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Oscillation criteria for a certain second-order nonlinear perturbed differential equations

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Abstract

In this paper, a class of second-order nonlinear perturbed differential equation and its special cases are studied. By using the generalized Riccati transformation and well-known techniques, some new oscillation criteria are established. The results obtained essentially generalize and improve some known results and can be applied to the well-known half-linear and damped half-linear-type equations.

1 Introduction

This paper is concerned with the problem of oscillatory behavior of the perturbed second-order nonlinear differential equation

$$\left(r(t)\psi(u)\left|u'(t)\right|^{\alpha-1}u'(t)\right)' + Q(t,u) = P\left(t,u,u'\right), \tag{1}$$

where $r \in C(I,\mathbb{R}^+)$, $\psi \in C(\mathbb{R},\mathbb{R}^+)$, $\mathbb{R} = (-\infty,\infty)$, $I = [t_0,\infty)$, and α is a positive real number. Throughout the paper, according to the results, we shall impose the following conditions:

- (H_1) Let $f \in C'(\mathbb{R},\mathbb{R})$, and there exists a constant $k > 0$ such that $\frac{f'(x)}{(\psi(x)|f(x)|^{\alpha-1})^{\frac{1}{\alpha}}} \geq k$ and $f(x)x > 0$ for $x \neq 0$,
- (H_2) $Q \in C(I \times \mathbb{R},\mathbb{R})$, and there exists a continuous function $q(t)$ such that $\frac{Q(t,x)}{f(x)} \geq q(t)$ for $x \neq 0$,
- (H_3) $P \in C(I \times \mathbb{R} \times \mathbb{R},\mathbb{R})$, and there exists a continuous function $p(t)$ such that $\frac{P(t,x,y)}{f(x)} \leq p(t)$ for $x \neq 0$, $y \neq 0$,
- (H_4) $P \in C(I \times \mathbb{R} \times \mathbb{R},\mathbb{R})$, and there exists a continuous function $p(t)$ such that $\frac{P(t,x,y)}{f(x)} \leq p(t)\frac{|y|^{\alpha-1}y}{f(x)}$ for $x \neq 0$, $y \neq 0$,
- (H_5) $\int_\epsilon^\infty (\frac{\psi(y)}{f(y)})^{\frac{1}{\alpha}}\,dy < \infty$ and $\int_{-\epsilon}^{-\infty} (\frac{\psi(y)}{f(y)})^{\frac{1}{\alpha}}\,dy < \infty$ for every $\epsilon > 0$.

By a solution of (1), we mean a function $u \in C'([T_u,\infty),\mathbb{R})$, $T_u \geq t_0$, which has the property $r(t)|u'(t)|^{\alpha-1}u'(t) \in C'([T_u,\infty),\mathbb{R})$ and satisfies (1) on $[T_u,\infty)$. We consider only those solutions $u(t)$ of (1), which satisfy $\sup\{|u(t)| : t \geq T_u\} > 0$ for all $T_u \geq t_0$. We assume that (1) possesses such a solution. A nontrivial solution of (1) is said to be oscillatory if it has a sequence of zeros tending to infinity, otherwise, it is called nonoscillatory. An equation is said to be oscillatory if all its solutions are oscillatory.

In the last century, oscillation theory of differential equations has developed quickly and played an important role in qualitative theory of differential equations and theory of

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boundary value problem. The study of oscillation theory plays an important role in physical science and technology; for example in the oscillation of building or machine, electromagnetic vibration in radio technology and optical science, self-exited vibration in control system, sound vibration, beam vibration in synchrotron accelerator, the vibration sparked for burning rocket engine, the complicated oscillation in chemical reaction, and also in the research of a lossless high-speed computer network and physical sciences [1, 2]. All of this different phenomena can be unified into oscillation theory through an oscillation equation. There are many books on oscillation theory, we choose to refer to [3–6].

This problem has received the attention of many authors. Many criteria have been found, some of which involve the average behavior of the integral of the alternating coefficient. Among numerous papers dealing with this subject, we refer in particular to [7–35].

The first attempt for Equation (1) was due to Graef *et al.* [13], who investigated the case of (1) with $\alpha = 1$ and $\psi(u) \equiv 1$. In 1996, Wong and Agarwal studied the oscillatory behavior of (1) with $\psi(u) \equiv 1$ and the existence of a positive monotone solution of the damped equation given in [33]. Note that their paper contains a lot of new results and has been the motivation for the work for many others. It is the motivation for two recent papers and this work. In [2], Zhang and Wang studied the oscillation of Equation (1) with $\alpha = 1$. We should note that Wong's [33] result for $\alpha = 1$ corresponds to the special case with $\psi(u) \equiv 1$ in two main results in [2]. On the other hand, in another recent paper, Remili [22] studied the oscillation results for Equation (1) with $\alpha = 1$, $\psi(u) \equiv 1$. Two results of Remili, are also similar to Wong's results with addition of a suitable weighted function.

In this paper, motivated by the ideas in [33], we obtain several new oscillation criteria for Equation (1) and its special cases by using generalized Riccati transformation and well known techniques. The results obtained essentially generalize and improve some known results and can be applied to the well-known half-linear equation and damped half-linear-type equations.

2 Main results

In this section, we prove our main results.

Theorem 2.1 *Let conditions* (H_1), (H_2) *and* (H_3) *hold. If there exists a differentiable function* $\rho : I \to \mathbb{R}^+$ *such that* $\rho'(t) \geq 0$,

$$\lim_{t\to\infty} \int_{t_0}^{t} \left(\frac{1}{\rho^{\alpha}(s) r(s)} \right)^{\frac{1}{\alpha}} ds = \infty \tag{2}$$

and

$$\limsup_{t\to\infty} \int_{t_0}^{t} A(s)\, ds = \infty, \tag{3}$$

where

$$A(t) = \rho^{\alpha}(t) \left[q(t) - p(t) - \mu r(t) \left(\frac{\rho'(t)}{\rho(t)} \right)^{\alpha+1} \right] \quad \textit{and} \quad \mu := \left(\frac{\alpha}{\alpha+1} \right)^{\alpha+1} \left(\frac{\alpha}{k} \right)^{\alpha}, \tag{4}$$

then Equation (1) *is oscillatory.*

Proof Let $u(t)$ be a nonoscillatory solution of (1). We may assume that $u(t) > 0$ for $t \geq t_1 \geq t_0$. The proof in the case $u(t) < 0$ for $t \geq t_1$ is similar, and hence omitted. Define

$$w(t) = \rho^\alpha(t)\frac{r(t)\psi(u(t))|u'(t)|^{\alpha-1}u'(t)}{f(u(t))}. \tag{5}$$

Differentiating (5) and making use of (1) and from hypotheses (H_2) and (H_3), it follows that

$$w'(t) \leq \frac{\alpha\rho'(t)}{\rho(t)}\bigl|w(t)\bigr| - \rho^\alpha(t)\bigl(q(t)-p(t)\bigr) - \frac{\rho^\alpha(t)r(t)\psi(u(t))|u'(t)|^{\alpha-1}u'^2(t)f'(u(t))}{f^2(u(t))}. \tag{6}$$

It is easy to see that condition (H_1) implies that

$$w'(t) \leq \frac{\alpha\rho'(t)}{\rho(t)}\bigl|w(t)\bigr| - \rho^\alpha(t)\bigl(q(t)-p(t)\bigr) - k\frac{|w(t)|^{\frac{\alpha+1}{\alpha}}}{\rho(t)(r(t))^{\frac{1}{\alpha}}}. \tag{7}$$

By using the extremum of one variable function it can be easily proved that

$$DX - EX^{\frac{\alpha+1}{\alpha}} \leq \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}D^{\alpha+1}E^{-\alpha}, \quad D \geq 0, E > 0, X \geq 0. \tag{8}$$

Using the above inequality, we get

$$w'(t) \leq -\rho^\alpha(t)\left[q(t) - p(t) - \mu r(t)\left(\frac{\rho'(t)}{\rho(t)}\right)^{\alpha+1}\right]. \tag{9}$$

Integrating this inequality from t_1 to t, we get

$$w(t) \leq w(t_1) - \int_{t_1}^{t}\rho^\alpha(s)\left[q(s) - p(s) - \mu r(s)\left(\frac{\rho'(s)}{\rho(s)}\right)^{\alpha+1}\right]ds, \quad t \geq t_1 \geq t_0. \tag{10}$$

Letting $\limsup_{t\to\infty}$, we get in view of (3) that $w(t) \to -\infty$. Hence there exists $t_2 \geq t_1$ such that $u'(t) < 0$ for $t \geq t_2$. Condition (3) also implies

$$\int_{t_0}^{\infty}\rho^\alpha(s)\bigl(q(s)-p(s)\bigr)\,ds = \infty, \tag{11}$$

and there exists $t_3 \geq t_2$ such that

$$\int_{t_3}^{t}\rho^\alpha(s)\bigl(q(s)-p(s)\bigr)\,ds \geq 0$$

for $t \geq t_3$. Now multiplying (1) by $\rho^\alpha(t)$ and integrating by parts, we obtain

$$\begin{aligned}
&-\rho^\alpha(t)\bigl(r(t)\psi\bigl(u(t)\bigr)\bigl(-u'(t)\bigr)^\alpha\bigr)' \leq -\rho^\alpha(t)\bigl(q(t)-p(t)\bigr)f\bigl(u(t)\bigr), \\
&-\rho^\alpha(t)r(t)\psi\bigl(u(t)\bigr)\bigl(-u'(t)\bigr)^\alpha + c_{t_3} \leq -\int_{t_3}^{t}\bigl(\rho^\alpha(s)\bigr)'r(s)\psi\bigl(u(s)\bigr)\bigl(-u'(s)\bigr)^\alpha\,ds \\
&\qquad\qquad - \int_{t_3}^{t}\rho^\alpha(s)\bigl(q(s)-p(s)\bigr)f\bigl(u(s)\bigr)\,ds,
\end{aligned}$$

where $c_{t_3} := \rho^\alpha(t_3)r(t_3)\psi(u(t_3))(-u'(t_3))^\alpha > 0$. Then

$$\begin{aligned}
&\rho^\alpha(t)r(t)\psi\big(u(t)\big)\big(-u'(t)\big)^\alpha \\
&\quad \geq c_{t_3} + \int_{t_3}^t \big(\rho^\alpha(s)\big)' r(s)\psi\big(u(s)\big)\big(-u'(s)\big)^\alpha\,ds \\
&\qquad + \int_{t_3}^t \rho^\alpha(s)\big(q(s)-p(s)\big)f\big(u(s)\big)\,ds \\
&\quad \geq c_{t_3} + \int_{t_3}^t \rho^\alpha(s)\big(q(s)-p(s)\big)f\big(u(s)\big)\,ds \\
&\quad = c_{t_3} + f\big(u(t)\big)\int_{t_3}^t \rho^\alpha(s)\big(q(s)-p(s)\big)\,ds \\
&\qquad - \int_{t_3}^t f'\big(u(s)\big)u'(s)\int_{t_3}^s \rho^\alpha(v)\big(q(v)-p(v)\big)\,dv\,ds, \\
&\rho^\alpha(t)r(t)\psi\big(u(t)\big)\big(-u'(t)\big)^\alpha \geq c_{t_3}, \\
&\psi\big(u(t)\big)\big(-u'(t)\big)^\alpha \geq \frac{c_{t_3}}{\rho^\alpha(t)r(t)}, \\
&\big(\psi\big(u(t)\big)\big)^{\frac{1}{\alpha}}\big(-u'(t)\big) \geq \left(\frac{c_{t_3}}{\rho^\alpha(t)r(t)}\right)^{\frac{1}{\alpha}}, \\
&\int_{t_3}^t \big(\psi\big(u(s)\big)\big)^{\frac{1}{\alpha}}u'(s)\,ds \leq -(c_{t_3})^{\frac{1}{\alpha}}\int_{t_3}^t \left(\frac{1}{\rho^\alpha(s)r(s)}\right)^{\frac{1}{\alpha}}ds, \\
&\int_{u(t_3)}^{u(t)} \big(\psi(y)\big)^{\frac{1}{\alpha}}\,dy \leq -(c_{t_3})^{\frac{1}{\alpha}}\int_{t_3}^t \left(\frac{1}{\rho^\alpha(s)r(s)}\right)^{\frac{1}{\alpha}}ds.
\end{aligned}$$

Noting condition (2), and the fact that $0 < u(t) \leq u(t_3)$, this implies that the left-hand side of this inequality, that is, $\int_{u(t_3)}^{u(t)}(\psi(y))^{\frac{1}{\alpha}}\,dy$ is lower bounded. But the right-hand side of it tend towards mines infinite, so contradiction exists. The proof is complete. □

Example 2.1 Consider the differential equations of the form

$$\begin{aligned}
&\big(r(t)\psi(u)\big|u'\big|^{\alpha-1}u'\big)' + \left[\frac{3}{2}t^{\frac{-3}{2}}(2+\cos t) + \theta_1(t,u)\right]f(u) \\
&\quad = \big(t^{\frac{-1}{2}}\sin t + t^{-3}\theta_2\big(t,u,u'\big)\big)f(u), \quad t \geq t_0,
\end{aligned} \tag{E}$$

where θ_1 and θ_2 are continuous functions such that $\theta_1(t,u) \geq 0$ and $\theta_2(t,u,u') \leq 1$. If we take the functions r, ψ, f and the constant $\alpha > 0$ satisfying the condition of Theorem 2.1, then Equation (E) is oscillatory. In particular, for the equation

$$\big(\big|u'\big|u'\big)' + \left[\frac{3}{2}t^{\frac{-3}{2}}(2+\cos t) + te^u\right]u^3 = \left(t^{\frac{-1}{2}}\sin t + t^{-3}\frac{u^3\cos u'}{u^2+1}\right)u^3, \quad t \geq \frac{\pi}{2},$$

all conditions of Theorem 2.1 are satisfied. Hence it is oscillatory.

A close look at the proof of Theorem 2.1 reveals that condition (3) may be replaced by the conditions

$$\limsup_{t\to\infty}\int_{t_0}^{t}\rho^{\alpha}(s)\big[q(s)-p(s)\big]\,ds=\infty, \tag{12}$$

$$\limsup_{t\to\infty}\int_{t_0}^{t}\rho^{\alpha}(s)r(s)\left(\frac{\rho'(s)}{\rho(s)}\right)^{\alpha+1}ds<\infty. \tag{13}$$

Corollary 2.1 *Let the conditions of Theorem* 2.1 *be satisfied except that condition* (3) *is replaced by* (12) *and* (13). *Then Equation* (1) *is oscillatory.*

Note that there is no restriction on the sign of functions $p(t)$ and $q(t)$. But we have the condition

$$\int_{t_0}^{\infty}\rho^{\alpha}(s)\big(q(s)-p(s)\big)\,ds=\infty. \tag{14}$$

Now, supposing that this condition is not an establishment, we discuss the oscillatory behavior of (1). We have the following result.

Theorem 2.2 *Let conditions* (H_1), (H_2), (H_3) *and* (H_5) *hold. Suppose that* ρ *is a positive continuously differentiable function on the interval I such that* $\rho'\geq 0$ *on I and* (2) *hold. If*

$$\int_{t_0}^{\infty}\rho^{\alpha}(t)\big[q(t)-p(t)\big]\,dt<\infty, \tag{15}$$

$$\liminf_{t\to\infty}\left[\int_{t_0}^{t}A(s)\,ds\right]\geq 0 \tag{16}$$

and

$$\lim_{t\to\infty}\int_{t_0}^{t}\left(\frac{1}{\rho^{\alpha}(s)r(s)}\int_{s}^{\infty}A(v)\,dv\right)^{\frac{1}{\alpha}}ds=\infty, \tag{17}$$

then Equation (1) *is oscillatory.*

Remark 2.1 Condition (15) implies that

$$\int_{t_0}^{\infty}A(s)\,ds<\infty\quad\text{and}\quad\lim_{t\to\infty}\int_{t_0}^{t}A(s)\,ds=\int_{t_0}^{\infty}A(s)\,ds,$$

hence (16) takes the form $\int_{t_0}^{\infty}A(s)\,ds\geq 0$ for all large T.

Proof Let $u(t)$ be a nonoscillatory solution on the interval I of the differential Equation (1). We suppose, as in Theorem 2.1, that $u(t)$ is positive on I. We consider the following three cases for the behavior of $u'(t)$.

Case 1: $u'(t)>0$ for $t\geq t_1$ for some $t_1\geq t_0$. From (10), we have

$$\int_{t_1}^{t}A(s)\,ds\leq w(t_1)-w(t).$$

Hence, for all $t \geq t_1$

$$\int_t^\infty A(s)\,ds \leq \rho^\alpha(t)\frac{r(t)\psi(u(t))(u'(t))^\alpha}{f(u(t))}.$$

Using (H_5), we obtain

$$\begin{aligned}\int_{t_1}^t \left(\frac{1}{\rho^\alpha(s)r(s)}\int_s^\infty A(v)\,dv\right)^{\frac{1}{\alpha}} ds &\leq \int_{t_1}^t \left(\frac{\psi(u(s))}{f(u(s))}\right)^{\frac{1}{\alpha}} u'(s)\,ds \\ &\leq \int_{u(t_1)}^\infty \left(\frac{\psi(y)}{f(y)}\right)^{\frac{1}{\alpha}} dy < \infty.\end{aligned}$$

This contradicts condition (17).

Case 2: $u'(t)$ changes signs, then there exists a sequence $(\alpha_n) \to \infty$ in I such that $u'(\alpha_n) < 0$. Choose N large enough, so that

$$\int_{\alpha_N}^\infty A(s)\,ds \geq 0.$$

Then from (10), we have

$$\begin{aligned}&\frac{-\rho^\alpha(t)r(t)\psi(u(t))(-u'(t))^\alpha}{f(u(t))} \leq -C_{\alpha_N} - \int_{\alpha_N}^t A(s)\,ds, \\ &\liminf_{t\to\infty}\frac{\rho^\alpha(t)r(t)\psi(u(t))(-u'(t))^\alpha}{f(u(t))} \geq C_{\alpha_N} + \liminf_{t\to\infty}\int_{\alpha_N}^t A(s)\,ds > 0,\end{aligned}$$

which contradicts the fact that $u'(t)$ oscillates.

Case 3: $u'(t) < 0$ for $t \geq t_1$. Condition (16) implies that for any $t_0 \geq t_1$, there exists $t_1 \geq t_0$ such that

$$\int_t^\infty \rho^\alpha(s)\big[q(s) - p(s)\big]\,ds \geq 0$$

for all $t \geq t_1$ as it was shown in [28]. The remaining part of the proof is similar to that of Theorem 2.1. □

Remark 2.2 When $\alpha = 1$ and $\psi(u) \equiv 1$, Theorem 2.1 and Theorem 2.2 reduce to Theorem 1 and 2 in [22], respectively.

Theorem 2.3 *Let conditions* (H_1), (H_2) *and* (H_3) *hold. If there exists a differentiable function* $\rho : I \to \mathbb{R}^+$ *such that* $\rho'(t) \geq 0$,

$$\lim_{t\to\infty}\int_{t_0}^t \left(\frac{1}{R(s,t_0)\rho^\alpha(s)r(s)}\right)^{\frac{1}{\alpha}} ds = \infty \tag{18}$$

and

$$\limsup_{t\to\infty}\int_{t_0}^t B(s)\,ds = \infty, \tag{19}$$

where

$$R(t,t_0) = \int_{t_0}^{t} \frac{ds}{r(s)\rho^\alpha(s)}, \tag{20}$$

$$B(t) = R(t,t_0)\rho^\alpha(t)\left[q(t) - p(t) - \mu_1 r(t)\left(\alpha\frac{\rho'(t)}{\rho(t)} + \frac{1}{\rho^\alpha(t)r(t)R(t,t_0)}\right)^{\alpha+1}\right], \tag{21}$$

then Equation (1) *is oscillatory.*

Proof Otherwise, $u(t) \neq 0$ for all $t \geq t_1 \geq t_0$. Define

$$w(t) = \rho^\alpha(t)\frac{r(t)\psi(u(t))|u'(t)|^{\alpha-1}u'(t)}{f(u(t))}R(t,t_0). \tag{22}$$

Differentiating (22) and making use of (1) and conditions (H_2) and (H_5), it follows that

$$\begin{aligned} w'(t) \leq{}& \frac{\alpha\rho'(t)}{\rho(t)}\bigl|w(t)\bigr| - \rho^\alpha(t)\bigl(q(t) - p(t)\bigr)R(t,t_0) \\ &- \frac{\rho^\alpha(t)r(t)\psi(u(t))|u'(t)|^{\alpha-1}u'^2(t)f'(u(t))}{f^2(u(t))}R(t,t_0) + \frac{\psi(u(t))|u'(t)|^{\alpha-1}u'(t)}{f(u(t))}. \end{aligned} \tag{23}$$

From (H_1) and using the equality

$$\frac{\psi(u(t))|u'(t)|^\alpha}{f(u(t))} = \frac{|w(t)|}{\rho^\alpha(t)r(t)R(t,t_0)}, \tag{24}$$

we get

$$\begin{aligned} w'(t) \leq{}& \left(\frac{\alpha\rho'(t)}{\rho(t)} + \frac{1}{\rho^\alpha(t)r(t)R(t,t_0)}\right)\bigl|w(t)\bigr| - \rho^\alpha(t)\bigl(q(t) - p(t)\bigr)R(t,t_0) \\ &- k\frac{|w(t)|^{1+\frac{1}{\alpha}}}{\rho(t)r(t)^{\frac{1}{\alpha}}(R(t,t_0))^{\frac{1}{\alpha}}}. \end{aligned}$$

The rest of the proof can be made as in the proof of Theorem 2.1. □

Theorem 2.4 *Let conditions* (H_1), (H_2), (H_3) *and* (H_5) *hold. Suppose that there exists a differentiable function* $\rho : I \to \mathbb{R}^+$ *such that* $\rho'(t) \geq 0$ *on* I *and* (18) *hold. If*

$$\limsup_{t\to\infty} \int_{t_0}^{t} A(s)R(s,t_0)\,ds = \infty, \tag{25}$$

then any solution $u(t)$ *of Equation* (1) *such that* $u'(t)$ *is bounded is oscillatory.*

Proof Let $u(t)$ be nonoscillatory solution of (1), say $u(t) > 0$ for $t \geq t_0$. Further, assume that $|u'(t)| \leq L$ for some $L > 0$. Beginning as in the proof of Theorem 2.3, we have

$$\begin{aligned} w'(t) \leq{}& \frac{\alpha\rho'(t)}{\rho(t)}\bigl|w(t)\bigr| - \rho^\alpha(t)\bigl(q(t) - p(t)\bigr)R(t,t_0) \\ &- \frac{\rho^\alpha(t)r(t)\psi(u(t))|u'(t)|^{\alpha-1}u'^2(t)f'(u(t))}{f^2(u(t))}R(t,t_0) + \frac{\psi(u(t))|u'(t)|^{\alpha-1}u'(t)}{f(u(t))}. \end{aligned}$$

From (H_1) and the definition of $w(t)$, we have

$$w'(t) \le \frac{\alpha\rho'(t)}{\rho(t)}\big|w(t)\big| - k\frac{|w(t)|^{1+\frac{1}{\alpha}}}{\rho(t)(r(t))^{\frac{1}{\alpha}}(R(t,t_0))^{\frac{1}{\alpha}}} - \rho^{\alpha}(t)\big(q(t)-p(t)\big)R(t,t_0) + \frac{\psi(u(t))|u'(t)|^{\alpha-1}u'(t)}{f(u(t))}.$$

Applying inequality (8), first two term in the second hand side, we get

$$w'(t) \le -\rho^{\alpha}(t)R(t,t_0)\left[q(t)-p(t)-\mu r(t)\left(\frac{\rho'(t)}{\rho(t)}\right)^{\alpha+1}\right] + \frac{\psi(u(t))|u'(t)|^{\alpha-1}u'(t)}{f(u(t))}. \tag{26}$$

Case 1: Suppose that $u'(t) > 0$ for $t \ge t_1 \ge t_0$. An integration of (26) from t_1 to t yields

$$w(t) \le w(t_1) - \int_{t_1}^{t} A(s)R(s,t_0)\,ds + L^{\alpha-1}\int_{u(t_1)}^{u(t)} \frac{\psi(y)}{f(y)}\,dy. \tag{27}$$

By (25) and (H_5), the right side of (27) tends to $-\infty$ as $t \to \infty$. However, the left side of (27) is nonnegative.

Case 2: Suppose that $u'(t)$ oscillates. Then there exists a sequence $(\alpha_n) \to \infty$ in I such that $u'(\alpha_n) = 0$. Choose N large enough, so that $\alpha_N \ge t_0$. With no loss of generality, we assume that $u'(t) > 0$ for the (α_N, α_{N+1}). Further, in view of (18), we have

$$\int_{\alpha_N}^{\alpha_{N+1}} A(s)R(s,t_0)\,ds > \epsilon > 0. \tag{28}$$

Now, an integration of (26) from α_N to α_{N+1}, provides

$$\int_{\alpha_N}^{\alpha_{N+1}} A(s)R(s,t_0)\,ds \le L^{\alpha-1}\int_{\alpha_N}^{\alpha_{N+1}} \frac{\psi(u(s))u'(s)}{f(u(s))}\,ds. \tag{29}$$

There are infinite number of N's such that $u'(t) > 0$ for $t \in (\alpha_N, \alpha_{N+1})$. Summing all these inequalities (29), we have

$$\sum_{k=1}^{\infty}\int_{\alpha_N}^{\alpha_{N+1}} A(s)R(s,t_0)\,ds \le L^{\alpha-1}\sum_{k=1}^{\infty}\int_{u(N_k)}^{u(N_{k+1})} \frac{\psi(y)}{f(y)}\,dy. \tag{30}$$

In view of (29), the left side of (30) is infinite, whereas the right side of (30) is finite by (H_5).

Case 3: Suppose that $u'(t) < 0$ for $t \ge t_1 \ge t_0$. In view of (25), we may assume that there exists $t_2 \ge t_1$, so that

$$\int_{t_2}^{t} \rho^{\alpha}(s)\big(q(s)-p(s)\big)R(s,t_1)\,ds \ge 0, \quad t \ge t_2. \tag{31}$$

Multiplying (1) by $\rho^{\alpha}(t)R(t,t_1)$ and using (H_3), we get

$$\rho^{\alpha}(t)R(t,t_1)\big(r(t)\big|u'(t)\big|^{\alpha-1}u'(t)\big)' \le -\rho^{\alpha}(t)R(t,t_1)\big(q(t)-p(t)\big)f\big(u(t)\big).$$

The rest of the proof is as in the proof of Theorem 2.1. Hence we omit it. □

Note that it can be easily seen from the proof of Theorem 2.4 when $\alpha = 1$, it is not necessary to assume that $u'(t)$ is bounded. In this case, conclusion of Theorem 2.4 leads to the following results.

Corollary 2.2 *Let conditions* (H_1), (H_2), (H_3) *and* (H_5) *hold. Suppose that there exists a differentiable function* $\rho : I \to \mathbb{R}^+$ *such that* $\rho'(t) \geq 0$ *on I. If*

$$\lim_{t\to\infty} \int_{t_0}^{t} \frac{ds}{\rho(s)r(s)} = \infty$$

and

$$\limsup_{t\to\infty} \int_{t_0}^{t} \rho(s)\left(\int_{t_0}^{s} \frac{d\tau}{\rho(\tau)r(\tau)}\right)\left[q(s) - p(s) - \frac{1}{4k}r(s)\left(\frac{\rho'(s)}{\rho(s)}\right)^2\right] ds = \infty,$$

then Equation (1) *is oscillatory.*

Remark 2.3 If we take $\rho(t) = 1$ in our results, then condition (H_1) may include weaker conditions. In fact, if we replace condition (H_1) with the condition $f'(x) \geq 0$, then all oscillation criteria above are valid with $\rho(t) = 1$. Hence when $\rho(t) = 1$ and $\psi(u) \equiv 1$, Theorem 2.1, Theorem 2.2 and Corollary 2.2 reduce to Theorem 2.1 and Theorem 2.2 and Theorem 2.9 in [33], respectively. When $\alpha = 1$, $\psi(u) \equiv 1$ and $\rho(t) = 1$, Theorem 2.2 and Corollary 2.2 give Theorem 1 and Theorem 2 in [22], respectively. When $\alpha = 1$ and $\rho(t) = 1$, Theorem 2.2 and Corollary 2.2 reduce Theorem 1 and Theorem 2 in [2], respectively.

By taking (H_4) instead of (H_3), we obtain the following interesting result, which can be applied, for example, to damped half-linear-type equations:

$$\left(r(t)\left|u'(t)\right|^{\alpha-1}u'(t)\right)' + b(t)\left|u'(t)\right|^{\alpha-1}u'(t) + q(t)f(u) = 0, \quad t \geq t_0 \geq 0. \tag{32}$$

Theorem 2.5 *Let conditions* (H_1), (H_2) *and* (H_4) *hold. If there exists a differentiable function* $\rho : I \to \mathbb{R}^+$ *such that* $\rho'(t) \geq 0$,

$$\lim_{t\to\infty} \int_{t_0}^{t} \left(\frac{e^{-\int_{t_0}^{s} \frac{p(\tau)}{r(\tau)}\,d\tau}}{\rho^\alpha(s)r(s)}\right)^{\frac{1}{\alpha}} ds = \infty \tag{33}$$

and

$$\limsup_{t\to\infty} \int_{t_0}^{t} C(s)\,ds = \infty, \tag{34}$$

where

$$C(t) = \rho^\alpha(t)\left[q(t) - \mu_1 r(t)\left|\frac{\alpha\rho'(t)}{\rho(t)} + \frac{p(t)}{r(t)}\right|^{\alpha+1}\right] \quad \textit{and} \quad \mu_1 := \frac{1}{(\alpha+1)^{\alpha+1}}\left(\frac{\alpha}{k}\right)^\alpha,$$

then Equation (1) *with* $\psi(u) \equiv 1$ *is oscillatory.*

Proof To the contrary, let $u(t)$ be a nonoscillatory solution of (1). We may assume that $u(t) > 0$ for $t \geq t_1 \geq t_0$. The proof in the case $u(t) < 0$ for $t \geq t_1$ is similar and hence omitted. Differentiating (5) and making use of (1) and from hypotheses (H_1), (H_2), (H_4) and (8), as in the proof of Theorem 2.1, we can obtain easily that

$$w'(t) \leq -\rho^\alpha(t)\left[q(t) - \mu_1 r(t)\left|\frac{\alpha\rho'(t)}{\rho(t)} + \frac{p(t)}{r(t)}\right|^{\alpha+1}\right]. \tag{35}$$

Integrating this inequality from t_1 to t, we get

$$w(t) \leq w(t_1) - \int_{t_1}^{t} \rho^\alpha(s)\left[q(s) - \mu_1 r(s)\left|\frac{\alpha\rho'(s)}{\rho(s)} + \frac{p(s)}{r(s)}\right|^{\alpha+1}\right] ds, \quad t \geq t_1 \geq t_0. \tag{36}$$

Letting $\limsup_{t\to\infty}$, we get in view of (3) that $w(t) \to -\infty$. Hence, there exists $t_2 \geq t_1$ such that $u'(t) < 0$ for $t \geq t_2$. Condition (34) also implies that

$$\int_{t_0}^{\infty} \rho^\alpha(s)q(s)\,ds = \infty, \tag{37}$$

and there exists $t_3 \geq t_2$ such that

$$\int_{t_3}^{t} \rho^\alpha(s)q(s)\,ds \geq 0$$

for $t \geq t_3$. Proceeding similarly as in Theorem 2.1, using condition (33), we obtain a contradiction. □

As an immediate consequence of Theorem 2.5, we have the following interesting criteria for the oscillation of (1).

Corollary 2.3 *Let condition* (34) *in Theorem* 2.5 *be replaced by*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \rho^\alpha(s)q(s)\,ds = \infty \tag{38}$$

and

$$\int_{t_0}^{\infty} \rho^\alpha(s)r(s)\left|\frac{\alpha\rho'(s)}{\rho(s)} + \frac{p(s)}{r(s)}\right|^{\alpha+1} ds < \infty. \tag{39}$$

Then the conclusion Theorem 2.5 *holds.*

It is clear that condition (38) is a necessary condition for (34) to hold.

In case (34) failed to be satisfied that the following theorem may be applicable.

Theorem 2.6 *Let conditions* (H_1), (H_2), (H_4) *and* (H_5) *hold. Suppose that* ρ *is a positive continuously differentiable function on the interval* I *such that* $\rho'(t) \geq 0$ *on* I *and* (34) *hold. If*

$$\int_{t_0}^{\infty} \rho^\alpha(t)q(t)\,dt < \infty, \tag{40}$$

and

$$\liminf_{t\to\infty}\int_{t_0}^{t} C(s)\,ds \geq 0, \tag{41}$$

$$\lim_{t\to\infty}\int_{t_0}^{t}\left(\frac{1}{\rho^{\alpha}(s)r(s)}\int_{s}^{\infty} C(u)\,du\right)^{\frac{1}{\alpha}} ds = \infty, \tag{42}$$

then Equation (1) *with* $\psi(u) \equiv 1$ *is oscillatory.*

Proof Proof is a similar to the proof of Theorem 2.2. □

Remark 2.4 If we take $f(u) = |u|^{\alpha-1}u$, then $\frac{f'(u)}{|f(u)|^{1-\frac{1}{\alpha}}} = k = \alpha$ is satisfied. Therefore, all above oscillation criteria are valid for the half-linear equation

$$\left(r(t)\left|u'\right|^{\alpha-1}u'\right)' + q(t)|u|^{\alpha-1}u = 0.$$

Remark 2.5 More recently, Ouyang and *et al.* [17] gave some oscillation criteria for Equation (32) under the main condition $f'(x) \geq 0$. However, their results impose sign conditions on function $b(t)$ and $q(t)$. In our results, we assume that hypothesis (H_1) is stronger than $f'(x) \geq 0$. But our results do not depend on signs of the functions $b(t)$ and $q(t)$. Note that Theorem 3.6 given in [26], which also does not depend on the signs of $b(t)$ and $q(t)$, is different from Theorem 2.5 above, because it contains suitable averaging functions.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors completed the paper together. All authors read and approved the final manuscript.

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