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Finite summation formulas involving binomial coefficients, harmonic numbers and generalized harmonic numbers

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Abstract
A variety of identities involving harmonic numbers and generalized harmonic numbers have been investigated since the distant past and involved in a wide range of diverse fields such as analysis of algorithms in computer science, various branches of number theory, elementary particle physics and theoretical physics. Here we show how one can obtain further interesting identities about certain finite series involving binomial coefficients, harmonic numbers and generalized harmonic numbers by applying the usual differential operator to a known identity.
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1 Introduction and preliminaries

The generalized harmonic numbers $H_n^{(s)}$ of order s are defined by (*cf.* [1]; see also [2, 3], [4, p.156] and [5, Section 3.5])

$$H_n^{(s)} := \sum_{j=1}^{n} \frac{1}{j^s} \quad (n \in \mathbb{N}; s \in \mathbb{C}), \tag{1.1}$$

and

$$H_n := H_n^{(1)} = \sum_{j=1}^{n} \frac{1}{j} \quad (n \in \mathbb{N}) \tag{1.2}$$

are the harmonic numbers. Here $\mathbb{N}$ and $\mathbb{C}$ denote the set of positive integers and the set of complex numbers, respectively, and we assume that

$$H_0 := 0, \qquad H_0^{(s)} := 0 \quad \bigl(s \in \mathbb{C} \setminus \{0\}\bigr) \quad \text{and} \quad H_0^{(0)} := 1.$$

The generalized harmonic functions $H_n^{(s)}(z)$ are defined by (see [2, 6]; see also [7, 8])

$$H_n^{(s)}(z) := \sum_{j=1}^{n} \frac{1}{(j+z)^s} \quad \bigl(n \in \mathbb{N}; s \in \mathbb{C} \setminus \mathbb{Z}^-; \mathbb{Z}^- := \{-1, -2, -3, \ldots\}\bigr) \tag{1.3}$$

so that, obviously,

$$H_n^{(s)}(0) = H_n^{(s)}.$$

Equation (1.1) can be written in the following form:

$$H_n^{(s)} = \zeta(s) - \zeta(s, n+1) \quad \bigl(\Re(s) > 1; n \in \mathbb{N}\bigr) \tag{1.4}$$

by recalling the well-known (easily-derivable) relationship between the Riemann zeta function $\zeta(s)$ and the Hurwitz (or generalized) zeta function $\zeta(s,a)$ (see [4, Eq. 2.3(9)])

$$\zeta(s) = \zeta(s, n+1) + \sum_{k=1}^{n} k^{-s} \quad \bigl(n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}\bigr). \tag{1.5}$$

The polygamma functions $\psi^{(n)}(s)$ ($n \in \mathbb{N}$) are defined by

$$\psi^{(n)}(s) := \frac{d^{n+1}}{dz^{n+1}} \log\Gamma(s) = \frac{d^n}{ds^n}\psi(s) \quad \bigl(n \in \mathbb{N}_0; s \in \mathbb{C} \setminus \mathbb{Z}_0^- := \mathbb{Z}^- \cup \{0\}\bigr), \tag{1.6}$$

where $\Gamma(s)$ is the familiar gamma function, and the psi-function ψ is defined by

$$\psi(s) := \frac{d}{ds}\log\Gamma(s) \quad \text{and} \quad \psi^{(0)}(s) = \psi(s) \quad \bigl(s \in \mathbb{C} \setminus \mathbb{Z}_0^-\bigr).$$

A well-known (and potentially useful) relationship between the polygamma functions $\psi^{(n)}(s)$ and the generalized zeta function $\zeta(s,a)$ is given by

$$\psi^{(n)}(s) = (-1)^{n+1} n! \sum_{k=0}^{\infty} \frac{1}{(k+s)^{n+1}} = (-1)^{n+1} n! \zeta(n+1, s) \quad \bigl(n \in \mathbb{N}; s \in \mathbb{C} \setminus \mathbb{Z}_0^-\bigr). \tag{1.7}$$

It is also easy to have the following expression (*cf.* [4, Eq. 1.2(54)]):

$$\psi^{(m)}(s+n) - \psi^{(m)}(s) = (-1)^m m! H_n^{(m+1)}(s-1) \quad (m, n \in \mathbb{N}_0), \tag{1.8}$$

which immediately gives $H_n^{(s)}$ another expression for $H_n^{(s)}$ as follows (*cf.* [9, Eq. (20)]):

$$H_n^{(m)} = \frac{(-1)^{m-1}}{(m-1)!}\bigl[\psi^{(m-1)}(n+1) - \psi^{(m-1)}(1)\bigr] \quad (m \in \mathbb{N}; n \in \mathbb{N}_0). \tag{1.9}$$

By using finite differences, Spivey [10] presented many summation formulas involving binomial coefficients, the Stirling numbers of the first and second kind and harmonic numbers, two of which are chosen to be recalled here: [10, Identity 14]

$$\sum_{k=0}^{n} \binom{n}{k} H_k = 2^n \left(H_n - \sum_{k=1}^{n} \frac{1}{k 2^k} \right) \quad (n \in \mathbb{N}_0), \tag{1.10}$$

which was also given by Paule and Schneider [11, Eq. (39)] by deriving it automatically by means of the *Sigma* package in [12], together with the following identity [10, Identity 20]:

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}H_k = -\frac{1}{n} \quad (n \in \mathbb{N}). \tag{1.11}$$

Paule and Schneider [11] proved five conjectured harmonic number identities similar to those arising in the context of supercongruences for Apéry numbers, one of which is recalled here as follows [11, Eq. (5)]:

$$\sum_{j=0}^{n}(1 - 5jH_j + 5jH_{n-j})\binom{n}{j}^5 = (-1)^n\sum_{j=0}^{n}\binom{n}{j}^2\binom{n+j}{j}. \tag{1.12}$$

Greene and Knuth [13, p.10] recorded six commonly used identities that involve both binomial coefficients and harmonic numbers, two of which are recalled here:

$$\sum_{j=1}^{n}H_j = (n+1)H_n - n \quad (n \in \mathbb{N}_0); \tag{1.13}$$

$$\sum_{j=1}^{n}\binom{j}{m}H_j = \binom{n+1}{m+1}\left(H_{n+1} - \frac{1}{m+1}\right) \quad (m, n \in \mathbb{N}_0). \tag{1.14}$$

Alzer *et al.* [14, Eq. (3.62)] proved, by using the principle of mathematical induction, that

$$\sum_{j=1}^{n}\frac{H_j}{j} = \frac{1}{2}\left[(H_n)^2 + H_n^{(2)}\right] \quad (n \in \mathbb{N}). \tag{1.15}$$

By using (1.15) in conjunction with the following elementary identity (see [2]):

$$H_{j+1} = H_j + \frac{1}{j+1}, \tag{1.16}$$

we obtain

$$\sum_{j=1}^{n}\frac{H_j}{j+1} = \frac{1}{2}\left[(H_{n+1})^2 - H_{n+1}^{(2)}\right] \quad (n \in \mathbb{N}). \tag{1.17}$$

Chu and De Donno [15] made use of the classical hypergeometric summation theorems to derive several striking identities for harmonic numbers other than those discovered recently by Paule and Schneider [11], one of which is recalled below [15, Thereoem 1].

$$\begin{aligned}&\sum_{k=0}^{n}\binom{n+\mu n}{k}\binom{n+\lambda n}{n-k}H_{\lambda n+k}\\&\quad = \binom{2n+\lambda n+\mu n}{n}(H_{\lambda n+n} + H_{\lambda n+\mu n+n} - H_{\lambda n+\mu n+2n}) \quad (\lambda, \mu \in \mathbb{N}_0).\end{aligned} \tag{1.18}$$

One interesting special case of (1.18) is when we set $\mu = 0$. We thus find that

$$\sum_{k=0}^{n}\binom{n}{k}\binom{n+\lambda n}{n-k}H_{\lambda n+k} = \binom{2n+\lambda n}{n}(2H_{\lambda n+n} - H_{\lambda n+2n}), \tag{1.19}$$

which can be further specialized, with $\lambda = 0$, to the following form:

$$\sum_{k=0}^{n} \binom{n}{k}^2 H_k = \binom{2n}{n} (2H_n - H_{2n}). \tag{1.20}$$

Dattolli and Srivastava [16] proposed several generating functions involving harmonic numbers by making use of an interesting approach based on the umbral calculus. Subsequently, Cvijović [17] showed the truth of the conjectured relations in [16] by using simple analytical arguments.

For a concise and beautiful description of these numbers, we refer also to Wolfram-MathWorld's website [18].

As we have seen in the above brief eclectic review, harmonic and generalized harmonic numbers are involved in a variety of useful identities. Of course, certain interesting properties of harmonic and generalized harmonic numbers have been studied (see, *e.g.*, [19]). Here we aim at presenting further interesting identities about certain interesting finite series associated with binomial coefficients, harmonic numbers and generalized harmonic numbers.

2 Finite-series involving binomial coefficients, harmonic numbers and generalized harmonic numbers

As the illustrative identities in Section 1, we consider certain interesting identities about finite-series involving binomial coefficients, harmonic numbers and generalized harmonic numbers. We begin by recalling a known formula (*cf.* [20, p.362, Entry (55.4.8)]; see also [2, Eq. (2.6)]):

$$\begin{aligned} &\sum_{j=1}^{n} (-1)^j \frac{(a+1)_j}{(b+1)_j} \binom{n}{j} \bigl[\psi(a+1+j) - \psi(a+1)\bigr] \\ &\quad = \frac{(b-a)_n}{(b+1)_n} \bigl[\psi(b-a) - \psi(b-a+n)\bigr] \\ &\quad \bigl(n \in \mathbb{N}_0; a, b \in \mathbb{C} \setminus \mathbb{Z}^-; b-a \in \mathbb{C} \setminus \mathbb{Z}_0^-\bigr), \end{aligned} \tag{2.1}$$

where $(\alpha)_n$ denotes the Pochhammer symbol defined (for $\alpha \in \mathbb{C}$) by

$$(\alpha)_n := \begin{cases} 1 & (n = 0), \\ \alpha(\alpha+1)\cdots(\alpha+n-1) & (n \in \mathbb{N}). \end{cases} \tag{2.2}$$

Differentiating each side of (2.1) with respect to the variables a and b, respectively, using (1.8) and considering the following easily derivable identities:

$$\frac{d}{d\alpha} (\alpha)_n = (\alpha)_n H_n^{(1)}(\alpha - 1) \quad \bigl(n \in \mathbb{N}_0; \alpha \in \mathbb{C} \setminus \mathbb{Z}_0^-\bigr) \tag{2.3}$$

and

$$\frac{d}{d\alpha} \frac{1}{(\alpha)_n} = -\frac{H_n^{(1)}(\alpha - 1)}{(\alpha)_n} \quad \bigl(n \in \mathbb{N}_0; \alpha \in \mathbb{C} \setminus \mathbb{Z}_0^-\bigr), \tag{2.4}$$

we obtain the following formulas in Theorem 1.

Theorem 1 *Each of the following identities holds true*:

$$\begin{aligned}&\sum_{j=1}^{n}(-1)^j\frac{(a+1)_j}{(b+1)_j}\binom{n}{j}\bigl[\bigl\{H_j^{(1)}(a)\bigr\}^2-H_j^{(2)}(a)\bigr]\\&\quad=\frac{(b-a)_n}{(b+1)_n}\bigl[\bigl\{H_n^{(1)}(b-a-1)\bigr\}^2-H_n^{(2)}(b-a-1)\bigr]\quad\bigl(n\in\mathbb{N}_0;a,b\in\mathbb{C}\setminus\mathbb{Z}^-\bigr)\end{aligned}\tag{2.5}$$

and

$$\begin{aligned}&\sum_{j=1}^{n}(-1)^{j+1}\frac{(a+1)_j}{(b+1)_j}\binom{n}{j}H_j^{(1)}(a)H_j^{(1)}(b)\\&\quad=\frac{(b-a)_n}{(b+1)_n}\bigl[H_n^{(2)}(b-a-1)-H_n^{(1)}(b-a-1)\bigl\{H_n^{(1)}(b-a-1)-H_n^{(1)}(b)\bigr\}\bigr]\\&\quad\bigl(n\in\mathbb{N}_0;a,b\in\mathbb{C}\setminus\mathbb{Z}^-\bigr).\end{aligned}\tag{2.6}$$

Setting $a = b - 1 = 0$ in (2.1), (2.5) and (2.6) and using (1.3) and (1.8), we get certain interesting finite-sum identities involving binomial coefficients and harmonic numbers, respectively, asserted by Corollary 1.

Corollary 1 *Each of the following identities holds true*:

$$\sum_{j=1}^{n}\frac{(-1)^{j+1}}{j+1}\binom{n}{j}H_j=\frac{H_n}{n+1}\quad(n\in\mathbb{N}_0);\tag{2.7}$$

$$\sum_{j=1}^{n}\frac{(-1)^{j+1}}{j+1}\binom{n}{j}\bigl[H_j^{(2)}-(H_j)^2\bigr]=\frac{1}{n+1}\bigl[H_n^{(2)}-(H_n)^2\bigr]\quad(n\in\mathbb{N}_0)\tag{2.8}$$

and

$$\sum_{j=1}^{n}\frac{(-1)^{j+1}}{j+1}\binom{n}{j}H_j(H_{j+1}-1)=\frac{1}{n+1}\left[H_n^{(2)}-\frac{n}{n+1}H_n\right]\quad(n\in\mathbb{N}_0).\tag{2.9}$$

Remark 1 In the course of presenting a closed-form evaluation of some useful series involving the generalized zeta function $\zeta(s,a)$, Choi *et al.* [21] made use of the identity (2.7) without its proof. Choi and Srivastava [2] proved Eq. (2.7) as a special case of (2.1) here and presented another illustrative proof.

We will try to express a class of the following finite sums involving harmonic numbers and binomial coefficients as given above:

$$\sum_{j=1}^{n}\frac{(-1)^{j+1}}{(j+1)^k}\binom{n}{j}H_j\quad(n\in\mathbb{N}_0;k\in\mathbb{N}).\tag{2.10}$$

Here we give the answers for $k = 2$ and $k = 3$ in (2.10) asserted by the following lemma.

Lemma 1 *Each of the following identities holds true*:

$$\sum_{j=1}^{n} \frac{(-1)^{j+1}}{(j+1)^2} \binom{n}{j} H_j = \frac{1}{2(n+1)} \bigl[(H_{n+1})^2 - H_{n+1}^{(2)}\bigr] \quad (n \in \mathbb{N}_0) \tag{2.11}$$

and

$$\sum_{j=1}^{n} \frac{(-1)^{j+1}}{(j+1)^3} \binom{n}{j} H_j = \frac{1}{2(n+1)} \sum_{j=1}^{n} \frac{1}{j+1} \bigl[(H_{j+1})^2 - H_{j+1}^{(2)}\bigr] \quad (n \in \mathbb{N}_0). \tag{2.12}$$

Proof We will prove only (2.11) by using the same method as in [2, pp.2224-2225]. A similar argument will establish (2.12). We first recall two basic relations for binomial coefficients:

$$\binom{n+1}{j} = \binom{n}{j} + \binom{n}{j-1} \quad \text{and} \quad \binom{n}{j-1} = \frac{j}{n+1} \binom{n+1}{j}. \tag{2.13}$$

We let the left-hand side of (2.11) be

$$f_n := \sum_{j=1}^{n} \frac{(-1)^{j+1}}{(j+1)^2} \binom{n}{j} H_j \tag{2.14}$$

so that, using the first one of (2.13),

$$\begin{aligned} f_{n+1} &= \frac{(-1)^n}{(n+2)^2} H_{n+1} \\ &\quad + \sum_{j=1}^{n} \frac{(-1)^{j+1}}{(j+1)^2} \left[\binom{n}{j} + \binom{n}{j-1} \right] H_j \\ &= \frac{(-1)^n}{(n+2)^2} H_{n+1} + f_n + \sum_{j=1}^{n} \frac{(-1)^{j+1}}{(j+1)^2} \binom{n}{j-1} H_j. \end{aligned} \tag{2.15}$$

We now see that, using the second one of (2.13),

$$\begin{aligned} &\sum_{j=1}^{n} \frac{(-1)^{j+1}}{(j+1)^2} \binom{n}{j-1} H_j \\ &\quad = \frac{1}{n+1} \sum_{j=1}^{n} (-1)^{j+1} \binom{n+1}{j} \frac{j}{(j+1)^2} H_j \\ &\quad = \frac{1}{n+1} \left[\sum_{j=1}^{n} \frac{(-1)^{j+1}}{j+1} \binom{n+1}{j} H_j - \sum_{j=1}^{n} \frac{(-1)^{j+1}}{(j+1)^2} \binom{n+1}{j} H_j \right]. \end{aligned} \tag{2.16}$$

By using the identity in (2.7), we find that

$$\sum_{j=1}^{n} \frac{(-1)^{j+1}}{j+1} \binom{n+1}{j} H_j = \frac{1-(-1)^n}{n+2} H_{n+1} \quad (n \in \mathbb{N}_0). \tag{2.17}$$

We also have

$$\sum_{j=1}^{n} \frac{(-1)^{j+1}}{(j+1)^2} \binom{n+1}{j} H_j = f_{n+1} - \frac{(-1)^n}{(n+2)^2} H_{n+1} \quad (n \in \mathbb{N}_0). \tag{2.18}$$

Thus, substituting from (2.17) and (2.18) into (2.16), we obtain

$$\begin{aligned} &\sum_{j=1}^{n} \frac{(-1)^{j+1}}{(j+1)^2} \binom{n}{j-1} H_j \\ &\quad = \frac{1}{n+1} \left[\frac{1-(-1)^n}{n+2} H_{n+1} - f_{n+1} + \frac{(-1)^n}{(n+2)^2} H_{n+1} \right] \quad (n \in \mathbb{N}_0). \end{aligned} \tag{2.19}$$

Finally, it follows from (2.15) and (2.19) that

$$(n+2) f_{n+1} - (n+1) f_n = \frac{H_{n+1}}{n+2}.$$

Let $a_n := (n+1) f_n$ so that we have

$$a_{n+1} - a_n = \frac{H_{n+1}}{n+2} \quad \text{and} \quad a_1 = \frac{1}{2} H_1 = \frac{1}{2}. \tag{2.20}$$

By telescoping this last sum (2.20), we obtain

$$a_n = (n+1) f_n = \sum_{j=1}^{n} \frac{H_j}{j+1}. \tag{2.21}$$

Applying (1.17) to (2.21), we get the desired identity (2.11). □

Applying (2.7) and (2.11) to (2.9) and considering (2.8), we obtain two interesting identities asserted by the following theorem.

Theorem 2 *Each of the following identities holds true*:

$$\sum_{j=1}^{n} \frac{(-1)^{j+1}}{j+1} \binom{n}{j} (H_j)^2 = \frac{1}{2(n+1)} \left[3H_n^{(2)} - (H_n)^2 \right] \quad (n \in \mathbb{N}_0) \tag{2.22}$$

and

$$\sum_{j=1}^{n} \frac{(-1)^{j+1}}{j+1} \binom{n}{j} H_j^{(2)} = \frac{1}{2(n+1)} \left[5H_n^{(2)} - 3(H_n)^2 \right] \quad (n \in \mathbb{N}_0). \tag{2.23}$$

Differentiating (2.5) and (2.6) with respect to a and observing the following identity:

$$\frac{d}{d\alpha} H_j^{(\ell)}(\alpha) = -\ell H_j^{(\ell)}(\alpha) \quad (\ell \in \mathbb{N}), \tag{2.24}$$

we obtain further interesting identities involving binomial coefficients and generalized harmonic functions asserted by the following theorem.

Theorem 3 *Each of the following identities holds true*:

$$\begin{aligned}
&\sum_{j=1}^{n} (-1)^{j+1} \frac{(a+1)_j}{(b+1)_j} \binom{n}{j} \bigl[\bigl\{ H_j^{(1)}(a) \bigr\}^3 - 3H_j^{(1)}(a) H_j^{(2)}(a) + 2H_j^{(3)}(a) \bigr] \\
&\quad = \frac{(b-a)_n}{(b+1)_n} \bigl[\bigl\{ H_n^{(1)}(b-a-1) \bigr\}^3 - 3H_n^{(1)}(b-a-1) H_n^{(2)}(b-a-1) + 2H_n^{(3)}(b-a-1) \bigr] \\
&\quad \bigl(n \in \mathbb{N}_0; a, b \in \mathbb{C} \setminus \mathbb{Z}^- \bigr)
\end{aligned} \tag{2.25}$$

and

$$\begin{aligned}
&\sum_{j=1}^{n} (-1)^{j+1} \frac{(a+1)_j}{(b+1)_j} \binom{n}{j} H_j^{(1)}(b) \bigl[\bigl\{ H_j^{(1)}(a) \bigr\}^2 - H_j^{(2)}(a) \bigr] \\
&\quad = \frac{(b-a)_n}{(b+1)_n} \bigl[2H_n^{(3)}(b-a-1) - 2H_n^{(1)}(b-a-1) H_n^{(2)}(b-a-1) \\
&\qquad + \bigl(\bigl\{ H_n^{(1)}(b-a-1) \bigr\}^2 - H_n^{(2)}(b-a-1) \bigr) \bigl\{ H_n^{(1)}(b-a-1) - H_n^{(1)}(b) \bigr\} \bigr] \\
&\quad \bigl(n \in \mathbb{N}_0; a, b \in \mathbb{C} \setminus \mathbb{Z}^- \bigr).
\end{aligned} \tag{2.26}$$

Setting $a = b - 1 = 0$ in (2.25) and (2.26), we find certain interesting identities and using (2.8), respectively, assert the following corollary.

Corollary 2 *Each of the following identities holds true*:

$$\begin{aligned}
&\sum_{j=1}^{n} \frac{(-1)^{j+1}}{j+1} \binom{n}{j} \bigl[(H_j)^3 - 3H_j H_j^{(2)} + 2H_j^{(3)} \bigr] \\
&\quad = \frac{1}{n+1} \bigl[(H_n)^3 - 3H_n H_n^{(2)} + 2H_n^{(3)} \bigr] \quad (n \in \mathbb{N}_0)
\end{aligned} \tag{2.27}$$

and

$$\begin{aligned}
&\sum_{j=1}^{n} \frac{(-1)^{j+1}}{j+1} \binom{n}{j} (H_{j+1} - 1) \bigl[(H_j)^2 - H_j^{(2)} \bigr] \\
&\quad = \frac{1}{n+1} \left[2H_n^{(3)} - 2H_n H_n^{(2)} + \frac{n}{n+1} \bigl\{ (H_n)^2 - H_n^{(2)} \bigr\} \right] \quad (n \in \mathbb{N}_0).
\end{aligned} \tag{2.28}$$

Remark 2 As in getting the results in Theorem 3, it is seen that a variety of interesting identities involving the generalized harmonic numbers can be obtained by applying the differential operator to the parameters of known formulas.

3 Inverse relations and a question

By using the known orthogonal relation

$$\sum_{k=j}^{n} (-1)^{k+j} \binom{n}{k} \binom{k}{j} = \delta_{nj} \quad (n \geq j; n, j \in \mathbb{N}_0), \tag{3.1}$$

with δ_{nj} the Kronecker delta ($\delta_{nn} = 1$, $\delta_{nj} = 0$ if $n \neq j$) and a manipulation of double series

$$\sum_{k=0}^{n} \sum_{j=0}^{k} A_{k,j} = \sum_{j=0}^{n} \sum_{k=j}^{n} A_{k,j}, \tag{3.2}$$

it is easy to find the following simplest inverse relation (see [22, Chapter 2]):

$$a_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} b_k \quad \Leftrightarrow \quad b_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} a_k. \tag{3.3}$$

Applying this inverse relation to the identities in Section 2, we obtain many formulas involving binomial coefficients, harmonic numbers and generalized harmonic numbers asserted by the following corollary.

Corollary 3 *Each of the following identities holds true*:

$$\sum_{j=1}^{n} \frac{(-1)^{j+1}}{j+1} \binom{n}{j} \left[H_j^{(2)} - \frac{j}{j+1} H_j \right] = \frac{H_n(H_{n+1} - 1)}{n+1} \quad (n \in \mathbb{N}_0); \tag{3.4}$$

$$\sum_{j=1}^{n} \frac{(-1)^{j+1}}{j+1} \binom{n}{j} \left[(H_{j+1})^2 - H_{j+1}^{(2)} \right] = \frac{2H_n}{(n+1)^2} \quad (n \in \mathbb{N}_0); \tag{3.5}$$

$$\sum_{j=1}^{n} \frac{(-1)^{j+1}}{j+1} \binom{n}{j} \left[3H_j^{(2)} - (H_j)^2 \right] = \frac{2(H_n)^2}{n+1} \quad (n \in \mathbb{N}_0); \tag{3.6}$$

$$\sum_{j=1}^{n} \frac{(-1)^{j+1}}{j+1} \binom{n}{j} \left[5H_j^{(2)} - 3(H_j)^2 \right] = \frac{2H_n^{(2)}}{n+1} \quad (n \in \mathbb{N}_0); \tag{3.7}$$

$$\begin{aligned} &\sum_{j=1}^{n} \frac{(-1)^{j+1}}{j+1} \binom{n}{j} (H_{j+1} - 1) \left[2H_j^{(3)} - 2H_j H_j^{(2)} + \frac{j}{j+1} \left\{ (H_j)^2 - H_j^{(2)} \right\} \right] \\ &\quad = \frac{(H_{n+1} - 1)[(H_n)^2 - H_n^{(2)}]}{n+1} \quad (n \in \mathbb{N}_0). \end{aligned} \tag{3.8}$$

It is observed that Eqs. (2.7), (2.8) and (2.27) are of the following form:

$$\sum_{j=1}^{n} (-1)^{j+1} \binom{n}{j} a_j = a_n \quad (n \in \mathbb{N}) \quad \text{and} \quad a_0 = 0. \tag{3.9}$$

By using the first one of (2.13), we find an identity in the following lemma.

Lemma 2 *If Eq.* (3.9) *holds true*, *then we obtain the following identity*:

$$\sum_{j=1}^{n} (-1)^{j+1} j \binom{n}{j} a_j = n(a_n - a_{n-1}) \quad (n \in \mathbb{N}). \tag{3.10}$$

Applying Eq. (3.10) to Eqs. (2.7), (2.8) and (2.27), we get some interesting identities asserted by the following corollary.

Corollary 4 *Each of the following identities holds true*:

$$\sum_{j=1}^{n}(-1)^{j+1}\frac{j}{j+1}\binom{n}{j}H_j = \frac{1}{n+1}(1-H_{n-1}) \quad (n \in \mathbb{N}); \tag{3.11}$$

$$\sum_{j=1}^{n}(-1)^{j+1}\frac{j}{j+1}\binom{n}{j}\bigl[H_j^{(2)} - (H_j)^2\bigr]$$

$$= \frac{1}{n+1}\bigl[(H_n)^2 - H_n^{(2)}\bigr] - \frac{2}{n}H_{n-1} \quad (n \in \mathbb{N}); \tag{3.12}$$

$$\sum_{j=1}^{n}(-1)^{j}\frac{j}{j+1}\binom{n}{j}\bigl[(H_j)^3 - 3H_jH_j^{(2)} + 2H_j^{(3)}\bigr]$$

$$= \frac{1}{n+1}\bigl[(H_n)^3 - 3H_nH_n^{(2)} + 2H_n^{(3)}\bigr] \quad (n \in \mathbb{N}_0). \tag{3.13}$$

Question We conclude this paper by posing a natural question: *Under what conditions does* Eq. (3.9) *hold true*?

Competing interests
The author declares that he has no competing interests.

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