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Almost periodic solutions for SICNNs with time-varying delays in the leakage terms

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Abstract

This paper is concerned with the shunting inhibitory cellular neural networks (SICNNs) with time-varying delays in the leakage (or forgetting) terms. Under proper conditions, we employ a novel argument to establish a criterion on the global exponential stability of almost periodic solutions by using Lyapunov functional method and differential inequality techniques. We also provide numerical simulations to support the theoretical result.

Keywords: shunting inhibitory cellular neural networks; almost periodic solution; exponential stability; time-varying delay; leakage term

1 Introduction

It is well known that shunting inhibitory cellular neural networks (SICNNs) have been introduced as new cellular neural networks (CNNs) in Bouzerdout and Pinter in [1–3], which can be described by

$$x'_{ij}(t) = -a_{ij}(t)x_{ij}(t) - \sum_{C_{kl} \in N_r(i,j)} C_{ij}^{kl}(t) f\big(x_{kl}\big(t - \tau(t)\big)\big) x_{ij}(t)$$

$$- \sum_{C_{kl} \in N_q(i,j)} B_{ij}^{kl}(t) \int_0^\infty K_{ij}(u) g\big(x_{kl}(t-u)\big) du \, x_{ij}(t) + L_{ij}(t),$$

$$i = 1, 2, \ldots, m, j = 1, 2, \ldots, n, \tag{1.1}$$

where C_{ij} denotes the cell at the (i,j) position of the lattice. The r-neighborhood $N_r(i,j)$ is given as

$$N_r(i,j) = \big\{ C_{kl} : \max\big(|k-i|, |l-j|\big) \le r, 1 \le k \le m, 1 \le l \le n \big\},$$

$N_q(i,j)$ is similarly specified. x_{ij} is the activity of the cell C_{ij}, $L_{ij}(t)$ is the external input to C_{ij}, the function $a_{ij}(t) > 0$ represents the passive decay rate of the cell activity, $C_{ij}^{kl}(t)$ and $B_{ij}^{kl}(t)$ are the connection or coupling strength of postsynaptic activity of the cell transmitted to the cell C_{ij}, and the activity functions $f(\cdot)$ and $g(\cdot)$ are continuous functions representing the output or firing rate of the cell C_{kl}, and $\tau(t) \ge 0$ corresponds to the transmission delay.

Recently, SICNNs have been extensively applied in psychophysics, speech, perception, robotics, adaptive pattern recognition, vision, and image processing. Hence, they have been the object of intensive analysis by numerous authors in recent years. In particular,

there have been extensive results on the problem of the existence and stability of the equilibrium point, periodic and almost periodic solutions of SICNNs with time-varying delays in the literature. We refer the reader to [4–9] and the references cited therein. Obviously, the first term in each of the right side of (1.1) corresponds to a stabilizing negative feedback of the system, which acts instantaneously without time delay; these terms are variously known as 'forgetting' or leakage terms (see, for instance, Kosko [10], Haykin [11]). It is known from the literature on population dynamics and neural networks dynamics (see Gopalsamy [12]) that time delays in the stabilizing negative feedback terms will have a tendency to destabilize a system. Therefore, the authors of [13–19] dealt with the existence and stability of equilibrium and periodic solutions for neuron networks model involving leakage delays. However, to the best of our knowledge, few authors have considered the existence and exponential stability of almost periodic solutions of SICNNs with time-varying delays in the leakage terms. Motivated by the discussions above, in this paper, we consider the following SICNNs with time-varying leakage delays:

$$x'_{ij}(t) = -a_{ij}(t)x_{ij}(t - \eta_{ij}(t)) - \sum_{C_{kl} \in N_r(i,j)} C^{kl}_{ij}(t)f(x_{kl}(t - \tau(t)))x_{ij}(t)$$

$$- \sum_{C_{kl} \in N_q(i,j)} B^{kl}_{ij}(t) \int_0^\infty K_{ij}(u)g(x_{kl}(t - u))\,du\,x_{ij}(t) + L_{ij}(t), \tag{1.2}$$

where $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$, $a_{ij} : R \to (0 + \infty)$, $\eta_{ij}, \tau : R \to [0 + \infty)$, and $L_{ij}, C^{kl}_{ij}, B^{kl}_{ij} : R \to R$ are almost periodic functions, $\eta_i(t)$ and $\tau(t)$ denote the leakage delay and transmission delay, respectively, the delay kernels $K_{ij} : [0, \infty) \to [0, \infty)$ are continuous and integrable, and η'_{ij} is a bounded continuous function.

The main purpose of this paper is to give the conditions for the existence and exponential stability of the almost periodic solutions for system (1.2). By applying Lyapunov functional method and differential inequality techniques, we derive some new sufficient conditions ensuring the existence, uniqueness and exponential stability of the almost periodic solution for system (1.2), which are new and complement previously known results. Moreover, an example with numerical simulations is also provided to illustrate the effectiveness of our results.

Throughout this paper, for $ij \in J := \{11, 12, \ldots, 1n, 21, 22, \ldots, 2n, \ldots, m1, m2, \ldots, mn\}$, from the theory of almost periodic functions in [20, 21], it follows that for all $\epsilon > 0$, it is possible to find a real number $l = l(\epsilon) > 0$, for any interval with length $l(\epsilon)$, there exists a number $\delta = \delta(\epsilon)$ in this interval such that

$$\begin{cases} |a_{ij}(t + \delta) - a_{ij}(t)| < \epsilon, & |\eta_{ij}(t + \delta) - \eta_{ij}(t)| < \epsilon, & |C^{kl}_{ij}(t + \delta) - C^{kl}_{ij}(t)| < \epsilon, \\ |B^{kl}_{ij}(t + \delta) - B^{kl}_{ij}(t)| < \epsilon, & |\tau(t + \delta) - \tau(t)| < \epsilon, & |L_{ij}(t + \delta) - L_{ij}(t)| < \epsilon, \end{cases} \tag{1.3}$$

for all $t \in R$, $kl, ij \in J$.

We set

$$\{x_{ij}(t)\} = (x_{11}(t), \ldots, x_{1n}(t), \ldots, x_{i1}(t), \ldots, x_{in}(t), \ldots, x_{m1}(t), \ldots, x_{mn}(t)) \in R^{m \times n}.$$

For $\forall x(t) = \{x_{ij}(t)\} \in R^{m \times n}$, we define the norm $\|x(t)\| = \max_{(i,j)}\{|x_{ij}(t)|\}$. For the convenience, we shall introduce the notations

$$h^+ = \sup_{t \in R}|h(t)|,$$

where $h(t)$ is a bounded continuous function.

We also make the following assumptions.

(T$_1$) There exist constants M_f, M_g, μ_f and μ_g such that

$$|f(u) - f(v)| \le \mu_f|u - v|, \qquad |f(u)| \le M_f,$$

$$|g(u) - g(v)| \le \mu_g|u - v|, \qquad |g(u)| \le M_g, \quad \text{for all } u, v \in R.$$

(T$_2$) For $ij \in J$, $a_{ij}^+ \eta_{ij}^+ < 1$,

$$\begin{aligned}
\delta_{ij}(t) = \Bigg\{ & a_{ij}(t)\big(1 - 2a_{ij}^+\eta_{ij}^+\big) - \big|a_{ij}(t) - \big(1 - \eta_{ij}'(t)\big)a_{ij}\big(t - \eta_{ij}(t)\big)\big| \\
& - \sum_{C_{kl} \in N_r(i,j)} \big|C_{ij}^{kl}(t)\big|M_f \\
& - \sum_{C_{kl} \in N_q(i,j)} \big|B_{ij}^{kl}(t)\big| \int_0^\infty \big|K_{ij}(u)\big|\,du\,M_g \Bigg\} \frac{1}{1 - a_{ij}^+\eta_{ij}^+} \\
& > 0,
\end{aligned} \tag{1.4}$$

and there exist positive constants $\eta > 0$ and λ such that

$$\lambda < a_{ij}(t), \qquad \int_0^\infty e^{\lambda u}\big|K_{ij}(u)\big|\,du < +\infty,$$

and

$$\begin{aligned}
-\Big[& \big(a_{ij}(t) - \lambda\big)\big(1 - 2a_{ij}^+\eta_{ij}^+\big) \\
& - \big|a_{ij}(t)e^{\lambda\eta_{ij}(t)} - \big(1 - \eta_{ij}'(t)\big)a_{ij}\big(t - \eta_{ij}(t)\big)\big|\Big]\frac{1}{1 - a_{ij}^+\eta_{ij}^+} \\
& + \sum_{C_{kl} \in N_r(i,j)} \big|C_{ij}^{kl}(t)\big|\mu_f e^{\lambda\tau(t)}\frac{1}{1 - a_{kl}^+\eta_{kl}^+}\frac{1}{1 - a_{ij}^+\eta_{ij}^+}\left(\frac{L_{ij}}{\delta_{ij}}\right)^+ \\
& + \sum_{C_{kl} \in N_r(i,j)} \big|C_{ij}^{kl}(t)\big|M_f\frac{1}{1 - a_{ij}^+\eta_{ij}^+} \\
& + \sum_{C_{kl} \in N_q(i,j)} \big|B_{ij}^{kl}(t)\big|\int_0^\infty e^{\lambda u}\big|K_{ij}(u)\big|\,du\,\mu_g\frac{1}{1 - a_{kl}^+\eta_{kl}^+}\frac{1}{1 - a_{ij}^+\eta_{ij}^+}\left(\frac{L_{ij}}{\delta_{ij}}\right)^+ \\
& + \sum_{C_{kl} \in N_q(i,j)} \big|B_{ij}^{kl}(t)\big|\int_0^\infty \big|K_{ij}(u)\big|\,du\,M_g\frac{1}{1 - a_{ij}^+\eta_{ij}^+} \\
& < -\eta, \quad \text{for all } t \ge 0.
\end{aligned} \tag{1.5}$$

The initial conditions associated with system (1.2) are of the form

$$x_{ij}(s) = \varphi_{ij}(s), \quad s \in (-\infty, 0], ij \in J, \tag{1.6}$$

where $\varphi_{ij}(\cdot)$ denotes a real-valued bounded continuous function defined on $(-\infty, 0]$.

Definition 1.1 (See [20, 21]) Let $u(t) : R \longrightarrow R^{m \times n}$ be continuous in t. $u(t)$ is said to be almost periodic on R if for any $\varepsilon > 0$, the set $T(u, \varepsilon) = \{\delta : \|u(t + \delta) - u(t)\| < \varepsilon, \forall t \in R\}$ is relatively dense, *i.e.*, for any $\varepsilon > 0$, it is possible to find a real number $l = l(\varepsilon) > 0$, for any interval with length $l(\varepsilon)$, there exists a number $\delta = \delta(\varepsilon)$ in this interval such that $\|u(t + \delta) - u(t)\| < \varepsilon$ for all $t \in R$.

2 Preliminary results

The following lemmas will be useful to prove our main results in Section 3.

Lemma 2.1 *Let* (T_1) *and* (T_2) *hold. Suppose that* $x(t) = \{x_{ij}(t)\}$ *is a solution of system* (1.2) *with initial conditions*

$$x_{ij}(s) = \varphi_{ij}(s), \quad \left| \varphi_{ij}(t) - \int_{t-\eta_{ij}(t)}^{t} a_{ij}(s)\varphi_{ij}(s)\,ds \right| < \left(\frac{L_{ij}}{\delta_{ij}} \right)^{+}, \quad s, t \in (-\infty, 0], ij \in J. \tag{2.1}$$

Then

$$\left| x_{ij}(t) - \int_{t-\eta_{ij}(t)}^{t} a_{ij}(s)x_{ij}(s)\,ds \right| \leq \left(\frac{L_{ij}}{\delta_{ij}} \right)^{+}, \quad \text{for all } t \geq 0, ij \in J, \tag{2.2}$$

and

$$\left| x_{ij}(t) \right| \leq \frac{1}{1 - a_{ij}^{+}\eta_{ij}^{+}} \left(\frac{L_{ij}}{\delta_{ij}} \right)^{+}, \quad \text{for all } t \geq 0, ij \in J. \tag{2.3}$$

Proof Assume, by way of contradiction, that (2.2) does not hold. Then, there exist $ij \in \Lambda$, $\gamma > (\frac{L_{ij}}{\delta_{ij}})^{+}$ and $t_* > 0$ such that

$$\left| X_{ij}(t_*) \right| = \gamma \quad \text{and} \quad \left| X_{ij}(t) \right| < \gamma \quad \text{for all } t \in (-\infty, t_*), \tag{2.4}$$

where

$$X_{ij}(t) = x_{ij}(t) - \int_{t-\eta_{ij}(t)}^{t} a_{ij}(s)x_{ij}(s)\,ds.$$

It follows that

$$
\begin{aligned}
\left| x_{ij}(t) \right| &\leq \left| x_{ij}(t) - \int_{t-\eta_{ij}(t)}^{t} a_{ij}(s)x_{ij}(s)\,ds \right| + \left| \int_{t-\eta_{ij}(t)}^{t} a_{ij}(s)x_{ij}(s)\,ds \right| \\
&\leq \gamma + a_{ij}^{+}\eta_{ij}^{+} \sup_{s \in (-\infty, t_*]} \left| x_{ij}(s) \right|, \quad \text{for all } t \in (-\infty, t_*].
\end{aligned}
\tag{2.5}
$$

Consequently, in view of (2.5) and the fact $a_{ij}^+ \eta_{ij}^+ < 1$ ($ij \in J$), we have

$$
\begin{aligned}
\left| x_{ij}(t) \right| &\leq \sup_{s \in (-\infty, t_*]} \left| x_{ij}(s) \right| \\
&\leq \frac{1}{1 - a_{ij}^+ \eta_{ij}^+} \gamma, \quad \text{for all } t \in (-\infty, t_*].
\end{aligned}
\tag{2.6}
$$

From system (1.2), we derive

$$
\begin{aligned}
\frac{d}{dt} & \left(x_{ij}(t) - \int_{t-\eta_{ij}(t)}^{t} a_{ij}(s) x_{ij}(s)\, ds \right) \\
&= x_{ij}'(t) - \left[a_{ij}(t) x_{ij}(t) - \left(1 - \eta_{ij}'(t)\right) a_{ij}\left(t - \eta_{ij}(t)\right) x_{ij}\left(t - \eta_{ij}(t)\right) \right] \\
&= -\left[a_{ij}(t) x_{ij}(t) - \left(1 - \eta_{ij}'(t)\right) a_{ij}\left(t - \eta_{ij}(t)\right) x_{ij}\left(t - \eta_{ij}(t)\right) \right] \\
&\quad + \left[-a_{ij}(t) x_i\left(t - \eta_{ij}(t)\right) - \sum_{C_{kl} \in N_r(i,j)} C_{ij}^{kl}(t) f\left(x_{kl}\left(t - \tau(t)\right)\right) x_{ij}(t) \right. \\
&\quad \left. - \sum_{C_{kl} \in N_q(i,j)} B_{ij}^{kl}(t) \int_0^\infty K_{ij}(u) g\left(x_{kl}(t-u)\right) du\, x_{ij}(t) + L_{ij}(t) \right] \\
&= -a_{ij}(t) x_{ij}(t) - \left[a_{ij}(t) - \left(1 - \eta_{ij}'(t)\right) a_{ij}\left(t - \eta_{ij}(t)\right) \right] x_{ij}\left(t - \eta_{ij}(t)\right) \\
&\quad - \sum_{C_{kl} \in N_r(i,j)} C_{ij}^{kl}(t) f\left(x_{kl}\left(t - \tau(t)\right)\right) x_{ij}(t) \\
&\quad - \sum_{C_{kl} \in N_q(i,j)} B_{ij}^{kl}(t) \int_0^\infty K_{ij}(u) g\left(x_{kl}(t-u)\right) du\, x_{ij}(t) + L_{ij}(t) \\
&= -a_{ij}(t) \left(x_{ij}(t) - \int_{t-\eta_{ij}(t)}^{t} a_{ij}(s) x_{ij}(s)\, ds \right) - a_{ij}(t) \int_{t-\eta_{ij}(t)}^{t} a_{ij}(s) x_{ij}(s)\, ds \\
&\quad - \left[a_{ij}(t) - \left(1 - \eta_{ij}'(t)\right) a_{ij}\left(t - \eta_{ij}(t)\right) \right] x_{ij}\left(t - \eta_i(t)\right) \\
&\quad - \sum_{C_{kl} \in N_r(i,j)} C_{ij}^{kl}(t) f\left(x_{kl}\left(t - \tau(t)\right)\right) x_{ij}(t) \\
&\quad - \sum_{C_{kl} \in N_q(i,j)} B_{ij}^{kl}(t) \int_0^\infty K_{ij}(u) g\left(x_{kl}(t-u)\right) du\, x_{ij}(t) + L_{ij}(t), \quad ij \in J.
\end{aligned}
\tag{2.7}
$$

Calculating the upper left derivative of $|X_{ij}(t)|$, together with (2.4), (2.6), (2.7), (T_1) and (T_2), we obtain

$$
\begin{aligned}
0 \leq D^- & \left| X_{ij}(t_*) \right| \\
\leq & -a_{ij}(t_*) \left| X_{ij}(t_*) \right| + \left| -a_{ij}(t_*) \int_{t_*-\eta_{ij}(t_*)}^{t_*} a_{ij}(s) x_{ij}(s)\, ds \right. \\
& - \left[a_{ij}(t_*) - \left(1 - \eta_{ij}'(t_*)\right) a_{ij}\left(t_* - \eta_{ij}(t_*)\right) \right] x_{ij}\left(t_* - \eta_{ij}(t_*)\right) \\
& - \sum_{C_{kl} \in N_r(i,j)} C_{ij}^{kl}(t_*) f\left(x_{kl}\left(t_* - \tau(t_*)\right)\right) x_{ij}(t_*) \\
& - \left. \sum_{C_{kl} \in N_q(i,j)} B_{ij}^{kl}(t_*) \int_0^\infty K_{ij}(u) g\left(x_{kl}(t_*-u)\right) du\, x_{ij}(t_*) + L_{ij}(t_*) \right|
\end{aligned}
$$

$$\leq -a_{ij}(t_*)\big|X_{ij}(t_*)\big| + a_{ij}(t_*)\int_{t_*-\eta_{ij}(t_*)}^{t_*} a_{ij}^+\big|x_{ij}(s)\big|\,ds$$

$$+ \big|a_{ij}(t_*) - \big(1-\eta_{ij}'(t_*)\big)a_{ij}\big(t_*-\eta_{ij}(t_*)\big)\big|\big|x_{ij}\big(t_*-\eta_{ij}(t_*)\big)\big|$$

$$+ \sum_{C_{kl}\in N_r(i,j)} \big|C_{ij}^{kl}(t_*)\big|\big|f\big(x_{kl}\big(t_*-\tau(t_*)\big)\big)\big|\big|x_{ij}(t_*)\big|$$

$$+ \sum_{C_{kl}\in N_q(i,j)} \big|B_{ij}^{kl}(t_*)\big|\int_0^\infty \big|K_{ij}(u)\big|\big|g\big(x_{kl}(t_*-u)\big)\big|\,du\big|x_{ij}(t_*)\big| + \big|L_{ij}(t_*)\big|$$

$$\leq -a_{ij}(t_*)\big|X_{ij}(t_*)\big| + a_{ij}(t_*)\int_{t_*-\eta_{ij}(t_*)}^{t_*} a_{ij}^+\big|x_{ij}(s)\big|\,ds$$

$$+ \big|a_{ij}(t_*) - \big(1-\eta_{ij}'(t_*)\big)a_{ij}\big(t_*-\eta_{ij}(t_*)\big)\big|\big|x_{ij}\big(t_*-\eta_{ij}(t_*)\big)\big|$$

$$+ \sum_{C_{kl}\in N_r(i,j)} \big|C_{ij}^{kl}(t_*)\big|M_f\big|x_{ij}(t_*)\big|$$

$$+ \sum_{C_{kl}\in N_q(i,j)} \big|B_{ij}^{kl}(t_*)\big|\int_0^\infty \big|K_{ij}(u)\big|\,du M_g\big|x_{ij}(t_*)\big| + \big|L_{ij}(t_*)\big|$$

$$\leq -a_{ij}(t_*)\gamma + a_{ij}(t_*)a_{ij}^+\eta_{ij}^+\frac{1}{1-a_{ij}^+\eta_{ij}^+}\gamma$$

$$+ \big|a_{ij}(t_*) - \big(1-\eta_{ij}'(t_*)\big)a_{ij}\big(t_*-\eta_{ij}(t_*)\big)\big|\frac{\gamma}{1-a_{ij}^+\eta_{ij}^+}$$

$$+ \sum_{C_{kl}\in N_r(i,j)} \big|C_{ij}^{kl}(t_*)\big|M_f\frac{1}{1-a_{ij}^+\eta_{ij}^+}\gamma$$

$$+ \sum_{C_{kl}\in N_q(i,j)} \big|B_{ij}^{kl}(t_*)\big|\int_0^\infty \big|K_{ij}(u)\big|\,du M_g\frac{1}{1-a_{ij}^+\eta_{ij}^+}\gamma + \big|L_{ij}(t_*)\big|$$

$$= \bigg\{-\big[a_{ij}(t_*)\big(1-2a_{ij}^+\eta_{ij}^+\big) - \big|a_{ij}(t_*) - \big(1-\eta_{ij}'(t_*)\big)a_{ij}\big(t_*-\eta_{ij}(t_*)\big)\big|\big]$$

$$+ \sum_{C_{kl}\in N_r(i,j)} \big|C_{ij}^{kl}(t_*)\big|M_f$$

$$+ \sum_{C_{kl}\in N_q(i,j)} \big|B_{ij}^{kl}(t_*)\big|\int_0^\infty \big|K_{ij}(u)\big|\,du M_g\bigg\}\frac{1}{1-a_{ij}^+\eta_{ij}^+}\gamma + \big|L_{ij}(t_*)\big|$$

$$\leq -\delta_{ij}(t_*)\bigg[\gamma - \bigg(\frac{L_{ij}}{\delta_{ij}}\bigg)^+\bigg]$$

$$< 0.$$

It is a contradiction, and it shows that (2.2) holds. Then, using a similar argument as in the proof of (2.5) and (2.6), we can show that (2.3) holds. The proof of Lemma 2.1 is now completed. □

Remark 2.1 In view of the boundedness of this solution, from the theory of functional differential equations with infinite delay in [22], it follows that the solution of system (1.2) with initial conditions (2.1) can be defined on $[0, +\infty)$.

Lemma 2.2 *Suppose that* (T_1) *and* (T_2) *hold. Moreover, assume that* $x(t) = \{x_{ij}(t)\}$ *is a solution of system* (1.2) *with initial function* $\varphi_{ij}(\cdot)$ *satisfying* (2.1), *and* $\varphi'_{ij}(\cdot)$ *is bounded continuous on* $(-\infty, 0]$. *Then for any* $\epsilon > 0$, *there exists* $l = l(\epsilon) > 0$, *such that every interval* $[\alpha, \alpha + l]$ *contains at least one number* δ, *for which there exists* $N > 0$ *satisfying*

$$\|x(t + \delta) - x(t)\| \leq \epsilon, \quad \text{for all } t > N. \tag{2.8}$$

Proof For $ij \in J$, set

$$
\begin{aligned}
\epsilon_{ij}(\delta, t) = &-a_{ij}(t)\big[x_{ij}\big(t + \delta - \eta_{ij}(t + \delta)\big) - x_{ij}\big(t - \eta_{ij}(t) + \delta\big)\big] \\
&- \big[a_{ij}(t + \delta) - a_{ij}(t)\big]x_{ij}\big(t + \delta - \eta_{ij}(t + \delta)\big) \\
&- \sum_{C_{kl} \in N_r(i,j)} \big[C_{ij}^{kl}(t + \delta) - C_{ij}^{kl}(t)\big]f\big(x_{kl}\big(t - \tau(t + \delta) + \delta\big)\big)x_{ij}(t + \delta) \\
&- \sum_{C_{kl} \in N_r(i,j)} C_{ij}^{kl}(t)\big[f\big(x_{kl}\big(t - \tau(t + \delta) + \delta\big)\big) - f\big(x_{kl}\big(t - \tau(t) + \delta\big)\big)\big]x_{ij}(t + \delta) \\
&- \sum_{C_{kl} \in N_q(i,j)} \big[B_{ij}^{kl}(t + \delta) - B_{ij}^{kl}(t)\big]\int_0^\infty K_{ij}(u)g\big(x_{kl}(t + \delta - u)\big)\,du\,x_{ij}(t + \delta) \\
&+ \big[L_{ij}(t + \delta) - L_{ij}(t)\big].
\end{aligned}
$$

By Lemma 2.1, the solution $x(t) = \{x_{ij}(t)\}$ is bounded and

$$|x_{ij}(t)| \leq \frac{1}{1 - a_{ij}^+ \eta_{ij}^+}\left(\frac{L_{ij}}{\delta_{ij}}\right)^+ \quad \text{for all } t \in [0, +\infty), ij \in J. \tag{2.9}$$

Thus, the right side of (1.2) is also bounded, which implies that $x(t)$ is uniformly continuous on R. From (1.3), for any $\epsilon > 0$, there exists $l = l(\epsilon) > 0$, such that every interval $[\alpha, \alpha + l]$, $\alpha \in R$, contains a δ, for which

$$|\epsilon_{ij}(\delta, t)| \leq \frac{1}{2}\eta\min_{ij \in J}\{1 - a_{ij}^+ \eta_{ij}^+\}\epsilon, \quad \text{where } ij \in J, t \in R. \tag{2.10}$$

Let $N_0 \geq 0$ be sufficiently large such that $t + \delta \geq 0$ for $t \geq N_0$, and denote $u_{ij}(t) = x_{ij}(t + \delta) - x_{ij}(t)$. We obtain

$$
\begin{aligned}
\frac{du_{ij}(t)}{dt} = &-a_{ij}(t)u_{ij}\big(t - \eta_{ij}(t)\big) - \sum_{C_{kl} \in N_r(i,j)} C_{ij}^{kl}(t)\big(f\big(x_{kl}\big(t - \tau(t) + \delta\big)\big) \\
&- f\big(x_{kl}\big(t - \tau(t)\big)\big)\big)x_{ij}(t + \delta) - \sum_{C_{kl} \in N_r(i,j)} C_{ij}^{kl}(t)f\big(x_{kl}\big(t - \tau(t)\big)\big) \\
&\cdot \big(x_{ij}(t + \delta) - x_{ij}(t)\big) - \sum_{C_{kl} \in N_q(i,j)} B_{ij}^{kl}(t)\int_0^\infty K_{ij}(u)\big(g\big(x_{kl}(t + \delta - u)\big) \\
&- g\big(x_{kl}(t - u)\big)\big)\,du\,x_{ij}(t + \delta) - \sum_{C_{kl} \in N_q(i,j)} B_{ij}^{kl}(t)\int_0^\infty K_{ij}(u)g\big(x_{kl}(t - u)\big)\,du \\
&\cdot \big(x_{ij}(t + \delta) - x_{ij}(t)\big) + \epsilon_{ij}(\delta, t), \quad \text{for all } t \geq N_0, ij \in J,
\end{aligned}
$$

which yields

$$\frac{d}{dt}\left(e^{\lambda t}u_{ij}(t) - \int_{t-\eta_{ij}(t)}^{t} a_{ij}(s)e^{\lambda s}u_{ij}(s)\,ds\right)$$

$$= \lambda e^{\lambda t}u_{ij}(t) + e^{\lambda t}u_{ij}'(t)$$

$$\quad - \left[a_{ij}(t)e^{\lambda t}u_{ij}(t) - \left(1 - \eta_{ij}'(t)\right)c_{ij}\left(t - \eta_{ij}(t)\right)e^{\lambda(t-\eta_{ij}(t))}u_{ij}\left(t - \eta_{ij}(t)\right)\right]$$

$$= \lambda e^{\lambda t}u_{ij}(t) - \left[a_{ij}(t)e^{\lambda t}u_{ij}(t) - \left(1 - \eta_{ij}'(t)\right)a_{ij}\left(t - \eta_{ij}(t)\right)e^{\lambda(t-\eta_{ij}(t))}u_{ij}\left(t - \eta_{ij}(t)\right)\right]$$

$$\quad + e^{\lambda t}\left\{-a_{ij}(t)u_{ij}\left(t - \eta_{ij}(t)\right) - \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}(t)\left(f\left(x_{kl}\left(t - \tau(t) + \delta\right)\right)\right.\right.$$

$$\quad \left. - f\left(x_{kl}\left(t - \tau(t)\right)\right)\right)x_{ij}(t + \delta) - \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}(t)f\left(x_{kl}\left(t - \tau(t)\right)\right)$$

$$\quad \cdot \left(x_{ij}(t + \delta) - x_{ij}(t)\right) - \sum_{C_{kl}\in N_q(i,j)} B_{ij}^{kl}(t)\int_0^\infty K_{ij}(u)\left(g\left(x_{kl}(t + \delta - u)\right)\right.$$

$$\quad \left. - g\left(x_{kl}(t - u)\right)\right)du\, x_{ij}(t + \delta) - \sum_{C_{kl}\in N_q(i,j)} B_{ij}^{kl}(t)\int_0^\infty K_{ij}(u)g\left(x_{kl}(t - u)\right)du$$

$$\quad \cdot \left(x_{ij}(t + \delta) - x_{ij}(t)\right) + \epsilon_{ij}(\delta, t)\right\}$$

$$= -\left(a_{ij}(t) - \lambda\right)e^{\lambda t}u_{ij}(t)$$

$$\quad - \left[a_{ij}(t) - \left(1 - \eta_{ij}'(t)\right)a_{ij}\left(t - \eta_{ij}(t)\right)e^{-\lambda\eta_{ij}(t)}\right]e^{\lambda t}u_{ij}\left(t - \eta_{ij}(t)\right)$$

$$\quad + e^{\lambda t}\left\{-\sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}(t)\left(f\left(x_{kl}\left(t - \tau(t) + \delta\right)\right)\right.\right.$$

$$\quad \left. - f\left(x_{kl}\left(t - \tau(t)\right)\right)\right)x_{ij}(t + \delta) - \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}(t)f\left(x_{kl}\left(t - \tau(t)\right)\right)$$

$$\quad \cdot \left(x_{ij}(t + \delta) - x_{ij}(t)\right) - \sum_{C_{kl}\in N_q(i,j)} B_{ij}^{kl}(t)\int_0^\infty K_{ij}(u)\left(g\left(x_{kl}(t + \delta - u)\right)\right.$$

$$\quad \left. - g\left(x_{kl}(t - u)\right)\right)du\, x_{ij}(t + \delta) - \sum_{C_{kl}\in N_q(i,j)} B_{ij}^{kl}(t)\int_0^\infty K_{ij}(u)g\left(x_{kl}(t - u)\right)du$$

$$\quad \cdot \left(x_{ij}(t + \delta) - x_{ij}(t)\right) + \epsilon_{ij}(\delta, t)\right\}$$

$$= -\left(a_{ij}(t) - \lambda\right)\left(e^{\lambda t}a_{ij}(t) - \int_{t-\eta_{ij}(t)}^{t} a_{ij}(s)e^{\lambda s}u_{ij}(s)\,ds\right)$$

$$\quad - \left(a_{ij}(t) - \lambda\right)\int_{t-\eta_{ij}(t)}^{t} a_{ij}(s)e^{\lambda s}u_{ij}(s)\,ds$$

$$\quad - \left[a_{ij}(t) - \left(1 - \eta_{ij}'(t)\right)a_{ij}\left(t - \eta_{ij}(t)\right)e^{-\lambda\eta_{ij}(t)}\right]e^{\lambda t}u_{ij}\left(t - \eta_{ij}(t)\right)$$

$$\quad + e^{\lambda t}\left\{-\sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}(t)\left(f\left(x_{kl}\left(t - \tau(t) + \delta\right)\right)\right.\right.$$

$$\quad \left. - f\left(x_{kl}\left(t - \tau(t)\right)\right)\right)x_{ij}(t + \delta) - \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}(t)f\left(x_{kl}\left(t - \tau(t)\right)\right)$$

$$\cdot \left(x_{ij}(t+\delta) - x_{ij}(t)\right) - \sum_{C_{kl} \in N_q(i,j)} B_{ij}^{kl}(t) \int_0^\infty K_{ij}(u) \left(g\left(x_{kl}(t+\delta - u)\right)\right.$$

$$- g\left(x_{kl}(t-u)\right)\right) du \, x_{ij}(t+\delta) - \sum_{C_{kl} \in N_q(i,j)} B_{ij}^{kl}(t) \int_0^\infty K_{ij}(u) g\left(x_{kl}(t-u)\right) du$$

$$\cdot \left(x_{ij}(t+\delta) - x_{ij}(t)\right) + \epsilon_{ij}(\delta,t)\bigg\}, \quad \text{for all } t \geq N_0, ij \in J. \tag{2.11}$$

Set

$$U(t) = \left\{U_{ij}(t)\right\},$$

where

$$U_{ij}(t) = e^{\lambda t} u_{ij}(t) - \int_{t-\eta_{ij}(t)}^t a_{ij}(s) e^{\lambda s} u_{ij}(s) \, ds, \quad ij \in J.$$

Let $(ij)_t$ be such an index that

$$\left|U_{(ij)_t}(t)\right| = \left\|U(t)\right\|. \tag{2.12}$$

Calculating the upper left derivative of $|U_{(ij)_s}(s)|$ along (2.11), we have

$$D^-\left(\left|U_{(ij)_s}(s)\right|\right)\big|_{s=t}$$

$$\leq -\left(a_{(ij)_t}(t) - \lambda\right)\left|U_{(ij)_t}(t)\right| + \left| -\left(a_{(ij)_t}(t) - \lambda\right) \int_{t-\eta_{(ij)_t}(t)}^t a_{(ij)_t}(s) e^{\lambda s} u_{(ij)_t}(s) \, ds\right.$$

$$- \left[a_{(ij)_t}(t) - \left(1 - \eta'_{(ij)_t}(t)\right) a_{(ij)_t}\left(t - \eta_{(ij)_t}(t)\right) e^{-\lambda \eta_{(ij)_t}(t)}\right] e^{\lambda t} u_{(ij)_t}\left(t - \eta_{(ij)_t}(t)\right)$$

$$+ e^{\lambda t}\bigg\{ -\sum_{C_{kl} \in N_r(i,j)_t} C_{(ij)_t}^{kl}(t) \left(f\left(x_{kl}\left(t - \tau(t) + \delta\right)\right)\right.$$

$$- f\left(x_{kl}\left(t - \tau(t)\right)\right)\right) x_{(ij)_t}(t+\delta) - \sum_{C_{kl} \in N_r(i,j)_t} C_{(ij)_t}^{kl}(t) f\left(x_{kl}\left(t - \tau(t)\right)\right)$$

$$\cdot \left(x_{(ij)_t}(t+\delta) - x_{(ij)_t}(t)\right) - \sum_{C_{kl} \in N_q(i,j)_t} B_{(ij)_t}^{kl}(t) \int_0^\infty K_{(ij)_t}(u) \left(g\left(x_{kl}(t+\delta - u)\right)\right.$$

$$- g\left(x_{kl}(t-u)\right)\right) du \, x_{(ij)_t}(t+\delta) - \sum_{C_{kl} \in N_q(i,j)_t} B_{(ij)_t}^{kl}(t) \int_0^\infty K_{(ij)_t}(u) g\left(x_{kl}(t-u)\right) du$$

$$\cdot \left(x_{(ij)_t}(t+\delta) - x_{(ij)_t}(t)\right) + \epsilon_{(ij)_t}(\delta,t)\bigg\}\bigg|. \tag{2.13}$$

Let

$$M(t) = \sup_{s \leq t}\left\{\left\|U(s)\right\|\right\}. \tag{2.14}$$

It is obvious that $\|U(t)\| \leq M(t)$, and $M(t)$ is non-decreasing. In particular,

$$
\begin{aligned}
e^{\lambda\rho}\left|u_{ij}(\rho)\right| &\leq \left|e^{\lambda\rho}u_{ij}(\rho) - \int_{\rho-\eta_{ij}(\rho)}^{\rho} a_{ij}(s)e^{\lambda s}u_{ij}(s)\,ds\right| + \left|\int_{\rho-\eta_{ij}(\rho)}^{\rho} a_{ij}(s)e^{\lambda s}u_{ij}(s)\,ds\right| \\
&\leq M(t) + a_{ij}^{+}\eta_{ij}^{+} \sup_{\theta\in(-\infty,t]} e^{\lambda\theta}\left|u_{ij}(\theta)\right|, \quad \text{for all } t \geq \rho, ij \in J.
\end{aligned}
\tag{2.15}
$$

Consequently, in view of (2.15) and the fact $a_{ij}^{+}\eta_{ij}^{+} < 1$ ($ij \in J$), we have

$$
e^{\lambda s}\left|u_{ij}(s)\right| \leq \sup_{\theta\in(-\infty,t]} e^{\lambda\theta}\left|u_{ij}(\theta)\right| \leq \frac{M(t)}{1 - a_{ij}^{+}\eta_{ij}^{+}}, \quad \text{where } s \in (-\infty,t], ij \in J.
\tag{2.16}
$$

Now, we consider two cases.
Case (i). If

$$
M(t) > \|U(t)\| \quad \text{for all } t \geq N_0.
\tag{2.17}
$$

Then, we claim that

$$
M(t) \equiv M(N_0) \text{ is a constant for all } t \geq N_0.
\tag{2.18}
$$

Assume, by way of contradiction, that (2.18) does not hold. Then, there exists $t_1 > N_0$ such that $M(t_1) > M(N_0)$. Since

$$
\|U(t)\| \leq M(N_0) \quad \text{for all } t \leq N_0.
$$

There must exist $\beta \in (N_0, t_1)$ such that

$$
\|U(\beta)\| = M(t_1) \geq M(\beta),
$$

which contradicts (2.17). This contradiction implies that (2.18) holds. It follows from (2.16) that there exists $t_2 > N_0$ such that

$$
\|u(t)\| = \max_{ij\in J}\left|u_{ij}(t)\right| \leq \max_{ij\in J}\frac{e^{-\lambda t}M(t)}{1 - a_{ij}^{+}\eta_{ij}^{+}} = \max_{ij\in J}\frac{e^{-\lambda t}M(N_0)}{1 - a_{ij}^{+}\eta_{ij}^{+}} < \epsilon, \quad \text{for all } t \geq t_2.
\tag{2.19}
$$

Case (ii). If there is such a point $t_0 \geq N_0$ that $M(t_0) = \|U(t_0)\|$. Then, in view of (1.5), (2.9), (2.10), (2.13), (2.16), (T$_1$) and (T$_2$), we get

$$
\begin{aligned}
0 \leq\ & D^{-}\left(\left|U_{(ij)_s}(s)\right|\right)\big|_{s=t_0} \\
\leq\ & -\left(a_{(ij)_{t_0}}(t_0) - \lambda\right)\left|U_{(ij)_{t_0}}(t_0)\right| + \left(a_{(ij)_{t_0}}(t_0) - \lambda\right)\int_{t_0-\eta_{(ij)_{t_0}}(t_0)}^{t_0} a_{(ij)_{t_0}}^{+}e^{\lambda s}\left|u_{(ij)_{t_0}}(s)\right|ds \\
& + \left|a_{(ij)_{t_0}}(t_0)e^{\lambda\eta_{(ij)_{t_0}}(t_0)} - \left(1 - \eta_{(ij)_{t_0}}'(t)\right)a_{(ij)_{t_0}}\left(t_0 - \eta_{(ij)_{t_0}}(t_0)\right)\right|e^{\lambda(t_0-\eta_{(ij)_{t_0}}(t_0))} \\
& \cdot \left|u_{(ij)_{t_0}}\left(t_0 - \eta_{(ij)_{t_0}}(t_0)\right)\right| + \sum_{C_{kl}\in N_r(i,j)_{t_0}}\left|C_{(ij)_{t_0}}^{kl}(t_0)\right|e^{\lambda t_0}\left|f\left(x_{kl}\left(t_0 - \tau(t_0) + \delta\right)\right)\right|
\end{aligned}
$$

$$- f\big(x_{kl}(t_0 - \tau(t_0))\big)\big|\big|x_{(ij)_{t_0}}(t_0 + \delta)\big| + \sum_{C_{kl} \in N_r(i,j)_{t_0}} \big|C_{(ij)_{t_0}}^{kl}(t_0)\big|\big|f\big(x_{kl}(t_0 - \tau(t_0))\big)\big|$$

$$\cdot e^{\lambda t_0}\big|x_{(ij)_{t_0}}(t_0 + \delta) - x_{(ij)_{t_0}}(t_0)\big|$$

$$+ \sum_{C_{kl} \in N_q(i,j)_{t_0}} \big|B_{(ij)_{t_0}}^{kl}(t_0)\big| \int_0^\infty \big|K_{(ij)_{t_0}}(u)\big| e^{\lambda t_0}\big|g\big(x_{kl}(t_0 + \delta - u)\big)$$

$$- g\big(x_{kl}(t_0 - u)\big)\big|\, du \big|x_{(ij)_{t_0}}(t_0 + \delta)\big|$$

$$+ \sum_{C_{kl} \in N_q(i,j)_{t_0}} \big|B_{(ij)_{t_0}}^{kl}(t_0)\big| \int_0^\infty \big|K_{(ij)_{t_0}}(u)\big|\big|g\big(x_{kl}(t_0 - u)\big)\big|\, du$$

$$\cdot e^{\lambda t_0}\big|x_{(ij)_{t_0}}(t_0 + \delta) - x_{(ij)_{t_0}}(t_0)\big| + \big|\epsilon_{(ij)_{t_0}}(\delta, t_0)\big|e^{\lambda t_0}$$

$$\leq -\big(a_{(ij)_{t_0}}(t_0) - \lambda\big)M(t_0) + \big(a_{(ij)_{t_0}}(t_0) - \lambda\big)a_{(ij)_{t_0}}^+ \eta_{(ij)_{t_0}}^+ \frac{M(t_0)}{1 - a_{(ij)_{t_0}}^+ \eta_{(ij)_{t_0}}^+}$$

$$+ \big|a_{(ij)_{t_0}}(t_0)e^{\lambda \eta_{(ij)_{t_0}}(t_0)} - \big(1 - \eta_{(ij)_{t_0}}'(t)\big)a_{(ij)_{t_0}}\big(t_0 - \eta_{(ij)_{t_0}}(t_0)\big)\big|$$

$$\cdot \frac{M(t_0)}{1 - a_{(ij)_{t_0}}^+ \eta_{(ij)_{t_0}}^+} + \sum_{C_{kl} \in N_r(i,j)_{t_0}} \big|C_{(ij)_{t_0}}^{kl}(t_0)\big|\mu_f e^{\lambda \tau(t_0)} e^{\lambda(t_0 - \tau(t_0))}\big|u_{kl}\big(t_0 - \tau(t_0)\big)\big|$$

$$\cdot \frac{1}{1 - a_{(ij)_{t_0}}^+ \eta_{(ij)_{t_0}}^+}\left(\frac{L_{(ij)_{t_0}}}{\delta_{(ij)_{t_0}}}\right)^+ + \sum_{C_{kl} \in N_r(i,j)_{t_0}} \big|C_{(ij)_{t_0}}^{kl}(t_0)\big|M_f e^{\lambda t_0}\big|u_{(ij)_{t_0}}(t_0)\big|$$

$$+ \sum_{C_{kl} \in N_q(i,j)_{t_0}} \big|B_{(ij)_{t_0}}^{kl}(t_0)\big| \int_0^\infty e^{\lambda u}\big|K_{(ij)_{t_0}}(u)\big|\mu_g e^{\lambda(t_0 - u)}\big|u_{kl}(t_0 - u)\big|\, du$$

$$\cdot \frac{1}{1 - a_{(ij)_{t_0}}^+ \eta_{(ij)_{t_0}}^+}\left(\frac{L_{(ij)_{t_0}}}{\delta_{(ij)_{t_0}}}\right)^+$$

$$+ \sum_{C_{kl} \in N_q(i,j)_{t_0}} \big|B_{(ij)_{t_0}}^{kl}(t_0)\big| \int_0^\infty \big|K_{(ij)_{t_0}}(u)\big|\, du\, M_g e^{\lambda t_0}\big|u_{(ij)_{t_0}}(t_0)\big| + \big|\epsilon_{(ij)_{t_0}}(\delta, t_0)\big|e^{\lambda t_0}$$

$$\leq \Bigg\{ -\Big[\big(a_{(ij)_{t_0}}(t_0) - \lambda\big)\big(1 - 2a_{(ij)_{t_0}}^+ \eta_{(ij)_{t_0}}^+\big) - \big|a_{(ij)_{t_0}}(t_0)e^{\lambda \eta_{(ij)_{t_0}}(t_0)}$$

$$- \big(1 - \eta_{(ij)_{t_0}}'(t)\big)a_{(ij)_{t_0}}\big(t_0 - \eta_{(ij)_{t_0}}(t_0)\big)\big|\Big]\frac{1}{1 - a_{(ij)_{t_0}}^+ \eta_{(ij)_{t_0}}^+}$$

$$+ \sum_{C_{kl} \in N_r(i,j)_{t_0}} \big|C_{(ij)_{t_0}}^{kl}(t_0)\big|\mu_f e^{\lambda \tau(t_0)}\frac{1}{1 - a_{kl}^+ \eta_{kl}^+}\frac{1}{1 - a_{(ij)_{t_0}}^+ \eta_{(ij)_{t_0}}^+}\left(\frac{L_{(ij)_{t_0}}}{\delta_{(ij)_{t_0}}}\right)^+$$

$$+ \sum_{C_{kl} \in N_r(i,j)_{t_0}} \big|C_{(ij)_{t_0}}^{kl}(t_0)\big|M_f \frac{1}{1 - a_{(ij)_{t_0}}^+ \eta_{(ij)_{t_0}}^+}$$

$$+ \sum_{C_{kl} \in N_q(i,j)_{t_0}} \big|B_{(ij)_{t_0}}^{kl}(t_0)\big| \int_0^\infty e^{\lambda u}\big|K_{(ij)_{t_0}}(u)\big|\, du$$

$$\cdot \mu_g \frac{1}{1 - a_{kl}^+ \eta_{kl}^+}\frac{1}{1 - a_{(ij)_{t_0}}^+ \eta_{(ij)_{t_0}}^+}\left(\frac{L_{(ij)_{t_0}}}{\delta_{(ij)_{t_0}}}\right)^+$$

$$+ \sum_{C_{kl} \in N_q(i,j)_{t_0}} \big|B_{(ij)_{t_0}}^{kl}(t_0)\big| \int_0^\infty \big|K_{(ij)_{t_0}}(u)\big|\, du\, M_g \frac{1}{1 - a_{(ij)_{t_0}}^+ \eta_{(ij)_{t_0}}^+}\Bigg\}M(t_0)$$

$$+ \eta \min_{ij \in J}\{1 - a_{ij}^+ \eta_{ij}^+\}\epsilon e^{\lambda t_0}$$

$$< -\eta M(t_0) + \eta \min_{ij \in J}\{1 - a_{ij}^+ \eta_{ij}^+\}\epsilon e^{\lambda t_0}, \tag{2.20}$$

which yields that

$$\|U(t_0)\| = M(t_0) < \min_{ij \in J}\{1 - a_{ij}^+ \eta_{ij}^+\}\epsilon e^{\lambda t_0}, \tag{2.21}$$

and

$$\|u(t_0)\| \leq \max_{ij \in J} \frac{e^{-\lambda t_0}M(t_0)}{1 - a_{ij}^+ \eta_{ij}^+} < \epsilon. \tag{2.22}$$

For any $t > t_0$, by the same approach used in the proof of (2.21) and (2.22), we have

$$\left.\begin{array}{l} \|U(t)\| = M(t) < \min_{ij \in J}\{1 - a_{ij}^+ \eta_{ij}^+\}\epsilon e^{\lambda t} \\ \|u(t)\| < \epsilon \end{array}\right\}, \quad \text{if } M(t) = \|U(t)\|. \tag{2.23}$$

On the other hand, if $M(t) > \|U(t)\|$ and $t > t_0$, we can choose $t_0 \leq t_3 < t$ such that

$$M(t_3) = \|U(t_3)\|, \quad \text{and} \quad M(s) > \|U(s)\| \quad \text{for all } s \in (t_3, t],$$

which, together with (2.23), yields that

$$M(t_3) = \|U(t_3)\| < \min_{ij \in J}\{1 - a_{ij}^+ \eta_{ij}^+\}\epsilon e^{\lambda t_3}, \quad \text{and} \quad \|u(t_3)\| < \epsilon.$$

Using a similar argument as in the proof of Case (i), we can show that

$$M(s) \equiv M(t_3) \text{ is a constant for all } s \in (t_3, t], \tag{2.24}$$

which implies that

$$\|u(t)\| \leq \max_{ij \in J} \frac{e^{-\lambda t}M(t)}{1 - a_{ij}^+ \eta_{ij}^+} = \max_{ij \in J} \frac{e^{-\lambda t}M(t_3)}{1 - a_{ij}^+ \eta_{ij}^+} < \max_{ij \in J} \frac{e^{-\lambda t}\min_{\bar{i}j \in J}\{1 - a_{\bar{i}j}^+ \eta_{\bar{i}j}^+\}\epsilon e^{\lambda t_3}}{1 - a_{ij}^+ \eta_{ij}^+} < \epsilon.$$

In summary, there must exist $N > \max\{t_0, N_0, t_2\}$ such that $\|u(t)\| \leq \epsilon$ holds for all $t > N$. The proof of Lemma 2.2 is now complete. □

3 Main results

In this section, we establish some results for the existence, uniqueness and exponential stability of the almost periodic solution of (1.2).

Theorem 3.1 *Suppose that* (T_1) *and* (T_2) *are satisfied. Then system* (1.2) *has exactly one almost periodic solution* $Z^*(t)$. *Moreover,* $Z^*(t)$ *is globally exponentially stable.*

Proof Let $v(t) = \{v_{ij}(t)\}$ be a solution of system (1.2) with initial function $\varphi_{ij}^v(\cdot)$ satisfying (2.1), and $(\varphi_{ij}^v(\cdot))'$ is bounded continuous on $(-\infty, 0]$.

Set

$$
\begin{aligned}
\epsilon_{ij,k}(t) = {} & -a_{ij}(t)\big[x_{ij}\big(t + t_k - \eta_{ij}(t + t_k)\big) - x_{ij}\big(t - \eta_{ij}(t) + t_k\big)\big] \\
& - \big[a_{ij}(t + t_k) - a_{ij}(t)\big]x_{ij}\big(t + t_k - \eta_{ij}(t + t_k)\big) \\
& - \sum_{C_{kl} \in N_r(i,j)} \big[C_{ij}^{kl}(t + t_k) - C_{ij}^{kl}(t)\big]f\big(x_{kl}\big(t - \tau(t + t_k) + t_k\big)\big)x_{ij}(t + t_k) \\
& - \sum_{C_{kl} \in N_r(i,j)} C_{ij}^{kl}(t)\big[f\big(x_{kl}\big(t - \tau(t + t_k) + t_k\big)\big) - f\big(x_{kl}\big(t - \tau(t) + t_k\big)\big)\big]x_{ij}(t + t_k) \\
& - \sum_{C_{kl} \in N_q(i,j)} \big[B_{ij}^{kl}(t + t_k) - B_{ij}^{kl}(t)\big]\int_0^\infty K_{ij}(u)g\big(x_{kl}(t + t_k - u)\big)\,du\,x_{ij}(t + t_k) \\
& + \big[L_{ij}(t + t_k) - L_{ij}(t)\big], \quad ij \in J,
\end{aligned}
\tag{3.1}
$$

where $\{t_k\}$ is any sequence of real numbers. By Lemma 2.1, the solution $v(t)$ is bounded and

$$
\big|v_{ij}(t)\big| \le \frac{1}{1 - a_{ij}^+ \eta_{ij}^+}\left(\frac{L_{ij}}{\delta_{ij}}\right)^+ \quad \text{for all } t \in R, ij \in J,
\tag{3.2}
$$

which implies that the right side of (1.2) is also bounded, and $v'(t)$ is a bounded function on R. Thus, $v(t)$ is uniformly continuous on R. Then, from the almost periodicity of a_{ij}, η_{ij}, τ, C_{ij}^{kl} and B_{ij}^{kl}, we can select a sequence $\{t_k\} \to +\infty$ such that

$$
\left.
\begin{aligned}
& |a_{ij}(t + t_k) - a_{ij}(t)| \le \tfrac{1}{k}, \quad |C_{ij}^{kl}(t + t_k) - C_{ij}^{kl}(t)| \le \tfrac{1}{k}, \quad |B_{ij}^{kl}(t + t_k) - B_{ij}^{kl}(t)| \le \tfrac{1}{k}, \\
& |\eta_{ij}(t + t_k) - \eta_{ij}(t)| \le \tfrac{1}{k}, \quad |\tau(t + t_k) - \tau(t)| \le \tfrac{1}{k}, \quad |\epsilon_{ij,k}(t)| \le \tfrac{1}{k}
\end{aligned}
\right\}, \tag{3.3}
$$

for all $ij, kl \in J$, $t \in R$.

Since $\{v(t + t_k)\}_{k=1}^{+\infty}$ is uniformly bounded and equiuniformly continuous, by Arzala-Ascoli lemma and diagonal selection principle, we can choose a subsequence $\{t_{k_j}\}$ of $\{t_k\}$, such that $v(t + t_{k_j})$ (for convenience, we still denote by $v(t + t_k)$) uniformly converges to a continuous function $Z^*(t) = \{x_{ij}^*(t)\}$ on any compact set of R, and

$$
\big|x_{ij}^*(t)\big| \le \frac{1}{1 - a_{ij}^+ \eta_{ij}^+}\left(\frac{L_{ij}}{\delta_{ij}}\right)^+ \quad \text{for all } t \in R, ij \in J.
\tag{3.4}
$$

Now, we prove that $Z^*(t)$ is a solution of (1.2). In fact, for any $t > 0$ and $\Delta t \in R$, from (3.3), we have

$$
\begin{aligned}
& x_{ij}^*(t + \Delta t) - x_{ij}^*(t) \\
& \quad = \lim_{k \to +\infty} \big[v_{ij}(t + \Delta t + t_k) - v_{ij}(t + t_k)\big] \\
& \quad = \lim_{k \to +\infty} \int_t^{t + \Delta t} \Big\{ -a_{ij}(\mu + t_k)v_{ij}\big(\mu + t_k - \eta_{ij}(\mu + t_k)\big) \\
& \qquad\qquad - \sum_{C_{kl} \in N_r(i,j)} C_{ij}^{kl}(\mu + t_k)f\big(v_{kl}\big(\mu + t_k - \tau(\mu + t_k)\big)\big)v_{ij}(\mu + t_k)
\end{aligned}
$$

$$- \sum_{C_{kl} \in N_q(i,j)} B_{ij}^{kl}(\mu + t_k) \int_0^\infty K_{ij}(u) g\big(v_{kl}(\mu + t_k - u)\big) du v_{ij}(\mu + t_k) + L_{ij}(\mu + t_k) \bigg\} d\mu$$

$$= \lim_{k \to +\infty} \int_t^{t+\Delta t} \bigg\{ -a_{ij}(\mu) v_{ij}\big(\mu - \eta_{ij}(\mu) + t_k\big)$$

$$- \sum_{C_{kl} \in N_r(i,j)} C_{ij}^{kl}(\mu) f\big(v_{kl}\big(\mu - \tau(\mu) + t_k\big)\big) v_{ij}(\mu + t_k)$$

$$- \sum_{C_{kl} \in N_q(i,j)} B_{ij}^{kl}(\mu) \int_0^\infty K_{ij}(u) g\big(v_{kl}(\mu + t_k - u)\big) du v_{ij}(\mu + t_k) + L_{ij}(\mu) + \epsilon_{ij,k}(\mu) \bigg\} d\mu$$

$$= \int_t^{t+\Delta t} \bigg\{ -a_{ij}(\mu) x_{ij}^*\big(\mu - \eta_{ij}(\mu)\big) - \sum_{C_{kl} \in N_r(i,j)} C_{ij}^{kl}(\mu) f\big(x_{kl}^*\big(\mu - \tau(\mu)\big)\big) x_{ij}^*(\mu)$$

$$- \sum_{C_{kl} \in N_q(i,j)} B_{ij}^{kl}(\mu) \int_0^\infty K_{ij}(u) g\big(x_{kl}^*(\mu - u)\big) du x_{ij}^*(\mu) + L_{ij}(\mu) \bigg\} d\mu$$

$$+ \lim_{k \to +\infty} \int_t^{t+\Delta t} \epsilon_{ij,k}(\mu) d\mu$$

$$= \int_t^{t+\Delta t} \bigg\{ -a_{ij}(\mu) x_{ij}^*\big(\mu - \eta_{ij}(\mu)\big) - \sum_{C_{kl} \in N_r(i,j)} C_{ij}^{kl}(\mu) f\big(x_{kl}^*\big(\mu - \tau(\mu)\big)\big) x_{ij}^*(\mu)$$

$$- \sum_{C_{kl} \in N_q(i,j)} B_{ij}^{kl}(\mu) \int_0^\infty K_{ij}(u) g\big(x_{kl}^*(\mu - u)\big) du x_{ij}^*(\mu) + L_{ij}(\mu) \bigg\} d\mu, \tag{3.5}$$

which implies that

$$\frac{d}{dt}\big\{x_{ij}^*(t)\big\} = -a_{ij}(t) x_{ij}^*\big(t - \eta_{ij}(t)\big) - \sum_{C_{kl} \in N_r(i,j)} C_{ij}^{kl}(t) f\big(x_{kl}^*\big(t - \tau(t)\big)\big) x_{ij}^*(t)$$

$$- \sum_{C_{kl} \in N_q(i,j)} B_{ij}^{kl}(t) \int_0^\infty K_{ij}(u) g\big(x_{kl}^*(t - u)\big) du x_{ij}^*(t) + L_{ij}(t), \quad ij \in J. \tag{3.6}$$

Therefore, $Z^*(t)$ is a solution of (1.2).

Secondly, we prove that $Z^*(t)$ is a almost periodic solution of (1.2). From Lemma 2.2, for any $\varepsilon > 0$, there exists $l = l(\varepsilon) > 0$, such that every interval $[\alpha, \alpha + l]$ contains at least one number δ for which there exists $N > 0$ satisfying

$$\big|v_{ij}(t + \delta) - v_{ij}(t)\big| \leq \varepsilon \quad \text{for all } t > N, ij \in J. \tag{3.7}$$

Then, for any fixed $s \in R$, we can find a sufficient large positive integer $N_1 > N$ such that for any $k > N_1$

$$s + t_k > N, \quad \big|v_{ij}(s + t_k + \delta) - v_{ij}(s + t_k)\big| \leq \varepsilon, \quad ij \in J. \tag{3.8}$$

Let $k \to +\infty$, we obtain

$$\big|x_{ij}^*(s + \delta) - x_{ij}^*(s)\big| \leq \varepsilon, \quad ij \in J, \tag{3.9}$$

which implies that $Z^*(t)$ is an almost periodic solution of (1.2).

Finally, we prove that $Z^*(t)$ is globally exponentially stable.

Let $Z^*(t) = \{x_{ij}^*(t)\}$ be the positive almost periodic solution of system (1.2) with initial value $\varphi^* = \{\varphi_{ij}^*(t)\}$, and let $Z(t) = \{x_{ij}(t)\}$ be an arbitrary solution of system (1.2) with initial value $\varphi = \{\varphi_{ij}(t)\}$, set $y(t) = \{y_{ij}(t)\} = \{x_{ij}(t) - x_{ij}^*(t)\} = Z(t) - Z^*(t)$. Then

$$
\begin{aligned}
y_{ij}'(t) = {}& -a_{ij}(t)y_{ij}\big(t - \eta_{ij}(t)\big) - \sum_{C_{kl} \in N_r(i,j)} C_{ij}^{kl}(t)\big[f\big(x_{kl}\big(t - \tau(t)\big)\big)x_{ij}(t) \\
& - f\big(x_{kl}^*\big(t - \tau(t)\big)\big)x_{ij}^*(t)\big] - \sum_{C_{kl} \in N_q(i,j)} B_{ij}^{kl}(t)\bigg[\int_0^\infty K_{ij}(u)g\big(x_{kl}(t-u)\big)\,du\,x_{ij}(t) \\
& - \int_0^\infty K_{ij}(u)g\big(x_{kl}^*(t-u)\big)\,du\,x_{ij}^*(t)\bigg],
\end{aligned}
$$

which yields

$$
\begin{aligned}
\frac{d}{dt}&\bigg(e^{\lambda t}y_{ij}(t) - \int_{t-\eta_{ij}(t)}^t a_{ij}(s)e^{\lambda s}y_{ij}(s)\,ds\bigg) \\
={}& \lambda e^{\lambda t}y_{ij}(t) + e^{\lambda t}y_{ij}'(t) \\
& - \big[a_{ij}(t)e^{\lambda t}y_{ij}(t) - \big(1 - \eta_{ij}'(t)\big)a_{ij}\big(t - \eta_{ij}(t)\big)e^{\lambda(t-\eta_{ij}(t))}y_{ij}\big(t - \eta_{ij}(t)\big)\big] \\
={}& \lambda e^{\lambda t}y_{ij}(t) - \big[a_{ij}(t)e^{\lambda t}y_{ij}(t) - \big(1 - \eta_{ij}'(t)\big)a_{ij}\big(t - \eta_{ij}(t)\big)e^{\lambda(t-\eta_{ij}(t))}y_{ij}\big(t - \eta_{ij}(t)\big)\big] \\
& + e^{\lambda t}\bigg\{-a_{ij}(t)y_{ij}\big(t - \eta_{ij}(t)\big) - \sum_{C_{kl} \in N_r(i,j)} C_{ij}^{kl}(t)\big[f\big(x_{kl}\big(t - \tau(t)\big)\big)x_{ij}(t) \\
& - f\big(x_{kl}^*\big(t - \tau(t)\big)\big)x_{ij}^*(t)\big] - \sum_{C_{kl} \in N_q(i,j)} B_{ij}^{kl}(t)\bigg[\int_0^\infty K_{ij}(u)g\big(x_{kl}(t-u)\big)\,du\,x_{ij}(t) \\
& - \int_0^\infty K_{ij}(u)g\big(x_{kl}^*(t-u)\big)\,du\,x_{ij}^*(t)\bigg]\bigg\} \\
={}& -\big(a_{ij}(t) - \lambda\big)e^{\lambda t}y_{ij}(t) \\
& - \big[a_{ij}(t) - \big(1 - \eta_{ij}'(t)\big)a_{ij}\big(t - \eta_{ij}(t)\big)e^{-\lambda\eta_{ij}(t)}\big]e^{\lambda t}y_{ij}\big(t - \eta_{ij}(t)\big) \\
& + e^{\lambda t}\bigg\{-\sum_{C_{kl} \in N_r(i,j)} C_{ij}^{kl}(t)\big[f\big(x_{kl}\big(t - \tau(t)\big)\big)x_{ij}(t) - f\big(x_{kl}^*\big(t - \tau(t)\big)\big)x_{ij}^*(t)\big] \\
& - \sum_{C_{kl} \in N_q(i,j)} B_{ij}^{kl}(t)\bigg[\int_0^\infty K_{ij}(u)g\big(x_{kl}(t-u)\big)\,du\,x_{ij}(t) \\
& - \int_0^\infty K_{ij}(u)g\big(x_{kl}^*(t-u)\big)\,du\,x_{ij}^*(t)\bigg]\bigg\} \\
={}& -\big(a_{ij}(t) - \lambda\big)\bigg(e^{\lambda t}y_{ij}(t) - \int_{t-\eta_{ij}(t)}^t a_{ij}(s)e^{\lambda s}y_{ij}(s)\,ds\bigg) \\
& - \big(a_{ij}(t) - \lambda\big)\int_{t-\eta_{ij}(t)}^t a_{ij}(s)e^{\lambda s}y_{ij}(s)\,ds \\
& - \big[a_{ij}(t) - \big(1 - \eta_{ij}'(t)\big)a_{ij}\big(t - \eta_{ij}(t)\big)e^{-\lambda\eta_{ij}(t)}\big]e^{\lambda t}y_{ij}\big(t - \eta_{ij}(t)\big)
\end{aligned}
$$

$$+ e^{\lambda t}\Bigg\{- \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}(t)\big[f\big(x_{kl}(t-\tau(t))\big) - f\big(x_{kl}^*(t-\tau(t))\big)\big]x_{ij}(t)$$

$$- \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}(t)f\big(x_{kl}^*(t-\tau(t))\big)\big[x_{ij}(t) - x_{ij}^*(t)\big]$$

$$- \sum_{C_{kl}\in N_q(i,j)} B_{ij}^{kl}(t)\int_0^\infty K_{ij}(u)\big[g\big(x_{kl}(t-u)\big) - g\big(x_{kl}^*(t-u)\big)\big]\,du\,x_{ij}(t)$$

$$- \sum_{C_{kl}\in N_q(i,j)} B_{ij}^{kl}(t)\int_0^\infty K_{ij}(u)g\big(x_{kl}^*(t-u)\big)\,du\big[x_{ij}(t) - x_{ij}^*(t)\big]\Bigg\}, \tag{3.10}$$

where $ij \in J$.

Let

$$Y_{ij}(t) = \left| e^{\lambda t}y_{ij}(t) - \int_{t-\eta_{ij}(t)}^t a_{ij}(s)e^{\lambda s}y_{ij}(s)\,ds \right|, \quad ij \in J.$$

We define a positive constant M as follows:

$$M = \max_{ij\in J}\Big\{\sup_{s\in(-\infty,0]} Y_{ij}(s)\Big\}.$$

Let K be a positive number such that

$$Y_{ij}(t) \le M < M + 1 = K \quad \text{for all } t \in (-\infty, 0], ij \in J. \tag{3.11}$$

We claim that

$$Y_{ij}(t) < K, \quad \text{for all } t > 0, i = 1, 2, \ldots, n. \tag{3.12}$$

Otherwise, there must exist $ij \in J$ and $\theta > 0$ such that

$$Y_{ij}(\theta) = K, \qquad Y_{\tilde{ij}}(t) < K, \quad \text{for all } t \in (-\infty, \theta), \tilde{ij} \in J. \tag{3.13}$$

It follows that

$$e^{\lambda t}\big|y_{\tilde{ij}}(t)\big| \le \left| e^{\lambda t}y_{\tilde{ij}}(t) - \int_{t-\eta_{\tilde{ij}}(t)}^t a_{\tilde{ij}}(s)e^{\lambda s}y_{\tilde{ij}}(s)\,ds \right|$$

$$+ \left| \int_{t-\eta_{\tilde{ij}}(t)}^t a_{\tilde{ij}}(s)e^{\lambda s}y_{\tilde{ij}}(s)\,ds \right|$$

$$\le K + a_{\tilde{ij}}^+\eta_{\tilde{ij}}^+ \sup_{s\in(-\infty,\theta]} e^{\lambda s}\big|y_{\tilde{ij}}(s)\big|, \quad \text{for all } t \in (-\infty, \theta], \tilde{ij} \in J. \tag{3.14}$$

Consequently, in view of (3.14) and the fact $a_{\tilde{ij}}^+\eta_{\tilde{ij}}^+ < 1$ $(\tilde{ij} \in J)$, we have

$$e^{\lambda t}\big|y_{\tilde{ij}}(t)\big| \le \sup_{s\in(-\infty,\theta]} e^{\lambda s}\big|y_{\tilde{ij}}(s)\big| \le \frac{K}{1 - a_{\tilde{ij}}^+\eta_{\tilde{ij}}^+} \quad \text{for all } t \in (-\infty, \theta], \tilde{ij} \in J. \tag{3.15}$$

Calculating the upper left derivative of $Y_{ij}(t)$, together with (2.3), (3.10), (3.13), (3.15), (T_1) and (T_2), we obtain

$$0 \leq D^- Y_{ij}(\theta)$$

$$\leq -\left(a_{ij}(\theta) - \lambda\right) Y_{ij}(\theta) + \left| -\left(a_{ij}(\theta) - \lambda\right) \int_{\theta - \eta_{ij}(\theta)}^{\theta} a_{ij}(s) e^{\lambda s} y_{ij}(s)\, ds \right.$$

$$- \left[a_{ij}(\theta) - \left(1 - \eta_{ij}'(\theta)\right) a_{ij}\left(\theta - \eta_{ij}(\theta)\right) e^{-\lambda \eta_{ij}(\theta)} \right] e^{\lambda \theta} y_{ij}\left(\theta - \eta_{ij}(\theta)\right)$$

$$+ e^{\lambda \theta} \left\{ - \sum_{C_{kl} \in N_r(i,j)} C_{ij}^{kl}(\theta) \left[f\left(x_{kl}\left(\theta - \tau(\theta)\right)\right) - f\left(x_{kl}^*\left(\theta - \tau(\theta)\right)\right) \right] x_{ij}(\theta) \right.$$

$$- \sum_{C_{kl} \in N_r(i,j)} C_{ij}^{kl}(\theta) f\left(x_{kl}^*\left(\theta - \tau(\theta)\right)\right) \left[x_{ij}(\theta) - x_{ij}^*(\theta) \right]$$

$$- \sum_{C_{kl} \in N_q(i,j)} B_{ij}^{kl}(\theta) \int_0^{\infty} K_{ij}(u) \left[g\left(x_{kl}(\theta - u)\right) - g\left(x_{kl}^*(\theta - u)\right) \right] du\, x_{ij}(\theta)$$

$$\left. \left. - \sum_{C_{kl} \in N_q(i,j)} B_{ij}^{kl}(\theta) \int_0^{\infty} K_{ij}(u) g\left(x_{kl}^*(\theta - u)\right) du \left[x_{ij}(\theta) - x_{ij}^*(\theta) \right] \right\} \right|$$

$$\leq -\left(a_{ij}(\theta) - \lambda\right) Y_{ij}(\theta) + \left(a_{ij}(\theta) - \lambda\right) \frac{K}{1 - a_{ij}^+ \eta_{ij}^+} a_{ij}^+ \eta_{ij}^+$$

$$+ \left| a_{ij}(\theta) e^{\lambda \eta_{ij}(\theta)} - \left(1 - \eta_{ij}'(\theta)\right) a_{ij}\left(\theta - \eta_{ij}(\theta)\right) \right| e^{\lambda(\theta - \eta_{ij}(\theta))} \left| y_{ij}\left(\theta - \eta_{ij}(\theta)\right) \right|$$

$$+ \sum_{C_{kl} \in N_r(i,j)} \left| C_{ij}^{kl}(\theta) \right| \mu_f e^{\lambda \tau(\theta)} e^{\lambda(\theta - \tau(\theta))} \left| y_{kl}\left(\theta - \tau(\theta)\right) \right| \left| x_{ij}(\theta) \right|$$

$$+ \sum_{C_{kl} \in N_r(i,j)} \left| C_{ij}^{kl}(\theta) \right| M_f e^{\lambda \theta} \left| y_{ij}(\theta) \right|$$

$$+ \sum_{C_{kl} \in N_q(i,j)} \left| B_{ij}^{kl}(\theta) \right| \int_0^{\infty} e^{\lambda u} \left| K_{ij}(u) \right| \mu_g e^{\lambda(\theta - u)} \left| y_{kl}(\theta - u) \right| du \left| x_{ij}(\theta) \right|$$

$$+ \sum_{C_{kl} \in N_q(i,j)} \left| B_{ij}^{kl}(\theta) \right| \int_0^{\infty} \left| K_{ij}(u) \right| du\, M_g e^{\lambda \theta} \left| y_{ij}(\theta) \right|$$

$$\leq \left\{ -\left[\left(a_{ij}(\theta) - \lambda\right)\left(1 - 2a_{ij}^+ \eta_{ij}^+\right) \right. \right.$$

$$\left. - \left| a_{ij}(\theta) e^{\lambda \eta_{ij}(\theta)} - \left(1 - \eta_{ij}'(\theta)\right) a_{ij}\left(\theta - \eta_{ij}(\theta)\right) \right| \right] \frac{1}{1 - a_{ij}^+ \eta_{ij}^+}$$

$$+ \sum_{C_{kl} \in N_r(i,j)} \left| C_{ij}^{kl}(\theta) \right| \mu_f e^{\lambda \tau(\theta)} \frac{1}{1 - a_{kl}^+ \eta_{kl}^+} \frac{1}{1 - a_{ij}^+ \eta_{ij}^+} \left(\frac{L_{ij}}{\delta_{ij}} \right)^+$$

$$+ \sum_{C_{kl} \in N_r(i,j)} \left| C_{ij}^{kl}(\theta) \right| M_f \frac{1}{1 - a_{ij}^+ \eta_{ij}^+}$$

$$+ \sum_{C_{kl} \in N_q(i,j)} \left| B_{ij}^{kl}(\theta) \right| \int_0^{\infty} e^{\lambda u} \left| K_{ij}(u) \right| du\, \mu_g \frac{1}{1 - a_{kl}^+ \eta_{kl}^+} \frac{1}{1 - a_{ij}^+ \eta_{ij}^+} \left(\frac{L_{ij}}{\delta_{ij}} \right)^+$$

$$\left. + \sum_{C_{kl} \in N_q(i,j)} \left| B_{ij}^{kl}(\theta) \right| \int_0^{\infty} \left| K_{ij}(u) \right| du\, M_g \frac{1}{1 - a_{ij}^+ \eta_{ij}^+} \right\} K$$

$$< -\eta K$$

$$< 0,$$

which is a contradiction, and it implies that (3.12) holds.

Consequently, using a similar argument as in (3.14)-(3.15), we know that

$$|y_{ij}(t)|e^{\lambda t} < \frac{K}{1 - a_{ij}^+ \eta_{ij}^+}, \quad \text{for all } t \in R, ij \in J.$$

That is,

$$|x_{ij}(t) - x_{ij}^*(t)| \leq \frac{K}{1 - a_{ij}^+ \eta_{ij}^+} e^{-\lambda t}, \quad \text{for all } t > 0, \text{ and } ij \in J.$$

This completes the proof. □

4 An example

In this section, we give an example with numerical simulations to demonstrate the results obtained in previous sections.

Example 4.1 Consider the following SICNNs with time-varying delays in the leakage terms:

$$\frac{dx_{ij}}{dt} = -a_{ij}(t)x_{ij}(t - \eta_{ij}(t)) - \sum_{C_{kl} \in N_r(i,j)} C_{ij}^{kl} f(x_{kl}(t - \sin^2 t))x_{ij}(t)$$

$$- \sum_{C_{kl} \in N_q(i,j)} B_{ij}^{kl} \int_0^\infty K_{ij}(u)g(x_{kl}(t - u)) \, du \, x_{ij} + L_{ij}(t), \quad i,j = 1,2,3, \tag{4.1}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 3 \\ 3 & 1 & 3 \\ 3 & 1 & 3 \end{bmatrix}, \tag{4.2}$$

$$\begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} = \begin{bmatrix} 0.1 & 0.2 & 0.1 \\ 0.2 & 0 & 0.2 \\ 0.1 & 0.2 & 0.1 \end{bmatrix}, \tag{4.3}$$

$$\begin{bmatrix} \eta_{11} & \eta_{12} & \eta_{13} \\ \eta_{21} & \eta_{22} & \eta_{23} \\ \eta_{31} & \eta_{32} & \eta_{33} \end{bmatrix} = 0.01 \begin{bmatrix} \sin^2 \sqrt{3}t & \cos^2 \sqrt{3}t & \sin^2 2t \\ \cos^2 \sqrt{5}t & \sin^2 \sqrt{5}t & \cos^2 2t \\ \sin^2 2t & \cos^2 3t & \sin^2 \sqrt{2}t \end{bmatrix}, \tag{4.4}$$

$$\begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{bmatrix}$$

$$= \begin{bmatrix} 0.7 + 0.24\sin^2 \sqrt{2}t & 0.41 + 0.5\cos^2 t & 0.74 + 0.2\sin^2 t \\ 0.61 + 0.2\cos^2 t & 0.67 + 0.2\sin^2 t & 0.75 + 0.2\sin^2 t \\ 0.59 + 0.4\cos^4 t & 0.5 + 0.41\sin^2 t & 0.76 + 0.2\cos^2 t \end{bmatrix}. \tag{4.5}$$

Set

$$\lambda = 0.001, \qquad r = q = 1, \qquad K_{ij}(u) = |\sin u|e^{-u}, \quad i = 1, 2, 3, j = 1, 2, 3,$$

and

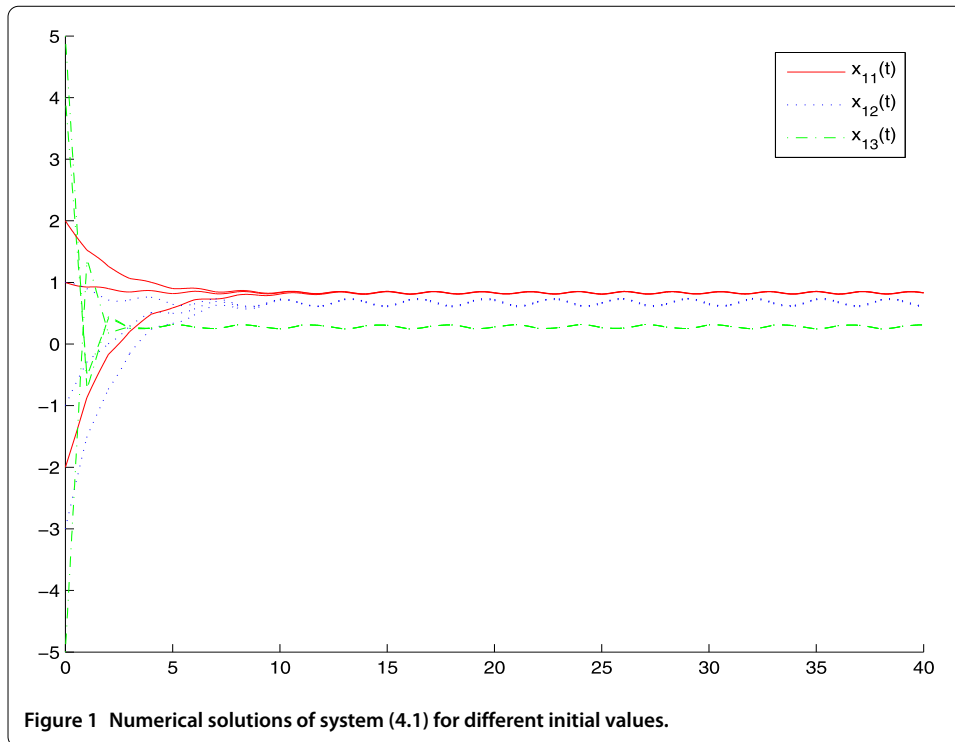$$f(x) = g(x) = \frac{1}{40}\big(|x - 1| - |x + 1|\big),$$

clearly,

$$M_f = M_g = 0.05, \qquad \mu_f = \mu_g = 0.05,$$

$$\sum_{C_{kl} \in N_1(1,1)} C_{11}^{kl} = \sum_{C_{kl} \in N_1(1,1)} B_{11}^{kl} = 0.5, \qquad \sum_{C_{kl} \in N_1(1,2)} C_{12}^{kl} = \sum_{C_{kl} \in N_1(1,2)} B_{12}^{kl} = 0.8,$$

$$\sum_{C_{kl} \in N_1(1,3)} C_{13}^{kl} = \sum_{C_{kl} \in N_1(1,3)} B_{13}^{kl} = 0.5, \qquad \sum_{C_{kl} \in N_1(2,1)} C_{21}^{kl} = \sum_{C_{kl} \in N_1(2,1)} B_{21}^{kl} = 0.8,$$

$$\sum_{C_{kl} \in N_1(2,2)} C_{22}^{kl} = \sum_{C_{kl} \in N_1(2,2)} B_{22}^{kl} = 1.2, \qquad \sum_{C_{kl} \in N_1(2,3)} C_{23}^{kl} = \sum_{C_{kl} \in N_1(2,3)} B_{23}^{kl} = 0.8,$$

$$\sum_{C_{kl} \in N_1(3,1)} C_{31}^{kl} = \sum_{C_{kl} \in N_1(3,1)} B_{31}^{kl} = 0.5, \qquad \sum_{C_{kl} \in N_1(3,2)} C_{32}^{kl} = \sum_{C_{kl} \in N_1(3,2)} B_{32}^{kl} = 0.8,$$

$$\sum_{C_{kl} \in N_1(3,3)} C_{33}^{kl} = \sum_{C_{kl} \in N_1(3,3)} B_{33}^{kl} = 0.5,$$

$$1 \le a_{ij}^+ \le 3, \qquad \eta_{ij}^+ \le 0.01, \qquad a_{ij}^+ \eta_{ij}^+ \le 0.03 < 1,$$

where $ij \in J = \{11, 12, 13, 21, 22, 23, 31, 32, 33\}$. Then,

$$\min_{ij \in J} \delta_{ij}(t) = \min_{ij \in J} \bigg\{ \bigg\{ a_{ij}(t)\big(1 - 2a_{ij}^+ \eta_{ij}^+\big) - \big|a_{ij}(t) - \big(1 - \eta_{ij}'(t)\big)a_{ij}\big(t - \eta_{ij}(t)\big)\big|$$

$$- \sum_{C_{kl} \in N_r(i,j)} \big|C_{ij}^{kl}(t)\big| M_f - \sum_{C_{kl} \in N_q(i,j)} \big|B_{ij}^{kl}(t)\big| \int_0^\infty \big|K_{ij}(u)\big|\, du M_g \bigg\} \frac{1}{1 - a_{ij}^+ \eta_{ij}^+} \bigg\}$$
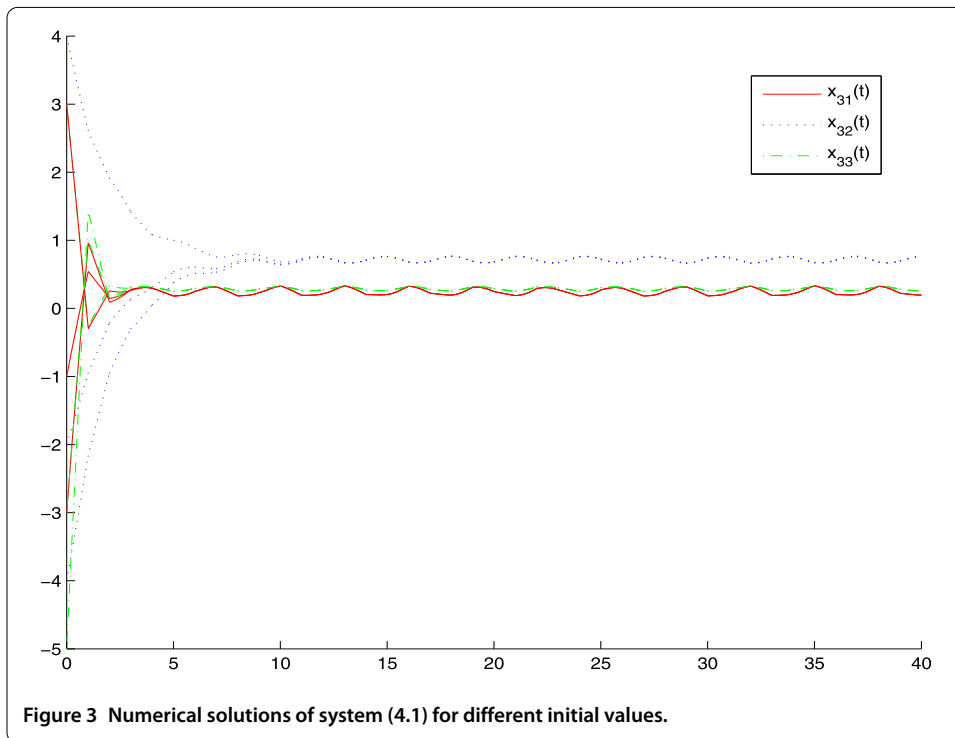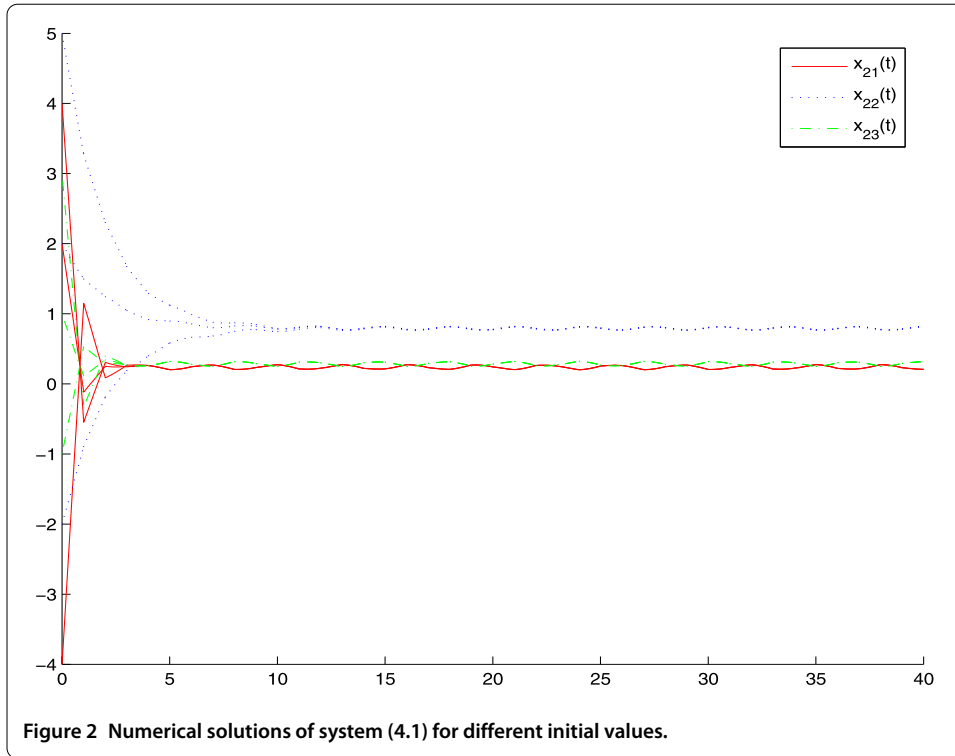
$$> 0.5 > 0, \quad \text{for all } t \ge 0, \tag{4.6}$$

$$\max_{ij \in J} \bigg\{ -\big[\big(a_{ij}(t) - \lambda\big)\big(1 - 2a_{ij}^+ \eta_{ij}^+\big) - \big|a_{ij}(t)e^{\lambda \eta_{ij}(t)} - \big(1 - \eta_{ij}'(t)\big)a_{ij}\big(t - \eta_{ij}(t)\big)\big|\big] \frac{1}{1 - a_{ij}^+ \eta_{ij}^+}$$

$$+ \sum_{C_{kl} \in N_r(i,j)} \big|C_{ij}^{kl}(t)\big| \mu_f e^{\lambda \tau(t)} \frac{1}{1 - a_{kl}^+ \eta_{kl}^+} \frac{1}{1 - a_{ij}^+ \eta_{ij}^+} \bigg(\frac{L_{ij}}{\delta_{ij}}\bigg)^+$$

$$+ \sum_{C_{kl} \in N_r(i,j)} \big|C_{ij}^{kl}(t)\big| M_f \frac{1}{1 - a_{ij}^+ \eta_{ij}^+}$$

$$+ \sum_{C_{kl} \in N_q(i,j)} \big|B_{ij}^{kl}(t)\big| \int_0^\infty e^{\lambda u} \big|K_{ij}(u)\big|\, du \mu_g \frac{1}{1 - a_{kl}^+ \eta_{kl}^+} \frac{1}{1 - a_{ij}^+ \eta_{ij}^+} \bigg(\frac{L_{ij}}{\delta_{ij}}\bigg)^+$$

$$+ \sum_{C_{kl} \in N_q(i,j)} \big|B_{ij}^{kl}(t)\big| \int_0^\infty \big|K_{ij}(u)\big|\, du M_g \frac{1}{1 - a_{ij}^+ \eta_{ij}^+} \bigg\}$$

$$< -(1 - \lambda) \times (1 - 2 \times 1 \times 0.01) + \big(1 \times e^{0.01\lambda} - 1 + 0.01\big) \frac{1}{1 - 0.01}$$

Figure 1 Numerical solutions of system (4.1) for different initial values.

$$+ 1.2 \times 0.05 \times e^{\lambda \times 1} \frac{1}{1 - 0.03} \frac{1}{1 - 0.03} \times 2 + 1.2 \times 0.05 \times \frac{1}{1 - 0.03}$$

$$+ 1.2 \times \frac{e^{\lambda}}{1 - \lambda} \times 0.05 \frac{1}{1 - 0.03} \frac{1}{1 - 0.03} + 1.2 \times 1 \times 0.05 \times \frac{1}{1 - 0.03}$$

$$\approx -0.6538 < -0.5, \quad \text{for all } t \geq 0. \tag{4.7}$$

It follows that system (4.1) satisfies all the conditions in Theorem 3.1. Hence, system (4.1) has exactly one positive almost periodic solution. Moreover, the almost periodic solution is globally exponentially stable. The fact is verified by the numerical simulation in Figures 1-3 and their three different initial values, which are $\varphi_{11} \equiv 1$, $\varphi_{12} \equiv -3$, $\varphi_{13} \equiv 4$, $\varphi_{21} \equiv 2$, $\varphi_{22} \equiv 5$, $\varphi_{23} \equiv 3$, $\varphi_{33} \equiv -1$, $\varphi_{32} \equiv -2$, $\varphi_{33} \equiv -5$; $\varphi_{11} \equiv 2$, $\varphi_{12} \equiv -1$, $\varphi_{13} \equiv 5$, $\varphi_{21} \equiv 4$, $\varphi_{22} \equiv 2$, $\varphi_{23} \equiv 1$, $\varphi_{33} \equiv -3$, $\varphi_{32} \equiv -4$, $\varphi_{33} \equiv 3$ and $\varphi_{11} \equiv -2$, $\varphi_{12} \equiv 1$, $\varphi_{13} \equiv -5$, $\varphi_{21} \equiv -4$, $\varphi_{22} \equiv -2$, $\varphi_{23} \equiv -1$, $\varphi_{33} \equiv 3$, $\varphi_{32} \equiv 4$, $\varphi_{33} \equiv -3$, respectively.

Remark 4.1 Since [4–9] only dealt with SICNNs without leakage delays, and [14–19, 23, 24] give no opinions about the problem of almost periodic solutions for SICNNs with time-varying leakage delays. One can observe that all the results in this literature and the references therein can not be applicable to prove the existence and exponential stability of almost periodic solutions for SICNNs (4.1). In this present paper, we employ a novel proof to establish some criteria to guarantee the existence and exponential stability of almost periodic solutions for SICNNs system with time-varying coefficients and leakage delays. What will happen when one can increase the neuron's order? Whether or not our results and method in this paper are available for this case, it is an interesting problem, and we leave it as our work in the future.

Figure 2 Numerical solutions of system (4.1) for different initial values.

Figure 3 Numerical solutions of system (4.1) for different initial values.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

BL gave the proof of Lemmas 2.1-2.2 and drafted the manuscript. JS proved Theorem 3.1 and gave the example to illustrate the effectiveness of the obtained results. All authors read and approved the final manuscript.

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