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General convergence analysis of projection methods for a system of variational inequalities in q-uniformly smooth Banach spaces

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Abstract

In this paper, we introduce and consider a system of variational inequalities involving two different operators in q-uniformly smooth Banach spaces. We suggest and analyze a new explicit projection method for solving the system under some more general conditions. Our results extend and unify the results of Verma (Appl. Math. Lett. 18:1286-1292, 2005) and Yao, Liou and Kang (J. Glob. Optim., 2011, doi:10.1007/s10898-011-9804-0) and some other previously known results.

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1 Introduction

Let E and E^* be a real Banach space and the dual space of E, respectively. Let C be a subset of E and $q > 1$ be a real number. The generalized duality mapping $J_q : E \to E^*$ is defined by

$$J_q(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^q, \|x^*\| = \|x\|^{q-1}\}$$

for all x in E. In particular, $J = J_2$ is called the normalized duality mapping and $J_q(x) = \|x\|^{q-2}J_2(x)$ for $x \neq 0$. If E is a Hilbert space, then $J = I$, the identity mapping. It is well known that if E is smooth, then J_q is single-valued, which is denoted by j_q.

Recall the variational inequality problem of finding $x^* \in C$ such that

$$\langle Tx^*, x - x^* \rangle \geq 0, \quad \forall x \in C, \tag{1.1}$$

where $T : C \to H$ is a nonlinear mapping. Variational inequalities theory, which was introduced by Stampacchia [1], emerged as an interesting and fascinating branch of mathematical and engineering sciences. The ideas and techniques of variational inequalities have been applied in structural analysis, economics, optimization, operations research fields. It has been shown that variational inequalities provide the most natural, direct, simple and efficient framework for a general treatment of some unrelated problems arising in various fields of pure and applied sciences. In recent years, there have been considerable activities in the development of numerical techniques including projection methods, Wiener-Hopf equations, auxiliary principle and descent framework for solving variational inequalities; see [1–14] and the references therein. These activities have motivated us to generalize and extend the variational inequalities and related optimization problems in several directions using novel techniques.

Recently, Verma [11] proved the strong convergence of two-step projection method for solving the following system of variational inequality problems in a Hilbert space: Find $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \rho T(y^*) + x^* - y^*, x - x^* \rangle \geq 0, & \forall x \in C, \rho > 0, \\ \langle \eta T(x^*) + y^* - x^*, x - y^* \rangle \geq 0, & \forall x \in C, \eta > 0. \end{cases} \tag{1.2}$$

In order to solve problem (1.2), Verma [11] introduced the following projection method:

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n P_C[y_n - \rho T(y_n)], \\ y_n = (1-\beta_n)x_n + \beta_n P_C[x_n - \eta T(x_n)], \quad n \geq 0, \end{cases} \tag{1.3}$$

where P_C is the projection of a Hilbert space H onto C. This method contains several previously known projection schemes as special cases, while some have been applied to problems arising, especially, from complementarity problems, convex quadratic programming and other variational problems; see [2–4, 11] and the references therein.

Very recently, Yao *et al.* [12] considered the following system of variational inequality problems in 2-uniformly smooth Banach spaces: Find $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \rho T_1(y^*) + x^* - y^*, j(x - x^*) \rangle \geq 0, & \forall x \in C, \rho > 0, \\ \langle \eta T_2(x^*) + y^* - x^*, j(x - y^*) \rangle \geq 0, & \forall x \in C, \eta > 0, \end{cases} \tag{1.4}$$

where T_1, T_2 are two different nonlinear operators. Moreover, they modified projection method to system (1.4) in Banach spaces and introduced the following iterative method:

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n \Pi_C[y_n - \rho T_1(y_n)], \\ y_n = \Pi_C[x_n - \eta T_2(x_n)], \quad n \geq 0, \end{cases} \tag{1.5}$$

where Π_C is a sunny nonexpansive retraction from E onto C.

One question arises naturally: Do Yao *et al.*'s new projection methods work for two bivariate nonlinear operators in 2-uniformly smooth Banach spaces, or more generally, in q-uniformly smooth Banach spaces with $q > 1$, under more general control conditions?

In order to give some affirmative answers to the question raised above, we introduce the following system of bivariate variational inequality problems in q-uniformly smooth Banach spaces: Find $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \rho T_1(y^*, x^*) + x^* - y^*, j_q(x - x^*) \rangle \geq 0, & \forall x \in C, \rho > 0, \\ \langle \eta T_2(x^*, y^*) + y^* - x^*, j_q(x - y^*) \rangle \geq 0, & \forall x \in C, \eta > 0. \end{cases} \tag{1.6}$$

The purpose of this paper is not only to show that the projection technique can be extended to the system of bivariate variational inequality problems in q-uniformly smooth Banach spaces, but also to suggest and analyze a new explicit iterative method, which includes the previously known projection methods as special cases, and whose convergence analysis is proved under some more general conditions. Our results extend and unify the corresponding results of [5, 7, 11, 12] and many others.

2 Preliminaries

Let E and E^* be a real Banach space and the dual space of E, respectively. Let C be a nonempty closed convex subset of E, and let $S(E) = \{x \in E : \|x\| = 1\}$. Then the norm of E is said to be Gâteaux differentiable if the following limit

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in S(E)$. In this case, E is called smooth. The norm of E is said to be uniformly Gâteaux differentiable if for each $y \in S(E)$, the limit above is attained uniformly for $x \in S(E)$. The norm of E is called Fréchet differentiable if for each $x \in S(E)$, the limit above is attained uniformly for $y \in S(E)$. The norm of E is called uniformly Fréchet differentiable if the limit above is attained uniformly for $x, y \in S(E)$. It is well known that (uniform) Fréchet differentiability of the norm of E implies (uniform) Gâteaux differentiability of the norm of E.

Recall that $\psi_E : [0,\infty) \to [0,\infty)$, the modulus of smoothness of E, is defined by

$$\psi_E(t) = \sup\left\{\frac{1}{2}\bigl(\|x + y\| + \|x - y\|\bigr) - 1 : x \in S(E), \|y\| \le t\right\}.$$

A Banach space E is said to be uniformly smooth if $\frac{\psi_E(t)}{t}$ as $t \to 0$. A Banach space E is said to be q-uniformly smooth, if there exists a fixed constant $c > 0$ such that $\psi_E(t) \le ct^q$ ($q > 1$). It is well known that E is uniformly smooth if and only if the norm of E is uniformly Fréchet differentiable. If E is q-uniformly smooth, then $q \le 2$ and E is uniformly smooth, and hence the norm of E is uniformly Fréchet differentiable, in particular, the norm of E is Fréchet differentiable. It is well known that Hilbert and Lebesgue L^p ($p > 1$) spaces are uniformly smooth. More precisely, L^p is $\min\{p, 2\}$-uniformly smooth for every $p > 1$.

In order to prove our main results, we also need the following concepts and lemmas.

Let C be a nonempty closed and convex subset of a real Banach space E, and let K be a nonempty subset of C. Let $\Pi_C : C \to K$ be a mapping, and Π_C is said to be:

(a) sunny if for each $x \in C$ and $t \in [0,1]$, we have $\Pi_C[tx + (1-t)\Pi_C] = \Pi_C x$;
(b) a retraction of C onto K if $\Pi_C x = x$, $\forall x \in K$;
(c) a sunny nonexpansive retraction if Π_C is sunny, nonexpansive and retraction onto K.

Definition 2.1 An operator $T : C \to E$ is said to be μ-Lipschitz continuous in the first variable if there exists a constant $\mu > 0$ such that

$$\bigl\|T(x,\cdot) - T(y,\cdot)\bigr\| \le \mu\|x - y\|, \quad \forall x, y \in C.$$

Definition 2.2 Let C be a nonempty closed convex subset of a smooth Banach space E, and $J_q : E \to E^*$ is a generalized duality mapping. A bivariate operator $T : C \times C \to E$ is said to be:

(i) r-strongly accretive in the first variable if there exists a constant $r > 0$ such that

$$\langle T(x,\cdot) - T(y,\cdot), j_q(x-y)\rangle \geq r\|x-y\|^q, \quad \forall x,y \in C.$$

(ii) γ-cocoercive in the first variable if there exists a constant $\gamma > 0$ such that

$$\langle T(x,\cdot) - T(y,\cdot), j_q(x-y)\rangle \geq -\gamma \|T(x,\cdot) - T(y,\cdot)\|^q, \quad \forall x,y \in C.$$

(iii) relaxed (γ, r)-cocoercive in the first variable if there exist constants $\gamma, r > 0$ such that

$$\langle T(x,\cdot) - T(y,\cdot), j_q(x-y)\rangle \geq -\gamma \|T(x,\cdot) - T(y,\cdot)\|^q + r\|x-y\|^q, \quad \forall x,y \in C.$$

Remark 2.1 In Definition 2.2, the r-strongly accretive operator includes the r-strongly monotone and the r-strongly accretive ones defined in [11, 12] as special cases.

Remark 2.2 Obviously, an r-strongly accretive operator must be a relaxed (γ, r)-cocoercive whenever $\gamma = 0$, but the converse is not true. Therefore the relaxed (γ, r)-cocoercive operator is more general than r-strongly accretive one.

Lemma 2.1 [15] *Let E be a real q-uniformly smooth Banach space, then there exists a constant $C_q > 0$ such that*

$$\|x+y\|^q \leq \|x\|^q + q\langle y, j_q(x)\rangle + C_q\|y\|^q, \quad \forall x,y \in E.$$

In particular, if E is a real 2-uniformly smooth Banach space, then there exists the best smooth constant $K > 0$ such that

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x)\rangle + 2\|Ky\|^2, \quad \forall x,y \in E.$$

Lemma 2.2 [16, 17] *Let C be a nonempty subset of a smooth Banach space E, and let $\Pi_C : E \to C$ be a retraction. Then Π_C is sunny and nonexpansive if and only if*

$$\langle u - \Pi_C[u], J(y - \Pi_C[u])\rangle \leq 0, \quad \forall u \in E, y \in C.$$

Lemma 2.3 [18] *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\gamma_n)a_n + \sigma_n, \quad n \geq 0,$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\sigma_n\}$ is a sequence in R such that

(i) $\sum_{n=0}^{\infty} \gamma_n = \infty$;

(ii) $\limsup_{n\to\infty} \sigma_n/\gamma_n \leq 0$ *or* $\sum_{n=0}^{\infty} |\sigma_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

3 Main results

By Lemma 2.2, we establish the equivalence between the system of variational inequalities (1.6) and the fixed point problem with projection technique, that is, $(x^*, y^*) \in C \times C$ is a solution of the system of variational inequalities (1.6) if and only if

$$x^* = \Pi_C\big[y^* - \rho T_1\big(y^*, x^*\big)\big], \tag{3.1a}$$

$$y^* = \Pi_C\big[x^* - \eta T_2\big(x^*, y^*\big)\big]. \tag{3.1b}$$

This alternative formula enables us to suggest and analyze a two-step explicit projection method for solving system (1.6), and this is the main motivation of our next result.

Theorem 3.1 *Let C be a nonempty closed convex subset of a q-uniformly smooth Banach space E. Let $T_i : C \times C \to E$ be relaxed (γ_i, r_i)-cocoercive and μ_i-Lipschitz continuous in the first variable, $i = 1, 2$. For arbitrarily chosen initial points $(x_0, y_0) \in C \times C$, define sequences $\{x_n\}$ and $\{y_n\}$ in the following manner*:

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n \Pi_C[y_n - \rho T_1(y_n, x_n)], \\ y_{n+1} = (1-\beta_n)x_{n+1} + \beta_n \Pi_C[x_{n+1} - \eta T_2(x_{n+1}, y_n)], \end{cases} \tag{3.2}$$

where Π_C is a sunny nonexpansive retraction from E onto C, the following conditions are satisfied:

(i) $\alpha_n, \beta_n \in [0,1]$, $\sum_{n=0}^{\infty} \alpha_n = \infty$ *and* $\sum_{n=0}^{\infty} \alpha_n(1-\beta_{n-1}) < \infty$;

(ii) $0 < \rho < (\frac{q(r_1-\gamma_1\mu_1^q)}{C_q\mu_1^q})^{\frac{1}{q-1}}$ *and* $0 < \eta < (\frac{q(r_2-\gamma_2\mu_2^q)}{C_q\mu_2^q})^{\frac{1}{q-1}}$.

Then the sequences $\{x_n\}$ and $\{y_n\}$ converge to x^* and y^*, respectively, where (x^*, y^*) is a solution of the system of variational inequalities (1.6).

Proof Let $(x^*, y^*) \in C \times C$ be a solution of (1.6). From (3.1a) and (3.2), we have

$$\begin{aligned} \|x_{n+1} - x^*\| &= \big\|(1-\alpha_n)x_n + \alpha_n \Pi_C\big[y_n - \rho T_1(y_n, x_n)\big] - x^*\big\| \\ &\le (1-\alpha_n)\|x_n - x^*\| + \alpha_n \big\|\Pi_C\big[y_n - \rho T_1(y_n, x_n)\big] - \Pi_C\big[y^* - \rho T_1\big(y^*, x^*\big)\big]\big\| \\ &\le (1-\alpha_n)\|x_n - x^*\| + \alpha_n \big\|y_n - y^* - \rho\big(T_1(y_n, x_n) - T_1\big(y^*, x^*\big)\big)\big\|. \end{aligned} \tag{3.3}$$

Since the operator T_1 is relaxed (γ_1, r_1)-cocoercive and μ_1-Lipschitz continuous definition in the first variable, it follows from Lemma 2.1 that

$$\begin{aligned} &\big\|y_n - y^* - \rho\big(T_1(y_n, x_n) - T_1\big(y^*, x^*\big)\big)\big\|^q \\ &\quad = \|y_n - y^*\|^q - \rho q\big\langle T_1(y_n, x_n) - T_1\big(y^*, x^*\big), j_q\big(y_n - y^*\big)\big\rangle \\ &\qquad + C_q\rho^q\big\|T_1(y_n, x_n) - T_1\big(y^*, x^*\big)\big\|^q \\ &\quad \le \|y_n - y^*\|^q - \rho q\big(-\gamma_1\big\|T_1(y_n, x_n) - T_1\big(y^*, x^*\big)\big\|^q + r_1\|y_n - y^*\|^q\big) \\ &\qquad + C_q\rho^q\big\|T_1(y_n, x_n) - T_1\big(y^*, x^*\big)\big\|^q \\ &\quad \le \big(1 - \rho q r_1 + \rho q\gamma_1\mu_1^q + C_q\rho^q\mu_1^q\big)\|y_n - y^*\|^q, \end{aligned}$$

which implies that

$$\|y_n - y^* - \rho(T_1(y_n, x_n) - T_1(y^*, x^*))\| \le \theta_1 \|y_n - y^*\|, \tag{3.4}$$

where $\theta_1 = \sqrt[q]{1 - \rho q r_1 + \rho q \gamma_1 \mu_1^q + C_q \rho^q \mu_1^q}$. We can obtain $\theta_1 \in (0,1)$ by condition (ii). Substituting (3.4) into (3.3), we have

$$\|x_{n+1} - x^*\| \le (1-\alpha_n)\|x_n - x^*\| + \alpha_n \theta_1 \|y_n - y^*\|. \tag{3.5}$$

Similarly, it follows from (3.1b) and (3.2) that

$$\begin{aligned} &\|y_{n+1} - y^*\| \\ &\quad = \|(1-\beta_n)x_{n+1} + \beta_n \Pi_C[x_{n+1} - \eta T_2(x_{n+1}, y_n)] - y^*\| \\ &\quad \le (1-\beta_n)\|x_{n+1} - y^*\| + \beta_n \|\Pi_C[x_{n+1} - \eta T_2(x_{n+1}, y_n)] - \Pi_C[x^* - \eta T_2(x^*, y^*)]\| \\ &\quad \le (1-\beta_n)\|x_{n+1} - y^*\| + \beta_n \|x_{n+1} - x^* - \eta(T_2(x_{n+1}, y_n) - T_2(x^*, y^*))\|. \end{aligned} \tag{3.6}$$

Since the operator T_2 is relaxed (γ_2, r_2)-cocoercive and μ_2-Lipschitz continuous definition in the first variable, it follows that

$$\begin{aligned} &\|x_{n+1} - x^* - \eta(T_2(x_{n+1}, y_n) - T_2(x^*, y^*))\|^q \\ &\quad = \|x_{n+1} - x^*\|^q - \eta q\langle T_2(x_{n+1}, y_n) - T_2(x^*, y^*), j_q(x_{n+1} - x^*)\rangle \\ &\qquad + C_q \eta^q \|T_2(x_{n+1}, y_n) - T_2(x^*, y^*)\|^q \\ &\quad \le \|x_{n+1} - x^*\|^q - \eta q(-\gamma_2 \|T_2(x_{n+1}, y_n) - T_2(x^*, y^*)\|^q + r_2 \|x_{n+1} - x^*\|^q) \\ &\qquad + C_q \eta^q \|T_2(x_{n+1}, y_n) - T_2(x^*, y^*)\|^q \\ &\quad \le (1 - \eta q r_2 + \eta q \gamma_2 \mu_2^q + C_q \eta^q \mu_2^q)\|x_{n+1} - x^*\|^q, \end{aligned}$$

which implies that

$$\|x_{n+1} - x^* - \eta(T_2(x_{n+1}, y_n) - T_2(x^*, y^*))\| \le \theta_2 \|x_{n+1} - x^*\|, \tag{3.7}$$

where $\theta_2 = \sqrt[q]{1 - \eta q r_2 + \eta q \gamma_2 \mu_2^q + C_q \eta^q \mu_2^q}$. We can obtain $\theta_2 \in (0,1)$ from condition (ii). Substituting (3.7) into (3.6), we have

$$\begin{aligned} \|y_{n+1} - y^*\| &\le (1-\beta_n)\|x_{n+1} - y^*\| + \beta_n \theta_2 \|x_{n+1} - x^*\| \\ &\le (1-\beta_n)\|x_{n+1} - x^*\| + (1-\beta_n)\|x^* - y^*\| + \beta_n \theta_2 \|x_{n+1} - x^*\| \\ &= [1 - (1-\theta_2)\beta_n]\|x_{n+1} - x^*\| + (1-\beta_n)\|x^* - y^*\| \\ &< \|x_{n+1} - x^*\| + (1-\beta_n)\|x^* - y^*\|, \end{aligned} \tag{3.8}$$

that is,

$$\|y_n - y^*\| \le \|x_n - x^*\| + (1-\beta_{n-1})\|x^* - y^*\|. \tag{3.9}$$

It follows from (3.5) and (3.9) that

$$\begin{aligned}\|x_{n+1}-x^*\| &\le (1-\alpha_n)\|x_n-x^*\| + \alpha_n\theta_1\big(\|x_n-x^*\| + (1-\beta_{n-1})\|x^*-y^*\|\big)\\ &\le \big[1-(1-\theta_1)\alpha_n\big]\|x_n-x^*\| + \alpha_n(1-\beta_{n-1})\|x^*-y^*\|. \end{aligned} \tag{3.10}$$

Since $1-\theta_1>0$, $\sum_{n=0}^{\infty}\alpha_n=\infty$ and $\sum_{n=0}^{\infty}\alpha_n(1-\beta_{n-1})<\infty$, we apply Lemma 2.3 to get

$$\lim_{n\to\infty}\|x_n-x^*\| = 0. \tag{3.11}$$

Combining condition (ii), (3.9) and (3.11), we have

$$\lim_{n\to\infty}\|y_n-y^*\| = 0. \tag{3.12}$$

It shows that $\lim_{n\to\infty}x_n=x^*$, $\lim_{n\to\infty}y_n=y^*$, respectively, satisfying the system of variational inequalities (1.6). This completes the proof. □

Theorem 3.2 *Let C be a nonempty closed convex subset of a q-uniformly smooth Banach space E. Let $T_i : C\times C\to E$ be r_i-strongly accretive and μ_i-Lipschitz continuous in the first variable, $i=1,2$. For arbitrarily chosen initial points $(x_0,y_0)\in C\times C$, define sequences $\{x_n\}$ and $\{y_n\}$ in the following manner*:

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n\Pi_C[y_n-\rho T_1(y_n,x_n)],\\ y_{n+1} = \Pi_C[x_{n+1}-\eta T_2(x_{n+1},y_n)], \end{cases} \tag{3.13}$$

where Π_C is a sunny nonexpansive retraction from E onto C, the following conditions are satisfied:

(i) $\alpha_n\in(0,1)$, $\sum_{n=0}^{\infty}\alpha_n=\infty$;

(ii) $0<\rho<(\frac{qr_1}{C_q\mu_1^q})^{\frac{1}{q-1}}$ *and* $0<\eta<(\frac{qr_2}{C_q\mu_2^q})^{\frac{1}{q-1}}$.

Then the sequences $\{x_n\}$ and $\{y_n\}$ converge to x^* and y^*, respectively, where (x^*,y^*) is a solution of the system of variational inequalities (1.6).

Proof As $\gamma=0$ and $\beta_n=1$, from Remark 2.2, we know that a relaxed $(0,r)$-cocoercive operator reduces to r-strongly accretive, and iterative algorithm (3.2) reduces to (3.13), respectively. Then the conclusion follows immediately from Theorem 3.1. This completes the proof. □

If $T_1,T_2 : C\to E$ are univariate operators, applying Theorem 3.1 to a 2-uniformly smooth Banach space with constant $C_q=2K^2$, we obtain the following result.

Theorem 3.3 *Let C be a nonempty closed convex subset of a 2-uniformly smooth Banach space E. Let $T_i : C\to E$ be relaxed (γ_i,r_i)-cocoercive and μ_i-Lipschitz continuous, $i=1,2$. For arbitrarily chosen initial points $x_0,y_0\in C$, define sequences $\{x_n\}$ and $\{y_n\}$ in the following manner*:

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n\Pi_C[y_n-\rho T_1(y_n)],\\ y_{n+1} = \Pi_C[x_{n+1}-\eta T_2(x_{n+1})], \end{cases} \tag{3.14}$$

where Π_C is a sunny nonexpansive retraction from E onto C, the following conditions are satisfied:

(i) $\alpha_n \in [0,1]$, $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(ii) $0 < \rho < \frac{r_1 - \gamma_1 \mu_1^2}{K^2 \mu_1^2}$ *and* $0 < \eta < \frac{r_2 - \gamma_2 \mu_2^2}{K^2 \mu_2^2}$.

Then the sequences $\{x_n\}$ *and* $\{y_n\}$ *converge to* x^* *and* y^*, *respectively, where* (x^*, y^*) *is a solution of the system of variational inequalities* (1.4).

Since the Hilbert space H is a 2-uniformly smooth Banach space with the best smooth constant $K = \frac{\sqrt{2}}{2}$, from Theorem 3.1 we obtain the following result.

Theorem 3.4 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T_i : C \to H$ be relaxed (γ_i, r_i)-cocoercive and μ_i-Lipschitz continuous, $i = 1, 2$. For arbitrarily chosen initial points $x_0, y_0 \in C$, define sequences $\{x_n\}$ and $\{y_n\}$ in the following manner*:

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n P_C[y_n - \rho T_1(y_n)], \\ y_{n+1} = (1-\beta_n)x_{n+1} + \beta_n P_C[x_{n+1} - \eta T_2(x_{n+1})], \end{cases} \tag{3.15}$$

where P_C is the projection from H onto C, the following conditions are satisfied:

(i) $\alpha_n, \beta_n \in [0,1]$, $\sum_{n=0}^{\infty} \alpha_n = \infty$ *and* $\sum_{n=0}^{\infty} \alpha_n(1-\beta_{n-1}) < \infty$;

(ii) $0 < \rho < \frac{2(r_1 - \gamma_1 \mu_1^2)}{\mu_1^2}$ *and* $0 < \eta < \frac{2(r_2 - \gamma_2 \mu_2^2)}{\mu_2^2}$.

Then the sequences $\{x_n\}$ *and* $\{y_n\}$ *converge to* x^* *and* y^*, *respectively, where* (x^*, y^*) *is a solution of the system of variational inequalities* (1.2).

Remark 3.1 Theorems 3.1 and 3.3 extend Theorem 3.1 of [12] from a 2-uniformly smooth Banach space to a q-uniformly smooth Banach space. Moreover, the underlying operator T is extended to a bivariate operator, and the property defined on T is more general than [12] in convergence analysis.

Remark 3.2 Theorems 3.1 and 3.4 extend Theorem 3.1 of [5, 7, 11] from a real Hilbert space to a q-uniformly smooth Banach space. Moreover, the property defined on the underlying operator T is extended from r-strongly monotone to relaxed (γ, r)-cocoercive, respectively.

Remark 3.3 Algorithm (3.2) includes the projection methods in [2, 5, 7, 11, 12] as special cases and unifies the previously known one-step and two-step projection-type methods in a q-uniformly smooth Banach space. Furthermore, the computation workload of the present explicit projection method is much less than the implicit algorithm in [5] at each iteration step.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
GQF carried out the primary studies for projection methods for a system of variational inequality, participated in its design and coordination. WDJ participated in the convergence analysis and drafted the manuscript. All authors read and approved the final manuscript.

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References
1. Stampacchia, G: Formes bilinearies coercivities sur les ensembles convexes. C. R. Acad. Sci. Paris **258**, 4413-4416 (1964)
2. Verma, RU: Projection methods, algorithms and a new system of nonlinear variational inequalities. Comput. Math. Appl. **41**, 1025-1031 (2001)
3. Nie, H, Liu, Z, Kim, KH, Kang, SM: A system of nonlinear variational inequalities involving strongly monotone and pseudocontractive mappings. Adv. Nonlinear Var. Inequal. **6**, 91-99 (2003)
4. Noor, MA: Some developments in general variational inequalities. Appl. Math. Comput. **152**, 199-277 (2004)
5. Verma, RU: Generalized system for relaxed cocoercive variational inequalities and projection methods. J. Optim. Theory Appl. **121**(1), 203-210 (2004)
6. Noor, MA: Projection methods for nonconvex variational inequalities. Optim. Lett. **3**, 411-418 (2009)
7. Huang, Z, Noor, MA: An explicit projection method for a system of nonlinear variational inequalities with different (γ, r)-cocoercive mappings. Appl. Math. Comput. **190**, 356-361 (2007)
8. Noor, MA, Noor, KI: Projection algorithms for solving system of general variational inequalities. Nonlinear Anal. **70**, 2700-2706 (2009)
9. Noor, MA: Some iterative methods for nonconvex variational inequalities. Comput. Math. Model. **21**(1), 97-108 (2010)
10. Wen, DJ: Projection methods for a generalized system of nonconvex variational inequalities with different nonlinear operators. Nonlinear Anal. **73**, 2292-2297 (2010)
11. Verma, RU: General convergence analysis for two-step projection methods and applications to variational problems. Appl. Math. Lett. **18**, 1286-1292 (2005)
12. Yao, Y, Liou, YC, Kang, SM: Two-step projection methods for a system of variational inequality problems in Banach spaces. J. Glob. Optim. (2011). doi:10.1007/s10898-011-9804-0,
13. Ceng, LC, Teboulle, M, Yao, JC: Weak convergence of an iterative method for pseudomonotone variational inequalities and fixed-point problems. J. Optim. Theory Appl. **146**, 19-31 (2010)
14. Wen, DJ, Long, XJ, Gong, QF: Convergence analysis of projection methods for a new system of general nonconvex variational inequalities. Fixed Point Theory Appl. (2012). doi:10.1186/1687-1812-2012-59
15. Xu, HK: Inequalities in Banach spaces with applications. Nonlinear Anal. **16**, 1127-1138 (1991)
16. Reich, S: Asymptotic behavior of contractions in Banach spaces. J. Math. Anal. Appl. **44**, 57-70 (1973)
17. Xu, HK: Viscosity approximation methods for nonexpansive mappings. J. Math. Anal. Appl. **298**, 279-291 (2004)
18. Xu, HK: Iterative algorithms for nonlinear operators. J. Lond. Math. Soc. **2**, 240-256 (2002)

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