|^{-\frac{1}{2}} u_-.$$

Thus we have

$$\|z_0\|_{E_0} = \|u_0\|_{L_0}, \qquad \|z_+\|_{E_+} = \|u_+\|_{L_+}, \qquad \|z_-\|_{E_-} = \|u_-\|_{L_-}$$

and that E_0, E_+, E_- are isomorphic to L_0, L_+, L_-, respectively.

For the sake of simplicity, from now on we shall denote the subset of $C^2([0, 2\pi], D)$, satisfying the 2π-periodic condition, by $C^2(S^1, D)$.

Let us introduce an open set of the Hilbert space E as follows:

$$X = \{ z \in E \mid z(t) \in D \subset R^{2n}, t \in S^1 \}.$$

Let

$$X_0 = E_0 \cap X, \qquad X_+ = E_+ \cap X, \qquad X_- = E_- \cap X.$$

Then $X = X_0 \oplus X_+ \oplus X_-$ and for $z \in X$, z has the decomposition $z = z_0 + z_+ + z_- \in X$, where

$$z_0 \in X_0, \qquad z_+ \in X_+, \qquad z_- \in X_-.$$

The associated functional of (1.2) on X is as follows:

$$I(z) = \frac{1}{2} \left(\left\| A_\epsilon^{\frac{1}{2}} z_+ \right\|_{L^2}^2 + \left\| A_\epsilon^{\frac{1}{2}} M_+ z_0 \right\|^2 - \left\| (-A_\epsilon)^{\frac{1}{2}} z_- \right\|_{L^2}^2 - \left\| (-A_\epsilon)^{\frac{1}{2}} M_- z_0 \right\|^2 \right) - \psi_\epsilon(z), \quad (2.1)$$

where $\psi_\epsilon(z) = \psi(z) + \frac{\epsilon}{2} \|z\|_{L^2}^2$, $\psi(z) = \int_0^{2\pi} G(t, z(t)) \, dt$. Let

$$F(z) = G_z(t, z(t)).$$

By $G \in C^2$, $\psi(z) = \int_0^{2\pi} G(t, z(t)) \in C^2(S^1 \times D, R^1)$. Let

$$F_\epsilon(z) = \epsilon I + F(z) = \epsilon I + G_z(t, z(t)).$$

System (1.2) can be rewritten as

$$A_\epsilon(z) = F_\epsilon(z). \tag{2.2}$$

The Euler equation of the functional $I(z)$ is the system

$$u_+ = |A_\epsilon|^{-\frac{1}{2}} P_+ F_\epsilon(z), \tag{2.3}$$

$$u_- = -|A_\epsilon|^{-\frac{1}{2}} P_- F_\epsilon(z), \tag{2.4}$$

$$M_+ u_0 = |A_\epsilon|^{-\frac{1}{2}} M_+ P_0 F_\epsilon(z), \qquad M_- u_0 = -|A_\epsilon|^{-\frac{1}{2}} M_- P_0 F_\epsilon(z). \tag{2.5}$$

Thus $z = z_0 + z_+ + z_-$ is a solution of (2.2) if and only if $u = u_0 + u_+ + u_-$ is a critical point of I. System (2.3)-(2.5) is reduced to

$$A_\epsilon z_+ = P_+ F_\epsilon(z_0 + z_+ + z_-) \quad \text{or} \quad z_+ = (A_\epsilon)^{-1} P_+ F_\epsilon(z_0 + z_+ + z_-), \tag{2.6}$$

$$A_\epsilon z_- = P_- F_\epsilon(z_0 + z_+ + z_-) \quad \text{or} \quad z_- = (A_\epsilon)^{-1} P_- F_\epsilon(z_0 + z_+ + z_-), \tag{2.7}$$

$$A_\epsilon M_+ z_0 = M_+ P_0 F_\epsilon(z_0 + z_+ + z_-), \qquad A_\epsilon M_- z_0 = M_- P_0 F_\epsilon(z_0 + z_+ + z_-). \tag{2.8}$$

By the condition $F(z) \in C^2(E, E)$ and (G2),

$$\left\| F_\epsilon(z) - F_\epsilon(w) \right\|_{L^2} \le (\epsilon + \beta) \| u - v \|_H \quad \forall u, v \in E. \tag{2.9}$$

By (G3) and (G4), there exists $\gamma > \min\{j_1 - \alpha + \epsilon, \beta - j_1 + \epsilon\}$ such that

$$\left\| A_\epsilon^{-1} |_{L_+ \oplus L_-} \right\| \le \frac{1}{\gamma}.$$

By the following lemma, the weak solutions of (1.2) coincide with the critical points of the functional $I(z)$.

Lemma 2.1 *Assume that G satisfies conditions* (G1)-(G2). *Then $I(z)$ is continuous and Fréchet differentiable in X with the Fréchet derivative*

$$DI(z)w = \int_0^{2\pi} \left(A_\epsilon z - F_\epsilon(z) \right) \cdot w \, dt \quad \text{for all } w \in X.$$

Moreover, $DI \in C$. That is, $I \in C^1$.

Proof First we prove that $I(z) = \int_0^{2\pi} [\frac{1}{2} A_\epsilon z - G(t, z(t)) - \frac{\epsilon}{2} z^2] \, dt$ is continuous and Fréchet differentiable in X. For $z, w \in X$,

$$\left| I(z + w) - I(z) \right|$$

$$= \left| \int_0^{2\pi} \frac{1}{2} A_\epsilon(z + w) \cdot (z + w) - \int_0^{2\pi} \left[G(t, z + w) + \frac{\epsilon}{2}(z + w)^2 \right] \right.$$

$$\left. - \int_0^{2\pi} \frac{1}{2} A_\epsilon(z) \cdot z + \int_0^{2\pi} \left[G(t, z) + \frac{\epsilon}{2} z^2 \right] \right|$$

$$= \left| \int_0^{2\pi} \frac{1}{2} \left[A_\epsilon(z) \cdot w + A_\epsilon(w) \cdot z + A_\epsilon(w) \cdot w \right] \right.$$

$$\left. - \int_0^{2\pi} \left[G(t, z + w) - G(t, z) + \frac{\epsilon}{2} (2z \cdot w + w^2) \right] \right|.$$

We have

$$\left| \int_0^{2\pi} \left[G(t, z + w) - G(t, z) \right] dt \right|$$

$$\le \left| \int_0^{2\pi} \left[G_z(t, z(t)) \cdot w + O(\| w \|_{R^{2n}}) \right] dt \right| = O(\| w \|_{R^{2n}}). \tag{2.10}$$

Thus we have

$$\left| I(z + w) - I(z) \right| = O(\| w \|_{R^{2n}}).$$

Next we shall prove that $I(z)$ is Fréchet differentiable in X. For $z, w \in X$,

$$
\begin{aligned}
&\left| I(z + w) - I(z) - DI(z)w \right| \\
&= \left| \int_0^{2\pi} \frac{1}{2} A_\epsilon (z + w) \cdot (z + w) - \int_0^{2\pi} \left[G(t, z + w) + \frac{\epsilon}{2}(z + w)^2 \right] \right. \\
&\quad - \int_0^{2\pi} \frac{1}{2} A_\epsilon (z) \cdot z + \int_0^{2\pi} \left[G(t, z) + \frac{\epsilon}{2} z^2 \right] \\
&\quad \left. - \int_0^{2\pi} A_\epsilon (z) \cdot w + \int_0^{2\pi} \left[G_z(t, z) + \epsilon z \cdot w \right] \right| \\
&= \left| \int_0^{2\pi} \frac{1}{2} \left[A_\epsilon (w) \cdot z + A_\epsilon (w) \cdot w \right] - \int_0^{2\pi} \left[G(t, z + w) - G(t, z) - G_z(t, z) + \frac{\epsilon}{2} w^2 \right] \right|.
\end{aligned}
$$

By (2.10), we have

$$
\int_0^{2\pi} \left[G(t, z + w) - G(t, z) - G_z(t, z) \right] dt = O\left(\|w\|_{R^{2n}} \right).
$$

Thus

$$
\left| I(z + w) - I(z) - DI(z)w \right| = O\left(\|w\|_{R^{2n}} \right). \qquad \square
$$

Lemma 2.2 *Assume that G satisfies conditions* (G1)-(G2). *Let $\{z_k\} \subset X_-$ and $z_k \rightharpoonup z$ weakly in X with $z \in \partial X$. Then $I(z_k) \to -\infty$.*

Proof To prove the conclusion, it suffices to prove that

$$
\int_0^{2\pi} G\left(t, z_k(t) \right) dt \longrightarrow +\infty.
$$

Since $G(t, z(t))$ is bounded from below, it suffices to prove that there is an interval $[a, b] \subset [0, 2\pi]$ such that

$$
\int_a^b G\left(t, z_k(t) \right) dt \longrightarrow +\infty.
$$

By definition, $z \in \partial X$ means that there exists $t^* \in [0, 2\pi]$ such that $z(t^*) \in \partial D$. By G(2), there exists a constant $B > 0$ such that

$$
G(x, t, U) \geq \frac{A}{d^2(z, C)} - B.
$$

Thus we have

$$
\int_{t^*}^{t^* + \delta} G\left(t, z(t) \right) dt \geq \int_{t^*}^{t^* + \delta} \left(\frac{A}{\|z(t) - z(t^*)\|_{R^{2n}}^2} - B \right) dt
$$

for all $\delta > 0$. By Schwarz's inequality, we have

$$
\left\| z(t) - z_k(t) \right\|_{R^{2n}} \leq \left| t - t^* \right|^{\frac{1}{2}} \left(\int_0^{2\pi} \left\| \dot{z}(t) \right\|_{R^{2n}}^2 dt \right)^{\frac{1}{2}}.
$$

Thus we have

$$\int_{t^*}^{t^*+\delta} G(t, z(t))\, dt = +\infty.$$

Since the embedding $H^1(S^1, R^{2n}) \hookrightarrow C(S^1, R^{2n})$ is compact, we have

$$\max\left\{\left\|z(t) - z_k(t)\right\|_{R^{2n}}^2 \mid t \in S^1\right\} \longrightarrow 0 \quad \text{as } k \to \infty.$$

Thus by Fatou's lemma, we have

$$\liminf \int_{t^*}^{t^*+\delta} G(t, z_k(t))\, dt \geq \int_{t^*}^{t^*+\delta} \liminf G(t, z_k(t))$$

$$= \int_{t^*}^{t^*+\delta} G(t, z(t))\, dt = +\infty.$$

Thus

$$\liminf \int_{t^*}^{t^*+\delta} G(t, z_k(t)) = +\infty.$$

Thus if $\{z_k\} \subset X_-$,

$$I(z_k) = \int_0^{2\pi} \left[\frac{1}{2} A_\epsilon z_k(t) - \psi_\epsilon(z_k)\right] dx\, dt$$

$$= \frac{1}{2}\left(\left\|A_\epsilon^{\frac{1}{2}}(z_k)_+\right\|_{L^2}^2 + \left\|A_\epsilon^{\frac{1}{2}} M_+(z_k)_0\right\|^2 - \left\|(-A_\epsilon)^{\frac{1}{2}}(z_k)_-\right\|_{L^2}^2\right.$$

$$\left. - \left\|(-A_\epsilon)^{\frac{1}{2}} M_-(z_k)_0\right\|^2\right) - \psi_\epsilon(z_k)$$

$$\leq -\left\|(-A_\epsilon)^{\frac{1}{2}}(z_k)_-\right\|_{L^2}^2 - \psi_\epsilon(z_k) \longrightarrow -\infty,$$

so we prove the lemma. □

Now, we recall the critical point theorem for indefinite functional (*cf.* [5]).
Let

$$B_r = \left\{u \in X \mid \|u\| \leq r\right\},$$

$$S_r = \left\{u \in X \mid \|u\| = r\right\}.$$

Theorem 2.1 (Critical point theorem for indefinite functional) *Let X be a real Hilbert space with $X = X_1 \oplus X_2$ and $X_2 = X_1^\perp$. Suppose that $I \in C^1(X, R)$ satisfies (PS) and*

(I1) $I(u) = \frac{1}{2}(Lu, u) + bu$, *where $Lu = L_1 P_1 u + L_2 P_2 u$ and $L_i : X_i \to X_i$ is bounded and self-adjoint, $i = 1, 2$,*

(I2) *b' is compact, and*

(I3) *there exists a subspace $\tilde{X} \subset X$ and sets $S \subset X$, $Q \subset \tilde{X}$ and constants $\alpha > \omega$ such that*

   (i) *$S \subset X_1$ and $I|_S \geq \alpha$,*

   (ii) *Q is bounded and $I|_{\partial Q} \leq \omega$,*

   (iii) *S and ∂Q link.*

Then I possesses a critical value $c \geq \alpha$.

3 Proof of Theorem 1.1

We shall show that the functional $I(z)$ satisfies the geometric assumptions of the critical point theorem for indefinite functional.

Lemma 3.1 (Palais-Smale condition) *Assume that G satisfies conditions* (G1) *and* (G2). *Then there exists a constant γ_0 depending on the C^1 norm of the function G on $S^1 \times (R^{2n} \backslash B_{R_0})$ such that $I(z)$ satisfies the* (P.S.)$_\gamma$ *condition in X for $\gamma < \gamma_0$.*

Proof We shall prove the lemma by contradiction. We suppose that there exists a sequence $\{z_k\} \subset X$ satisfying

$$I(z_k) \to \gamma \tag{3.1}$$

and

$$DI(z_k) = z_k - A_\epsilon^{-1}\big(G_z(t, z_k) + \epsilon z_k\big) \longrightarrow \theta, \tag{3.2}$$

where A_ϵ^{-1} is a compact operator and $\theta = (0, \ldots, 0)$. Since $G(t, z(t))$ is bounded from below, (3.1) implies that there exists a constant $C_1 > 0$ such that

$$\int_0^{2\pi} A_\epsilon z_k(t)\, dt \geq C_1. \tag{3.3}$$

We claim that $\{z_k\}$ has a convergent subsequence. It suffices to prove that the sequence $\{z_k\}$ is bounded in X. By contradiction, we suppose that up to a subsequence, $\{z_k\}$ satisfies $\|z_k\|_{R^{2n}} \to \infty$. Then, for large k, we have

$$\|z_k\|_{R^{2n}} \geq R_0. \tag{3.4}$$

It follows from (3.4) that

$$\left| \int_0^{2\pi} G(t, z_k)\, dt \right| \leq 2\pi \sup\big\{ \big| G(t, z_k) \big| \,\big|\, (t, z_k) \in S^1 \times \big(R^{2n} \backslash B_{R_0}\big) \big\}. \tag{3.5}$$

By (3.3) and (3.5),

$$
\begin{aligned}
I(z_k) &= \int_0^{2\pi} \left[\frac{1}{2} A_\epsilon(z_k) - G\big(t, z_k(t)\big) - \frac{\epsilon}{2} z_k^2 \right] dt \\
&\geq \frac{C_1}{2} - 2\pi \sup\big\{ \big| G(t, z_k) \big| \,\big|\, (t, z_k) \in S^1 \times \big(R^{2n} \backslash B_{R_0}\big) \big\} - \frac{\epsilon}{2} \|z_k\|_{L^2}^2.
\end{aligned}
$$

Letting

$$\gamma_0 = \frac{C_1}{2} - 2\pi \sup\big\{ \big| G(t, z_k) \big| \,\big|\, (t, z_k) \in S^1 \times \big(R^{2n} \backslash B_{R_0}\big) \big\} - \frac{\epsilon}{2} \|z_k\|_{L^2}^2,$$

we have $I(z_k) \geq \gamma_0$, which is a contradiction. $\qquad\square$

Let

$$Q = \big(\bar{B}_r \cap X^-\big) \oplus \big\{ re \mid e \in B_1 \cap X^+, 0 < r < R \big\}.$$

Lemma 3.2 *Assume that G satisfies conditions* (G1) *and* (G2). *Then there exist sets* $S_\rho \subset X^+$ *with radius* $\rho > 0$, $Q \subset X$ *and a constant* $\alpha > 0$ *such that*

(i) $S_\rho \subset X^+$ *and* $I|_{S_\rho} \geq \alpha$,
(ii) Q *is bounded and* $I|_{\partial Q} \leq 0$,
(iii) S_ρ *and* ∂Q *link.*

Proof (i) Let us choose $z \in X^+ \subset X$. Then $z(t) \in D$. By (G1), $G(t,z)$ is bounded above and there exists a constant $\tau > 0$

$$I(z) = \frac{1}{2}\left(\left\|A_\epsilon^{\frac{1}{2}} z_+\right\|_{L^2}^2 + \left\|A_\epsilon^{\frac{1}{2}} M_+ z_0\right\| - \left\|(-A_\epsilon)^{\frac{1}{2}} z_-\right\|_{L^2}^2 - \left\|(-A_\epsilon)^{\frac{1}{2}} M_- z_0\right\|^2\right)$$

$$- \int_0^{2\pi} G(t,z)\,dt - \frac{\epsilon}{2}\|z\|_{L^2}^2$$

$$\geq \frac{1}{2}\left\|A_\epsilon^{\frac{1}{2}} z_+\right\|_{L^2}^2 - \tau - \frac{\epsilon}{2}\|z\|_{L^2}^2$$

for $\tau > 0$. Then there exist constants $\rho > 0$ and $\alpha > 0$ such that if $z \in S_\rho \cap X^+$, then $I(z) \geq \alpha$.

(ii) Let us choose $e \in B_1 \cap X^+$. Let $z \in \bar{B}_r \cap X^- \oplus \{re \mid 0 < r\}$. Then $z = w + y$, $w \in \bar{B}_r \cap X^-$, $y = re$. We note that

$$\text{If } w \in \bar{B}_r \cap X^-, \text{ then } \int_0^{2\pi} A_\epsilon(z) \cdot z\,dt = -\left\|(-A_\epsilon)^{\frac{1}{2}} z_-\right\|_{L^2}^2 \leq 0.$$

By (G2), $G(t, w + re)$ is bounded from below. Thus, by Lemma 2.2, there exists a constant $\tau > 0$ such that if $z = w + re$, then we have

$$I(z) = \frac{1}{2} r^2 - \left\|(-A_\epsilon)^{\frac{1}{2}} z_-\right\|_{L^2}^2 - \int_0^{2\pi} G(t, w + re)\,dt - \frac{\epsilon}{2}\|z\|_{L^2}^2$$

$$\leq \frac{1}{2} r^2 - \left\|(-A_\epsilon)^{\frac{1}{2}} z_-\right\|_{L^2}^2 - \int_\Omega \frac{\tau}{d^2(V + re, C)}\,dx\,dt - \frac{\epsilon}{2}\|z\|_{L^2}^2.$$

We can choose a constant $R > r$ such that if $z = w + re \in Q = (\bar{B}_r \cap X^-) \oplus \{re \mid e \in B_1 \cap X^+, 0 < r < R\}$, then $I(z) < 0$. Thus we prove the lemma. □

Proof of Theorem 1.1 By Lemma 2.1, $I(z)$ is continuous and Fréchet differentiable in X and, moreover, $DI \in C$. By Lemma 2.2, if $\{z_k\} \subset X^-$ and $z_k \rightharpoonup z$ weakly in X with $z \in \partial X$, then $I(z_k) \to -\infty$. By Lemma 3.1, $I(z)$ satisfies the $(P.S.)_\gamma$ condition for $\gamma < \gamma_0$. By Lemma 3.2, there exist sets $S_\rho \subset X^+$ with radius $\rho > 0$, $Q \subset X$ and a constant $\alpha > 0$ such that $I|_{S_\rho} \geq \alpha$, Q is bounded and $I|_{\partial Q} \leq 0$, and S_ρ and ∂Q link. By the critical point theorem, $I(z)$ possesses a critical value $c \geq \alpha$. Thus (1.1) has at least one nontrivial weak solution. Thus we prove Theorem 1.1. □

Author details
[1]Department of Mathematics, Kunsan National University, Kunsan, 573-701, Korea. [2]Department of Mathematics Education, Inha University, Incheon, 402-751, Korea.

References
1. Chang, KC: Infinite Dimensional Morse Theory and Multiple Solution Problems. Birkhäuser, Boston (1993)
2. Jung, T, Choi, QH: On the number of the periodic solutions of the nonlinear Hamiltonian system. Nonlinear Anal. TMA **71**, e1100-e1108 (2009)
3. Jung, T, Choi, QH: Periodic solutions for the nonlinear Hamiltonian systems. Korean J. Math. **17**, 331-340 (2009)
4. Rabinowitz, PH: Minimax Methods in Critical Point Theory with Applications to Differential Equations. CBMS. Regional Conf. Ser. Math., vol. 65. Am. Math. Soc., Providence (1986)
5. Benci, V, Rabinowitz, PH: Critical point theorems for indefinite functionals. Invent. Math. **52**, 241-273 (1979)