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A note on rough singular integrals in Triebel-Lizorkin spaces and Besov spaces

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Abstract
This paper is concerned with the singular integral operators along polynomial curves. The boundedness for such operators on Triebel-Lizorkin spaces and Besov spaces is established, provided the kernels satisfy rather weak size conditions both on the unit sphere and in the radial direction. Moreover, the corresponding results for the singular integrals associated to the compound curves formed by polynomial with certain smooth functions are also given.

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1 Introduction

Let $\mathbb{R}^n$, $n \geq 2$, be the n-dimensional Euclidean space and S^{n-1} denote the unit sphere in $\mathbb{R}^n$ equipped with the induced Lebesgue measure $d\sigma$. Let $\Omega \in L^1(S^{n-1})$ be a homogeneous function of degree zero and satisfy

$$\int_{S^{n-1}} \Omega(u)\, d\sigma(u) = 0. \tag{1.1}$$

For a suitable function h defined on $\mathbb{R}^+ = \{t \in \mathbb{R} : t \geq 0\}$ and a polynomial P_N with $P_N(0) = 0$, where N is the degree of P_N, we define the singular integral operators T_{h,Ω,P_N} along polynomial curves in $\mathbb{R}^n$ by

$$T_{h,\Omega,P_N}(f)(x) := \text{p.v.} \int_{\mathbb{R}^n} f\bigl(x - P_N\bigl(|y|\bigr)y'\bigr) \frac{\Omega(y)h(|y|)}{|y|^n}\, dy. \tag{1.2}$$

For $P_N(t) = t$, we denote T_{h,Ω,P_N} by T_h. Fefferman [1] first proved that T_h is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$ provided that Ω satisfies a Lipschitz condition of positive order on S^{n-1} and $h \in L^\infty(\mathbb{R})$. Subsequently, Namazi [2] improved Fefferman's result to the case $\Omega \in L^q(S^{n-1})$. Later on, Duoandikoetxea and Francia [3] showed that T_h is of type (p,p) for $1 < p < \infty$ provided that $\Omega \in L^q(S^{n-1})$ and $h \in \Delta_2(\mathbb{R}^+)$, where $\Delta_\gamma(\mathbb{R}^+)$, $\gamma > 0$, denotes the set of all measurable functions h on $\mathbb{R}^+$ satisfying the condition

$$\|h\|_{\Delta_\gamma(\mathbb{R}^+)} = \sup_{R>0} \left(R^{-1} \int_0^R \bigl|h(t)\bigr|^\gamma\, dt \right)^{1/\gamma} < \infty.$$

It is easy to check that $\Delta_\infty(\mathbb{R}^+) = L^\infty(\mathbb{R}^+) \subsetneq \Delta_{\gamma_2}(\mathbb{R}^+) \subsetneq \Delta_{\gamma_1}(\mathbb{R}^+)$ for $0 < \gamma_1 < \gamma_2 < \infty$. In 1997, Fan and Pan [4] extended the result of [3] to the singular integrals along polynomial mappings provided that $\Omega \in H^1(S^{n-1})$ and $h \in \Delta_\gamma(\mathbb{R}^+)$ for $\gamma > 1$ with $|1/p - 1/2| < \min\{1/2, 1/\gamma'\}$, where $H^1(S^{n-1})$ denotes the Hardy spaces on the unit sphere (see [5, 6]). In 2009, Fan and Sato [7] showed that T_h is bounded on $L^p(\mathbb{R}^n)$ for some $\beta > \max\{\gamma', 2\}$ with $|1/p - 1/2| < \min\{1/\gamma', 1/2\} - 1/\beta$, provided that $h \in \Delta_\gamma(\mathbb{R}^+)$ for $\gamma > 1$, and Ω satisfies the following size condition:

$$\sup_{\xi' \in S^{n-1}} \iint_{S^{n-1}\times S^{n-1}} \left|\Omega(\theta)\Omega(u')\right| \left(\log^+ \frac{1}{|(\theta - u')\cdot \xi'|}\right)^\beta d\sigma(\theta)\, d\sigma(u') < \infty. \tag{1.3}$$

For the sake of simplicity, we denote

$$\tilde{\mathcal{F}}_\beta(S^{n-1}) := \{\Omega \in L^1(S^{n-1}) : \Omega \text{ satisfies (1.3)}\}, \quad \forall \beta > 0.$$

On the other hand, for $h(t) = 1$, Fan *et al.* [8] showed that T_{h,Ω,P_N} is bounded on $L^p(\mathbb{R}^n)$ for $2\beta/(2\beta - 1) < p < 2\beta$ provided $\beta > 1$ and $\Omega \in \mathcal{F}_\beta(S^{n-1})$, where

$$\mathcal{F}_\beta(S^{n-1}) := \left\{\Omega \in L^1(S^{n-1}) : \sup_{\xi \in S^{n-1}} \int_{S^{n-1}} \left|\Omega(y')\right| \left(\log \frac{1}{|\xi \cdot y'|}\right)^\beta d\sigma(y') < \infty\right\}, \quad \forall \beta > 0.$$

Moreover, see [9, 10] for the corresponding results of the singular integrals in the mixed homogeneity setting.

Remark 1.1 It should be pointed out that the functions class $\mathcal{F}_\beta(S^{n-1})$ was originally introduced in Walsh's paper [11] and developed by Grafakos and Stefanov [12] in the study of L^p-boundedness of singular integrals with rough kernels. It follows from [12] that $\mathcal{F}_{\beta_1}(S^{n-1}) \subsetneq \mathcal{F}_{\beta_2}(S^{n-1})$ for $0 < \beta_2 < \beta_1$, and $\bigcup_{q>1} L^q(S^{n-1}) \subsetneq \mathcal{F}_\beta(S^{n-1})$ for any $\beta > 0$, moreover,

$$\bigcap_{\beta>1} \mathcal{F}_\beta(S^{n-1}) \nsubseteq H^1(S^{n-1}) \nsubseteq \bigcup_{\beta>1} \mathcal{F}_\beta(S^{n-1}). \tag{1.4}$$

We also remark that condition (1.3) was originally introduced by Fan and Sato in more general form in [7]. In addition, it follows from [7, Lemma 1] that

$$\mathcal{F}_\beta(S^1) \subset \tilde{\mathcal{F}}_\beta(S^1), \quad \text{for } \beta > 0. \tag{1.5}$$

In this paper, we consider the boundedness of T_{h,Ω,P_N} on the Triebel-Lizorkin spaces and the Besov spaces, which contain many important function spaces, such as Lebesgue spaces, Hardy spaces, Sobolev spaces and Lipschitz spaces. Let us recall some notations. The homogeneous Triebel-Lizorkin spaces $\dot{F}_\alpha^{p,q}(\mathbb{R}^n)$ and homogeneous Besov spaces $\dot{B}_\alpha^{p,q}(\mathbb{R}^n)$ are defined, respectively, by

$$\dot{F}_\alpha^{p,q}(\mathbb{R}^n) := \left\{f \in \mathscr{S}'(\mathbb{R}^n) : \|f\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^n)} = \left\|\left(\sum_{i\in\mathbb{Z}} 2^{-i\alpha q}|\Psi_i * f|^q\right)^{1/q}\right\|_{L^p(\mathbb{R}^n)} < \infty\right\} \tag{1.6}$$

and

$$\dot{B}_\alpha^{p,q}(\mathbb{R}^n) := \left\{f \in \mathscr{S}'(\mathbb{R}^n) : \|f\|_{\dot{B}_\alpha^{p,q}(\mathbb{R}^n)} = \left(\sum_{i\in\mathbb{Z}} 2^{-i\alpha q}\|\Psi_i * f\|_{L^p(\mathbb{R}^n)}^q\right)^{1/q} < \infty\right\}, \tag{1.7}$$

where $\alpha \in \mathbb{R}$, $0 < p, q \le \infty$ $(p \neq \infty)$, $\mathscr{S}'(\mathbb{R}^n)$ denotes the tempered distribution class on $\mathbb{R}^n$, $\widehat{\Psi}_i(\xi) = \phi(2^i\xi)$ for $i \in \mathbb{Z}$ and $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ satisfies the conditions: $0 \le \phi(x) \le 1$; $\operatorname{supp}(\phi) \subset \{x : 1/2 \le |x| \le 2\}$; $\phi(x) > c > 0$ if $3/5 \le |x| \le 5/3$. It is well known that

$$\dot{F}_0^{p,2}(\mathbb{R}^n) = L^p(\mathbb{R}^n) \tag{1.8}$$

for any $1 < p < \infty$, see [13–15], *etc.* for more properties of $\dot{F}_\alpha^{p,q}(\mathbb{R}^n)$ and $\dot{B}_\alpha^{p,q}(\mathbb{R}^n)$.

For $h(t) = 1$, the operator T_h is the classical Calderón-Zygmund singular integral operator denoted by T. In 2002, Chen *et al.* [16] proved that T is bounded on $\dot{F}_\alpha^{p,q}(\mathbb{R}^n)$ provided $\Omega \in L^r(S^{n-1})$ for some $r > 1$. Subsequently, Chen and Zhang [17] improved the result of [16] to the case $\Omega \in \mathcal{F}_\beta(S^{n-1})$ for some $\beta > 2$. Furthermore, in 2008, Chen and Ding [18] showed that T_h is bounded on $\dot{F}_\alpha^{p,q}(\mathbb{R}^n)$ for $1 < p, q < \infty$ and $\alpha \in \mathbb{R}$ if $\Omega \in H^1(S^{n-1})$ and $h \in L^\infty(\mathbb{R}^+)$. In 2010, Chen *et al.* [19] extended the result of [18] to the singular integrals along polynomial mappings provided that $h \in \Delta_\gamma(\mathbb{R}^+)$ for $\gamma > 1$ with $\max\{|1/p - 1/2|, |1/q - 1/2|\} < \min\{1/2, 1/\gamma'\}$.

In light of aforementioned facts, a natural question is the following.

Question Is T_{h,Ω,P_N} bounded on $\dot{F}_\alpha^{p,q}(\mathbb{R}^n)$ if $\Omega \in \tilde{\mathcal{F}}_\beta(S^{n-1})$ and $h \in \Delta_\gamma(\mathbb{R}^+)$ for some $\gamma > 1$?

In this paper, we will give an affirmative answer to this question. Our main results can be formulated as follows.

Theorem 1.1 *Let T_{h,Ω,P_N} be as in* (1.2) *and $h \in \Delta_\gamma(\mathbb{R}^+)$ for some $\gamma > 1$. Suppose that $\Omega \in \tilde{\mathcal{F}}_\beta(S^{n-1})$ for some $\beta > \max\{2, \gamma'\}$ and satisfies* (1.1). *Then for $\alpha \in \mathbb{R}$ and $\max\{|1/p - 1/2|, |1/q - 1/2|\} < \min\{1/2, 1/\gamma'\} - 1/\beta$, there exists a constant $C > 0$ such that*

$$\left\| T_{h,\Omega,P_N}(f) \right\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^n)} \le C\|f\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^n)},$$

where $C = C_{n,p,q,h,\alpha,N,\gamma,\beta}$ is independent of the coefficients of P_N.

Theorem 1.2 *Let T_{h,Ω,P_N} be as in* (1.2) *and $h \in \Delta_\gamma(\mathbb{R}^+)$ for some $\gamma > 1$. Suppose that $\Omega \in \tilde{\mathcal{F}}_\beta(S^{n-1})$ for some $\beta > \max\{2, \gamma'\}$ and satisfies* (1.1). *Then for $\alpha \in \mathbb{R}$, $1 < q < \infty$ and $|1/p - 1/2| < \min\{1/2, 1/\gamma'\} - 1/\beta$, there exists a constant $C > 0$ such that*

$$\left\| T_{h,\Omega,P_N}(f) \right\|_{\dot{B}_\alpha^{p,q}(\mathbb{R}^n)} \le C\|f\|_{\dot{B}_\alpha^{p,q}(\mathbb{R}^n)},$$

where $C = C_{n,p,q,h,\alpha,N,\gamma,\beta}$ is independent of the coefficients of P_N.

By (1.5) and Theorems 1.1-1.2, we get the following results immediately.

Theorem 1.3 *Let T_{h,Ω,P_N} be as in* (1.2) *and $h \in \Delta_\gamma(\mathbb{R}^+)$ for some $\gamma > 1$. Suppose that $\Omega \in \mathcal{F}_\beta(S^1)$ for some $\beta > \max\{2, \gamma'\}$ and satisfies* (1.1). *Then for $\alpha \in \mathbb{R}$ and $\max\{|1/p - 1/2|, |1/q - 1/2|\} < \min\{1/2, 1/\gamma'\} - 1/\beta$, there exists a constant $C > 0$ such that*

$$\left\| T_{h,\Omega,P_N}(f) \right\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^2)} \le C\|f\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^2)},$$

where $C = C_{p,q,h,\alpha,N,\gamma,\beta}$ is independent of the coefficients of P_N.

Theorem 1.4 *Let T_{h,Ω,P_N} be as in* (1.2) *and $h \in \Delta_\gamma(\mathbb{R}^+)$ for some $\gamma > 1$. Suppose that $\Omega \in \mathcal{F}_\beta(S^1)$ for some $\beta > \max\{2, \gamma'\}$ and satisfies* (1.1). *Then for $\alpha \in \mathbb{R}$, $1 < q < \infty$ and $|1/p - 1/2| < \min\{1/2, 1/\gamma'\} - 1/\beta$, there exists a constant $C > 0$ such that*

$$\|T_{h,\Omega,P_N}(f)\|_{\dot{B}_\alpha^{p,q}(\mathbb{R}^2)} \le C\|f\|_{\dot{B}_\alpha^{p,q}(\mathbb{R}^2)},$$

where $C = C_{p,q,h,\alpha,N,\gamma,\beta}$ is independent of the coefficients of P_N.

Remark 1.2 Obviously, by (1.5) and (1.8), our results can be regarded as the generalization of the results in [8] or [7], even in the special case $h(t) = 1$ or $P_N(t) = t$. Moreover, by (1.4)-(1.5), our results are also distinct from the ones in [18, 19].

Furthermore, by Theorems 1.1-1.4, and a switched method followed from [20], we can establish the following more general results.

Theorem 1.5 *Let $h \in \Delta_\gamma(\mathbb{R}^+)$ for some $\gamma > 1$ and $\Omega \in \tilde{\mathcal{F}}_\beta(S^{n-1})$ for some $\beta > \max\{2, \gamma'\}$ with satisfying* (1.1). *Suppose that φ is a nonnegative* (*or nonpositive*) *and monotonic C^1 function on $(0, \infty)$ such that $\Gamma(t) := \frac{\varphi(t)}{t\varphi'(t)}$ with $|\Gamma(t)| \le C$, where C is a positive constant which depends only on φ. Then*

(i) *for $\alpha \in \mathbb{R}$ and $\max\{|1/p - 1/2|, |1/q - 1/2|\} < \min\{1/2, 1/\gamma'\} - 1/\beta$, there exists a constant $C > 0$ such that*

$$\|T_{h,P_N,\varphi}(f)\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^n)} \le C\|f\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^n)},$$

where

$$T_{h,P_N,\varphi}(f)(x) := \text{p.v.} \int_{\mathbb{R}^n} f\bigl(x - P_N\bigl(\varphi(|y|)\bigr)y'\bigr) \frac{\Omega(y)h(|y|)}{|y|^n}\, dy.$$

(ii) *for $\alpha \in \mathbb{R}$, $1 < q < \infty$ and $|1/p - 1/2| < \min\{1/2, 1/\gamma'\} - 1/\beta$, there exists a constant $C > 0$ such that*

$$\|T_{h,P_N,\varphi}(f)\|_{\dot{B}_\alpha^{p,q}(\mathbb{R}^n)} \le C\|f\|_{\dot{B}_\alpha^{p,q}(\mathbb{R}^n)}.$$

The constant $C = C_{n,p,q,h,\alpha,\varphi,N,\gamma,\beta}$ is independent of the coefficients of P_N.

Theorem 1.6 *Let φ, h and $T_{h,P_N,\varphi}$ be as in Theorem* 1.5. *Suppose that $\Omega \in \mathcal{F}_\beta(S^1)$ for some $\beta > \max\{2, \gamma'\}$ with satisfying* (1.1). *Then*

(i) *for $\alpha \in \mathbb{R}$ and $\max\{|1/p - 1/2|, |1/q - 1/2|\} < \min\{1/2, 1/\gamma'\} - 1/\beta$, there exists a constant $C > 0$ such that*

$$\|T_{h,P_N,\varphi}(f)\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^2)} \le C\|f\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^2)};$$

(ii) *for $\alpha \in \mathbb{R}$, $1 < q < \infty$ and $|1/p - 1/2| < \min\{1/2, 1/\gamma'\} - 1/\beta$, there exists a constant $C > 0$ such that*

$$\|T_{h,P_N,\varphi}(f)\|_{\dot{B}_\alpha^{p,q}(\mathbb{R}^2)} \le C\|f\|_{\dot{B}_\alpha^{p,q}(\mathbb{R}^2)}.$$

The constant $C = C_{p,q,h,\alpha,\varphi,N,\gamma,\beta}$ is independent of the coefficients of P_N.

Remark 1.3 Under the assumptions on φ in Theorem 1.5, the following facts are obvious (see [20]):

(i) $\lim_{t\to 0}\varphi(t) = 0$ and $\lim_{t\to\infty}|\varphi(t)| = \infty$ if φ is nonnegative and increasing, or nonpositive and decreasing;
(ii) $\lim_{t\to 0}|\varphi(t)| = \infty$ and $\lim_{t\to\infty}\varphi(t) = 0$ if φ is nonnegative and decreasing, or nonpositive and increasing.

Moreover, the inhomogeneous versions of Triebel-Lizorkin spaces and Besov spaces, which are denoted by $F_\alpha^{p,q}(\mathbb{R}^n)$ and $B_\alpha^{p,q}(\mathbb{R}^n)$, respectively, are obtained by adding the term $\|\Phi * f\|_{L^p(\mathbb{R}^n)}$ to the right-hand side of (1.6) or (1.7) with $\sum_{i\in\mathbb{Z}}$ replaced by $\sum_{i\ge 1}$, where $\Phi\in\mathscr{S}(\mathbb{R}^n)$, $\operatorname{supp}(\hat{\Phi})\subset\{\xi:|\xi|\le 2\}$, $\hat{\Phi}(x) > c > 0$ if $|x|\le 5/3$. The following properties are well known (see [13, 14], for example):

$$\begin{aligned}&F_\alpha^{p,q}(\mathbb{R}^n)\sim \dot{F}_\alpha^{p,q}(\mathbb{R}^n)\cap L^p(\mathbb{R}^n)\quad\text{and}\\ &\|f\|_{F_\alpha^{p,q}(\mathbb{R}^n)}\sim\|f\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^n)}+\|f\|_{L^p(\mathbb{R}^n)}\quad(\alpha>0);\end{aligned}\tag{1.9}$$

$$\begin{aligned}&B_\alpha^{p,q}(\mathbb{R}^n)\sim \dot{B}_\alpha^{p,q}(\mathbb{R}^n)\cap L^p(\mathbb{R}^n)\quad\text{and}\\ &\|f\|_{B_\alpha^{p,q}(\mathbb{R}^n)}\sim\|f\|_{\dot{B}_\alpha^{p,q}(\mathbb{R}^n)}+\|f\|_{L^p(\mathbb{R}^n)}\quad(\alpha>0).\end{aligned}\tag{1.10}$$

Hence, by (1.8)-(1.10) and Theorems 1.5-1.6, we get the following conclusion immediately.

Corollary 1.7 *Under the same conditions of Theorems* 1.5 *and* 1.6 *with* $\alpha > 0$, *the operator* $T_{h,P_N,\varphi}$ *is bounded on* $F_\alpha^{p,q}(\mathbb{R}^n)$ *and* $B_\alpha^{p,q}(\mathbb{R}^n)$, *respectively.*

The paper is organized as follows. After recalling and establishing some auxiliary lemmas in Section 2, we give the proofs of our main results in Section 3. It should be pointed out that the methods employed in this paper follow from a combination of ideas and arguments in [3, 19, 20].

Throughout the paper, we let p' denote the conjugate index of p, which satisfies $1/p + 1/p' = 1$. The letter C or c, sometimes with certain parameters, will stand for positive constants not necessarily the same one at each occurrence, but are independent of the essential variables.

2 Auxiliary lemmas

For given polynomial $P_N(t) = \sum_{i=1}^N b_i t^i$, we let $P_\lambda(t) = \sum_{i=1}^\lambda b_i t^i$ for $\lambda\in\{1,2,\ldots,N\}$ and $P_0(t) = 0$ for all $t\in\mathbb{R}$. Without loss of generality, we may assume that $b_\lambda\neq 0$ for $\lambda\in\{1,2,\ldots,N\}$ (or there exist some positive integers $0 < l_1 < l_2 < \cdots < l_d \le N$ such that $P_N(t) = \sum_{i=1}^d b_{l_i} t^{l_i}$ with $b_{l_i}\neq 0$ for all $i\in\{1,2,\ldots,d\}$). Let $k\in\mathbb{Z}$ and $D_k = \{y\in\mathbb{R}^n : 2^k < |y|\le 2^{k+1}\}$. For $\lambda\in\{1,2,\ldots,N\}$ and $\xi\in\mathbb{R}^n$, we define the measures $\{\sigma_{k,\lambda}\}_{k\in\mathbb{Z}}$ by

$$\widehat{\sigma_{k,\lambda}}(\xi) = \int_{D_k} e^{-2\pi i P_\lambda(|y|)y'\cdot\xi}\frac{\Omega(y)h(|y|)}{|y|^n}\,dy.$$

It is clear that

$$T_{h,\Omega,P_N}(f) = \sum_{k\in\mathbb{Z}}\sigma_{k,N} * f.\tag{2.1}$$

We have the following estimates.

Lemma 2.1 *Let $h \in \Delta_\gamma(\mathbb{R}^+)$ for some $\gamma > 1$ and $\Omega \in \tilde{\mathcal{F}}_\beta(S^{n-1})$ for some $\beta > 0$. For $\lambda \in \{1, 2, \ldots, N\}$, $k \in \mathbb{Z}$ and $\xi \in \mathbb{R}^n$, there exists a constant $C > 0$ such that*

(i)

$$\left|\widehat{\sigma_{k,\lambda}}(\xi) - \widehat{\sigma_{k,\lambda-1}}(\xi)\right| \le C\left|2^{(k+1)\lambda} b_\lambda \xi\right|^{1/\lambda}; \tag{2.2}$$

(ii)

$$\left|\widehat{\sigma_{k,\lambda}}(\xi)\right| \le C\left(\log\left|2^{(k+1)\lambda} b_\lambda \xi\right|\right)^{-\beta/\tilde{\gamma}}, \quad \textit{for } \left|2^{(k+1)\lambda} b_\lambda \xi\right| > 1, \tag{2.3}$$

where $\tilde{\gamma} = \max\{2, \gamma'\}$. The constant C is independent of the coefficients of P_λ.

Proof By the change of the variables, we have

$$\begin{aligned} \left|\widehat{\sigma_{k,\lambda}}(\xi) - \widehat{\sigma_{k,\lambda-1}}(\xi)\right| &= \left|\int_{2^k}^{2^{k+1}} \int_{S^{n-1}} \Omega(y')\left(e^{-2\pi i P_\lambda(t) y' \cdot \xi} - e^{-2\pi i P_{\lambda-1}(t) y' \cdot \xi}\right) d\sigma(y') h(t) \frac{dt}{t}\right| \\ &\le C\|\Omega\|_{L^1(S^{n-1})} \|h\|_{\Delta_\gamma(\mathbb{R}^+)} \left|2^{(k+1)\lambda} b_\lambda \xi\right|. \end{aligned} \tag{2.4}$$

On the other hand, it is easy to check that

$$\left|\widehat{\sigma_{k,\lambda}}(\xi)\right| \le C\|\Omega\|_{L^1(S^{n-1})} \|h\|_{\Delta_\gamma(\mathbb{R}^+)}. \tag{2.5}$$

Interpolating between (2.4) and (2.5) implies (2.2). Next, we prove (2.3). Let

$$H_k(\xi, y', \theta) = \int_{2^k}^{2^{k+1}} e^{-2\pi i P_\lambda(t)(y' - \theta) \cdot \xi} \frac{dt}{t}.$$

By Van der Coupt lemma, there exists a constant $C > 0$, which is independent of the coefficients of P_λ and k such that

$$\left|H_k(\xi, y', \theta)\right| \le C \min\left\{1, \left|2^{(k+1)\lambda} b_\lambda \xi \cdot (y' - \theta)\right|^{-1/\lambda}\right\}.$$

For $|2^{(k+1)\lambda} b_\lambda \xi| > 1$, since $t/(\log t)^\beta$ is increasing in (e^β, ∞), we have

$$\left|H_k(\xi, y', \theta)\right| \le C \frac{(\log 2e^{\beta\lambda} |\eta \cdot (y' - \theta)|^{-1})^\beta}{(\log |2^{(k+1)\lambda} b_\lambda \xi|)^\beta}, \tag{2.6}$$

where $\eta = \xi/|\xi|$. Let $\tilde{\gamma}$ be as in Lemma 2.1, by the change of the variables and Hölder's inequality, we have

$$\begin{aligned} \left|\widehat{\sigma_{k,\lambda}}(\xi)\right| &= \left|\int_{2^k}^{2^{k+1}} \int_{S^{n-1}} \Omega(y') e^{-2\pi i P_\lambda(t) y' \cdot \xi} d\sigma(y') h(t) \frac{dt}{t}\right| \\ &\le C\|h\|_{\Delta_\gamma(\mathbb{R}^+)} \left(\int_{2^k}^{2^{k+1}} \left|\int_{S^{n-1}} \Omega(y') e^{-2\pi i P_\lambda(t) y' \cdot \xi} d\sigma(y')\right|^{\gamma'} \frac{dt}{t}\right)^{1/\gamma'} \\ &\le C\left(I_k(\xi)\right)^{1/\tilde{\gamma}}, \end{aligned} \tag{2.7}$$

where

$$I_k(\xi) = \int_{2^k}^{2^{k+1}} \left| \int_{S^{n-1}} \Omega(y') e^{-2\pi i P_\lambda(t) y' \cdot \xi} \, d\sigma(y') \right|^2 \frac{dt}{t}.$$

Note that

$$\begin{aligned} I_k(\xi) &= \int_{2^k}^{2^{k+1}} \left| \int_{S^{n-1}} \Omega(y') e^{-2\pi i P_\lambda(t) y' \cdot \xi} \, d\sigma(y') \right|^2 \frac{dt}{t} \\ &= \int_{2^k}^{2^{k+1}} \iint_{S^{n-1}\times S^{n-1}} \Omega(y') \overline{\Omega(\theta)} e^{-2\pi i P_\lambda(t)(y'-\theta)\cdot \xi} \, d\sigma(y') \, d\sigma(\theta) \frac{dt}{t} \\ &= \iint_{S^{n-1}\times S^{n-1}} H_k(\xi, y', \theta) \Omega(y') \overline{\Omega(\theta)} \, d\sigma(y') \, d\sigma(\theta). \end{aligned}$$

Combining (2.6)-(2.7) with the fact that $\Omega \in \tilde{\mathcal{F}}_\beta(S^{n-1})$, we get (2.3). This proves Lemma 2.1. □

Lemma 2.2 [19, Theorem 1.4] *Let $d \geq 2$ and $\mathcal{P} = (P_1, \ldots, P_d)$ with P_j being real-valued polynomials on $\mathbb{R}^n$. For $1 < p, q < \infty$, the operator $\mathcal{M}_{\mathcal{P}}$ given by*

$$\mathcal{M}_{\mathcal{P}}(f)(x) = \sup_{r>0} \frac{1}{r^n} \int_{|y|\leq r} \left| f\left(x - \mathcal{P}(y)\right) \right| dy$$

satisfies the following $L^p(\ell^q, \mathbb{R}^d)$ inequality

$$\left\| \left(\sum_{i\in\mathbb{Z}} \left| \mathcal{M}_{\mathcal{P}}(f_i) \right|^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^d)} \leq C_{p,q} \left\| \left(\sum_{i\in\mathbb{Z}} |f_i|^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^d)},$$

where $C_{p,q}$ is independent of the coefficients of P_j for all $1 \leq j \leq d$.

Lemma 2.3 [21, Proposition 2.3] *Let $0 < M \leq N$ and $H: \mathbb{R}^M \to \mathbb{R}^M$, $G: \mathbb{R}^N \to \mathbb{R}^N$ be two nonsingular linear transformations. Let $\{a_k\}_{k\in\mathbb{Z}}$ be a lacunary sequence of positive numbers satisfying $\inf_{k\in\mathbb{Z}} a_{k+1}/a_k \geq a > 1$. Let $\Phi(\xi) \in \mathscr{S}(\mathbb{R}^M)$ and $\Phi_k(\xi) = a_k^{-M}\Phi(\xi/a_k)$. Define the transformations J and X_k by*

$$J(f)(x) = f\left(G^t\left(H^t \otimes \mathrm{id}_{\mathbb{R}^{N-M}}\right)x\right)$$

and

$$X_k(f)(x) = J^{-1}\left((\Phi_k \otimes \delta_{\mathbb{R}^{N-M}}) * Jf\right)(x).$$

Here, we use $\delta_{\mathbb{R}^n}$ to denote the Dirac delta function on $\mathbb{R}^n$, J^{-1} denote the inverse transform of J and G^t denote the transpose of G. We have the following inequalities:

$$\left\| \left(\sum_{j\in\mathbb{Z}} \left(\sum_{k\in\mathbb{Z}} \left| X_k(f_j)(\cdot) \right|^2 \right)^{q/2} \right)^{1/q} \right\|_{L^p(\mathbb{R}^N)} \leq C \left\| \left(\sum_{j\in\mathbb{Z}} |f_j(\cdot)|^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^N)} \tag{2.8}$$

for arbitrary functions $\{f_j\} \in L^p(\ell^q, \mathbb{R}^N)$ *and* $1 < p, q < \infty$;

$$\left\| \left(\sum_{j\in\mathbb{Z}} \left(\sum_{k\in\mathbb{Z}} \left| X_k(g_{k,j})(\cdot) \right|^2 \right)^{q/2} \right)^{1/q} \right\|_{L^p(\mathbb{R}^N)} \leq \left\| \left(\sum_{j\in\mathbb{Z}} \left(\sum_{k\in\mathbb{Z}} \left| g_{k,j}(\cdot) \right|^2 \right)^{q/2} \right)^{1/q} \right\|_{L^p(\mathbb{R}^N)} \tag{2.9}$$

for arbitrary functions $\{g_{k,j}\}_{k,j} \in L^p(\ell^q(\ell^2), \mathbb{R}^N)$ *and* $1 < p, q < \infty$.

Lemma 2.4 *For any* $\lambda \in \{1, 2, \ldots, N\}$ *and arbitrary functions* $\{g_{k,j}\}_{k,j} \in L^p(\ell^q(\ell^2), \mathbb{R}^n)$, *there exists a constant* $C > 0$, *which is independent of the coefficients of* P_λ *such that*

$$\left\| \left(\sum_{j\in\mathbb{Z}} \left(\sum_{k\in\mathbb{Z}} |\sigma_{k,\lambda} * g_{k,j}|^2 \right)^{q/2} \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)} \leq C \left\| \left(\sum_{j\in\mathbb{Z}} \left(\sum_{k\in\mathbb{Z}} |g_{k,j}|^2 \right)^{q/2} \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)} \tag{2.10}$$

for $\max\{|1/p - 1/2|, |1/q - 1/2|\} < \min\{1/2, 1/\gamma'\}$.

Proof Since $\|h\|_{\Delta_\gamma(\mathbb{R}^+)} \leq C\|h\|_{\Delta_2(\mathbb{R}^+)}$ when $\gamma \geq 2$, we may assume that $1 < \gamma \leq 2$. By duality, it suffices to prove (2.10) for $2 < p, q < 2\gamma/(2-\gamma)$. Given functions $\{f_j\}$ with $\|\{f_j\}\|_{L^{(p/2)'}(\ell^{(q/2)'}, \mathbb{R}^n)} \leq 1$. It follows from the similar argument as in getting (7.7) in [4] that

$$\int_{\mathbb{R}^n} \left| \sigma_{k,\lambda} * g_{k,j}(x) \right|^2 f_j(x)\, dx \leq C \int_{\mathbb{R}^n} \left| g_{k,j}(x) \right|^2 \mathcal{M}_{P_\lambda}(f_j)(x)\, dx, \tag{2.11}$$

where

$$\mathcal{M}_{P_\lambda}(f_j)(x) = \int_{2^k}^{2^{k+1}} \int_{S^{n-1}} \left| f_j\left(x + P_\lambda(t)y'\right) \right| \left| \Omega\left(y'\right) \right| d\sigma\left(y'\right) \left| h(t) \right|^{2-\gamma} \frac{dt}{t}.$$

By Hölder's inequality, we have

$$\begin{aligned} \mathcal{M}_{P_\lambda}(f_j)(x) &\leq \|h\|_{\Delta_\gamma(\mathbb{R}^+)}^{2-\gamma} \int_{S^{n-1}} \left(\int_{2^k}^{2^{k+1}} \left| f\left(x + P_\lambda(t)y'\right) \right|^{\gamma'/2} \frac{dt}{t} \right)^{2/\gamma'} \left| \Omega\left(y'\right) \right| d\sigma\left(y'\right) \\ &\leq C \int_{S^{n-1}} \left| \Omega\left(y'\right) \right| \left(\sup_{r>0} \frac{1}{r} \int_{|t|<r} \left| f\left(x + P_\lambda(t)y'\right) \right|^{\gamma'/2} dt \right)^{2/\gamma'} d\sigma\left(y'\right). \end{aligned}$$

By Lemma 2.2 and Minkowski's inequality, we have for $\gamma'/2 < u, v < \infty$,

$$\left\| \left(\sum_{j\in\mathbb{Z}} \left| \mathcal{M}_{P_\lambda}(f_j) \right|^v \right)^{1/v} \right\|_{L^u(\mathbb{R}^n)} \leq C \left\| \left(\sum_{j\in\mathbb{Z}} |f_j|^v \right)^{1/v} \right\|_{L^u(\mathbb{R}^n)}. \tag{2.12}$$

Thus, by (2.11)-(2.12), we get

$$\begin{aligned} &\left\| \left(\sum_{j\in\mathbb{Z}} \left(\sum_{k\in\mathbb{Z}} |\sigma_{k,\lambda} * g_{k,j}|^2 \right)^{q/2} \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)}^2 \\ &\quad = \sup_{\|\{f_j\}\|_{L^{(p/2)'}(\ell^{(q/2)'}, \mathbb{R}^n)} \leq 1} \int_{\mathbb{R}^n} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \left| \sigma_{k,\lambda} * g_{k,j}(x) \right|^2 f_j(x)\, dx \\ &\quad \leq C \sup_{\|\{f_j\}\|_{L^{(p/2)'}(\ell^{(q/2)'}, \mathbb{R}^n)} \leq 1} \int_{\mathbb{R}^n} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \left| g_{k,j}(x) \right|^2 \mathcal{M}_{P_\lambda}(f_j)(x)\, dx \end{aligned}$$

$$\begin{aligned}
&\leq C \sup_{\|\{f_j\}\|_{L^{(p/2)'}(\ell^{(q/2)'},\mathbb{R}^n)}\leq 1} \left\| \left(\sum_{j\in\mathbb{Z}} \left| \mathcal{M}_{P_\lambda}(f_j)\right|^{\nu} \right)^{1/\nu} \right\|_{L^u(\mathbb{R}^n)} \\
&\quad \times \left\| \left(\sum_{j\in\mathbb{Z}} \left(\sum_{k\in\mathbb{Z}} |g_{k,j}|^2 \right)^{q/2} \right)^{1/q} \right\|^2_{L^p(\mathbb{R}^n)} \\
&\leq C \left\| \left(\sum_{j\in\mathbb{Z}} \left(\sum_{k\in\mathbb{Z}} |g_{k,j}|^2 \right)^{q/2} \right)^{1/q} \right\|^2_{L^p(\mathbb{R}^n)},
\end{aligned}$$

where we take $u = (p/2)'$ and $\nu = (q/2)'$. This completes the proof of Lemma 2.4. □

Lemma 2.5 [20, Lemma 2.1] *Let* Γ, φ *be as in Theorem* 1.5. *Suppose that* $h \in \Delta_\gamma(\mathbb{R}^+)$ *for some* $\gamma > 1$, *then we have* $h(\varphi^{-1})\Gamma(\varphi^{-1}) \in \Delta_\gamma(\mathbb{R}^+)$.

Lemma 2.6 *Let* $T_{h,P_N,\varphi}$ *be given as in Theorem* 1.5. *Then*

(i) *if* φ *is nonnegative and increasing,* $T_{h,P_N,\varphi}(f) = T_{h(\varphi^{-1})\Gamma(\varphi^{-1}),\Omega,P_N}(f)$;
(ii) *if* φ *is nonnegative and decreasing,* $T_{h,P_N,\varphi}(f) = -T_{h(\varphi^{-1})\Gamma(\varphi^{-1}),\Omega,P_N}(f)$;
(iii) *if* φ *is nonpositive and decreasing,* $T_{h,P_N,\varphi}(f) = T_{h(\varphi^{-1})\Gamma(\varphi^{-1}),\tilde{\Omega},P_N}(f)$;
(iv) *if* φ *is nonpositive and increasing,* $T_{h,P_N,\varphi}(f) = -T_{h(\varphi^{-1})\Gamma(\varphi^{-1}),\tilde{\Omega},P_N}(f)$,

where $\tilde{\Omega}(y) = \Omega(-y)$.

Proof We can get easily this lemma by Remark 1.3 and the similar arguments as in getting [20, Lemma 2.3]. The details are omitted. □

3 Proofs of main results

For a function $\phi \in C_0^\infty(\mathbb{R})$ such that $\phi(t) \equiv 1$ for $|t| \leq 1/2$ and $\phi(t) \equiv 0$ for $|t| \geq 1$. Let $\psi(t) = \phi(t^2)$, and define the measures $\{\tau_{k,\lambda}\}$ by

$$\widehat{\tau_{k,\lambda}}(\xi) = \widehat{\sigma_{k,\lambda}}(\xi) \prod_{j=\lambda+1}^{N} \psi\left(\left|2^{(k+1)j} b_j \xi\right|\right) - \widehat{\sigma_{k,\lambda-1}}(\xi) \prod_{j=\lambda}^{N} \psi\left(\left|2^{(k+1)j} b_j \xi\right|\right) \tag{3.1}$$

for $k \in \mathbb{Z}$ and $\lambda \in \{1, 2, \ldots, N\}$, where we use convention $\prod_{j\in\emptyset} a_j = 1$. It is easy to check that

$$\sigma_{k,N} = \sum_{\lambda=1}^{N} \tau_{k,\lambda}. \tag{3.2}$$

In addition, by Lemma 2.1, we can obtain the following estimates (see also in [4, (7.39)])

$$\left|\widehat{\tau_{k,\lambda}}(\xi)\right| \leq C\left|2^{(k+1)\lambda} b_\lambda \xi\right|^{1/\lambda}; \tag{3.3}$$

$$\left|\widehat{\tau_{k,\lambda}}(\xi)\right| \leq C\left(\log\left|2^{(k+1)\lambda} b_\lambda \xi\right|\right)^{-\beta/\tilde{\gamma}}, \quad \text{for } \left|2^{(k+1)\lambda} b_\lambda \xi\right| > 1, \tag{3.4}$$

where $\tilde{\gamma} = \max\{2, \gamma'\}$.

Now, we are in a position to prove our main results.

Proof of Theorem 1.1 It follows from (2.1) and (3.2) that

$$T_{h,\Omega,P_N}(f) = \sum_{\lambda=1}^{N} \sum_{k\in\mathbb{Z}} \tau_{k,\lambda} * f := \sum_{\lambda=1}^{N} B_\lambda(f). \tag{3.5}$$

By (3.5), to prove Theorem 1.1, it suffices to prove that for any $\lambda \in \{1, 2, \ldots, N\}$,

$$\|B_\lambda(f)\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^n)} \le C\|f\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^n)} \tag{3.6}$$

for $\max\{|1/p - 1/2|, |1/q - 1/2|\} < \min\{1/2, 1/\gamma'\} - 1/\beta$ and $\alpha \in \mathbb{R}$, where $C = C_{n,h,p,q,\alpha,\lambda,\gamma,\beta}$ is independent of the coefficients of P_λ for $\lambda \in \{1, 2, \ldots, N\}$.

For $\lambda \in \{1, 2, \ldots, N\}$, we choose a Schwartz function $\Upsilon \in \mathscr{S}(\mathbb{R}^+)$ such that

$$0 \le \Upsilon(t) \le 1, \qquad \operatorname{supp}(\Upsilon) \subset [2^{-\lambda}, 2^\lambda], \qquad \sum_{k\in\mathbb{Z}} \Upsilon_k(t)^2 = 1,$$

where $\Upsilon_k(t) = \Upsilon(2^{k\lambda}t)$. Define the operator S_k by

$$\widehat{S_k f}(\xi) := \Upsilon_k(|b_\lambda \xi|)\hat{f}(\xi).$$

Let $\widehat{\Theta_k}(\xi) = \Upsilon_k(|b_\lambda \xi|)$. It is clear that $\Theta_k \in \mathscr{S}(\mathbb{R}^n)$ and

$$S_k f(x) = \Theta_k * f(x).$$

Observe that we can write

$$B_\lambda(f) = \sum_{k\in\mathbb{Z}} \tau_{k,\lambda} * \left(\sum_{j\in\mathbb{Z}} S_{j+k} S_{j+k} f\right) = \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} S_{j+k}(\tau_{k,\lambda} * S_{j+k} f) := \sum_{j\in\mathbb{Z}} B_\lambda^j(f). \tag{3.7}$$

Invoking the Littlewood-Paley theory and Plancherel's theorem, we get

$$\|B_\lambda^j(f)\|_{L^2(\mathbb{R}^n)}^2 \le C\sum_{k\in\mathbb{Z}} \int_{E_{j+k}} \left|\widehat{\tau_{k,\lambda}}(\xi)\right|^2 \left|\hat{f}(\xi)\right|^2 d\xi,$$

where

$$E_{j+k} = \left\{\xi \in \mathbb{R}^n : 2^{-(j+k+1)\lambda} \le |b_\lambda \xi| \le 2^{-(j+k-1)\lambda}\right\}.$$

This together with (3.3)-(3.4) yields

$$\|B_\lambda^j(f)\|_{L^2(\mathbb{R}^n)} \le CC_j\|f\|_{L^2(\mathbb{R}^n)},$$

where

$$C_j = \begin{cases} |j|^{-\beta/\tilde{\gamma}}, & j \le -1; \\ 2^{-|j|}, & j > -1, \end{cases}$$

and $\tilde{\gamma} = \max\{2, \gamma'\}$. In other words (by (1.8)),

$$\|B_\lambda^j(f)\|_{\dot{F}_0^{2,2}(\mathbb{R}^n)} \le CC_j\|f\|_{\dot{F}_0^{2,2}(\mathbb{R}^n)}. \tag{3.8}$$

Next, we will show that

$$\|B_\lambda^j(f)\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^n)} \le C\|f\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^n)} \tag{3.9}$$

for $\max\{|1/p-1/2|,|1/q-1/2|\}<\min\{1/2,1/\gamma'\}$, $\alpha\in\mathbb{R}$, $j\in\mathbb{Z}$ and $\lambda\in\{1,2,\ldots,N\}$. To prove (3.9), it suffices to prove that

$$\left\|\left(\sum_{i\in\mathbb{Z}}\left|B_\lambda^j(g_i)\right|^q\right)^{1/q}\right\|_{L^p(\mathbb{R}^n)}\le C\left\|\left(\sum_{i\in\mathbb{Z}}|g_i|^q\right)^{1/q}\right\|_{L^p(\mathbb{R}^n)} \tag{3.10}$$

for $\max\{|1/p-1/2|,|1/q-1/2|\}<\min\{1/2,1/\gamma'\}$ and $\{g_i\}\in L^p(\ell^q,\mathbb{R}^n)$, where C is independent of the coefficients of P_λ. In fact, (3.10) implies (3.9), that is,

$$\begin{aligned}\|B_\lambda^j(f)\|_{\dot F_\alpha^{p,q}(\mathbb{R}^n)}&=\left\|\left(\sum_{i\in\mathbb{Z}}2^{-i\alpha q}\left|\Psi_i*B_\lambda^j(f)\right|^q\right)^{1/q}\right\|_{L^p(\mathbb{R}^n)}\\&\le\left\|\left(\sum_{i\in\mathbb{Z}}\left|B_\lambda^j\left(2^{-i\alpha}\Psi_i*f\right)\right|^q\right)^{1/q}\right\|_{L^p(\mathbb{R}^n)}\\&\le C\left\|\left(\sum_{i\in\mathbb{Z}}2^{-i\alpha q}|\Psi_i*f|^q\right)^{1/q}\right\|_{L^p(\mathbb{R}^n)}\\&=C\|f\|_{\dot F_\alpha^{p,q}(\mathbb{R}^n)},\end{aligned}$$

which leads to (3.9). Now, we return to the proof of (3.10). Using Lemmas 2.3-2.4, the definition of $\tau_{k,\lambda}$ and the similar argument in getting [19, Proposition 2.3], one can check that

$$\left\|\left(\sum_{i\in\mathbb{Z}}\left(\sum_{k\in\mathbb{Z}}|\tau_{k,\lambda}*g_{k,i}|^2\right)^{q/2}\right)^{1/q}\right\|_{L^p(\mathbb{R}^n)}\le C\left\|\left(\sum_{i\in\mathbb{Z}}\left(\sum_{k\in\mathbb{Z}}|g_{k,i}|^2\right)^{q/2}\right)^{1/q}\right\|_{L^p(\mathbb{R}^n)} \tag{3.11}$$

for $\max\{|1/p-1/2|,|1/q-1/2|\}<\min\{1/2,1/\gamma'\}$. Using Lemma 2.3 again, for $1<p,q<\infty$ and arbitrary functions $\{g_i\}_{i\in\mathbb{Z}}\in L^p(\ell^q,\mathbb{R}^n)$, we have

$$\left\|\left(\sum_{i\in\mathbb{Z}}\left(\sum_{k\in\mathbb{Z}}|S_kg_i|^2\right)^{q/2}\right)^{1/q}\right\|_{L^p(\mathbb{R}^n)}\le C\left\|\left(\sum_{i\in\mathbb{Z}}|g_i|^q\right)^{1/q}\right\|_{L^p(\mathbb{R}^n)}. \tag{3.12}$$

By duality and (3.11)-(3.12), we have

$$\begin{aligned}\left\|\left(\sum_{i\in\mathbb{Z}}\left|B_\lambda^j(g_i)\right|^q\right)^{1/q}\right\|_{L^p(\mathbb{R}^n)}&=\sup_{\|\{f_i\}\|_{L^{p'}(\ell^{q'},\mathbb{R}^n)}\le1}\left|\int_{\mathbb{R}^n}\sum_{i\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}S_{j+k}(\tau_{k,\lambda}*S_{j+k}g_i)(x)f_i(x)\,dx\right|\\&\le C\sup_{\|\{f_i\}\|_{L^{p'}(\ell^{q'},\mathbb{R}^n)}\le1}\left\|\left(\sum_{i\in\mathbb{Z}}\left(\sum_{k\in\mathbb{Z}}\left|S_{j+k}^*f_i\right|^2\right)^{q'/2}\right)^{1/q'}\right\|_{L^{p'}(\mathbb{R}^n)}\\&\quad\times\left\|\left(\sum_{i\in\mathbb{Z}}\left(\sum_{k\in\mathbb{Z}}|\tau_{k,\lambda}*S_{j+k}g_i|^2\right)^{q/2}\right)^{1/q}\right\|_{L^p(\mathbb{R}^n)}\\&\le C\left\|\left(\sum_{i\in\mathbb{Z}}\left(\sum_{k\in\mathbb{Z}}|S_{j+k}g_i|^2\right)^{q/2}\right)^{1/q}\right\|_{L^p(\mathbb{R}^n)}\\&\le C\left\|\left(\sum_{i\in\mathbb{Z}}|g_i|^q\right)^{1/q}\right\|_{L^p(\mathbb{R}^n)}.\end{aligned}$$

This proves (3.10). Interpolating between (3.8) and (3.9) (see [14, 22]), for $\max\{|1/p - 1/2|, |1/q - 1/2|\} < \min\{1/2, 1/\gamma'\} - 1/\beta$ and $\alpha \in \mathbb{R}$, we can obtain $\epsilon \in (0,1)$ such that $\epsilon\beta/\tilde{\gamma} > 1$ and

$$\left\| B_\lambda^j(f) \right\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^n)} \leq C C_j^\epsilon \|f\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^n)}, \tag{3.13}$$

which together with (3.7) implies (3.6) and completes the proof of Theorem 1.1. □

Proof of Theorem 1.2 The proof of Theorem 1.2 is to copy the arguments in proving [19, Theorem 1.2]. By Theorem 1.1 and (1.8), for $|1/p - 1/2| < \min\{1/2, 1/\gamma'\} - 1/\beta$, there exists a constant $C > 0$ such that

$$\left\| T_{h,\Omega,P_N}(f) \right\|_{L^p(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)}. \tag{3.14}$$

Then for $|1/p - 1/2| < \min\{1/2, 1/\gamma'\} - 1/\beta$, $1 < q < \infty$ and $\alpha \in \mathbb{R}$, we have

$$\begin{aligned} \left\| T_{h,\Omega,P_N}(f) \right\|_{\dot{B}_\alpha^{p,q}(\mathbb{R}^n)} &= \left(\sum_{i \in \mathbb{Z}} 2^{-i\alpha q} \left\| \Psi_i * T_{h,\Omega,P_N}(f) \right\|_{L^p(\mathbb{R}^n)}^q \right)^{1/q} \\ &= \left(\sum_{i \in \mathbb{Z}} \left\| T_{h,\Omega,P_N}\left(2^{-i\alpha} \Psi_i * f\right) \right\|_{L^p(\mathbb{R}^n)}^q \right)^{1/q} \\ &\leq C \left(\sum_{i \in \mathbb{Z}} 2^{-i\alpha q} \|\Psi_i * f\|_{L^p(\mathbb{R}^n)}^q \right)^{1/q} \\ &= C \|f\|_{\dot{B}_\alpha^{p,q}(\mathbb{R}^n)}. \end{aligned}$$

Theorem 1.2 is proved. □

Proofs of Theorems 1.5-1.6 Using Lemmas 2.5-2.6 and Theorems 1.1-1.2, we get Theorem 1.5. Theorem 1.6 follows from Theorem 1.5 and Remark 1.1. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Three authors worked jointly in drafting and approving the final manuscript.

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