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Existence of positive solutions of the Cauchy problem for a second-order differential equation

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Abstract

In this paper we consider the equation $u''(t) = f(t, u(t), u'(t))$ and prove the unique solvability of the Cauchy problem $u(0) = 0$, $u'(0) = \lambda$ with $\lambda > 0$.

1 Introduction

In [1], Knežević-Miljanović considered the Cauchy problem

$$\begin{cases} u''(t) = P(t)t^a u(t)^\sigma, & t \in (0,1], \\ u(0) = 0, & u'(0) = \lambda, \end{cases} \tag{1}$$

where P is continuous, $a, \sigma, \lambda \in \mathbb{R}$ with $\sigma < 0$ and $\lambda > 0$, and $\int_0^1 |P(t)|t^{a+\sigma}\,dt < \infty$. Moreover, in [2], Kawasaki and Toyoda considered the Cauchy problem

$$\begin{cases} u''(t) = f(t, u(t)), & \text{for almost every } t \in [0,1], \\ u(0) = 0, & u'(0) = \lambda, \end{cases} \tag{2}$$

where f is a mapping from $[0,1] \times (0,\infty)$ into $\mathbb{R}$ and $\lambda \in \mathbb{R}$ with $\lambda > 0$. They proved the unique solvability of Cauchy problem (2) using the Banach fixed point theorem. The theorem in [2] is as follows.

Theorem *Suppose that a mapping f from $[0,1] \times [0,\infty)$ into $\mathbb{R}$ satisfies the following.*

(a) *The mapping $t \longmapsto f(t,u)$ is measurable for any $u \in (0,\infty)$, and the mapping $u \longmapsto f(t,u)$ is continuous for almost every $t \in [0,1]$.*

(b) *$|f(t,u_1)| \geq |f(t,u_2)|$ for almost every $t \in [0,1]$ and for any $u_1, u_2 \in [0,\infty)$ with $u_1 \leq u_2$.*

(c) *There exists $\alpha \in \mathbb{R}$ with $0 < \alpha < \lambda$ such that*

$$\int_0^1 \big|f(t,\alpha t)\big|\,dt < \infty.$$

(d) *There exists $\beta \in \mathbb{R}$ with $\beta > 0$ such that*

$$\left|\frac{\partial f}{\partial u}(t,u)\right| \leq \frac{\beta |f(t,u)|}{u}$$

for almost every $t \in [0,1]$ and for any $u \in (0,\infty)$.

Then there exists $h \in \mathbb{R}$ with $0 < h \leq 1$ such that Cauchy problem (2) *has a unique solution in X, where X is a subset*

$$X = \left\{ u \;\middle|\; \begin{array}{l} u \in C[0,h], u(0) = 0, u'(0) = \lambda \\ \text{and } \alpha t \leq u(t) \text{ for any } t \in [0,h] \end{array} \right\}$$

of $C[0,h]$, which is the class of continuous mappings from $[0,h]$ into $\mathbb{R}$.

The case that $f(t,u(t)) = P(t)t^a u(t)^\sigma$ in the above theorem is the theorem of Knežević-Miljanović [1].

In this paper, we consider the Cauchy problem

$$\begin{cases} u''(t) = f(t,u(t),u'(t)), & \text{for almost every } t \in [0,1], \\ u(0) = 0, \qquad u'(0) = \lambda, \end{cases}$$

where f is a mapping from $[0,1] \times (0,\infty) \times \mathbb{R}$ into $\mathbb{R}$ and $\lambda \in \mathbb{R}$ with $\lambda > 0$. We prove the unique solvability of this Cauchy problem using the Banach fixed point theorem.

In Section 2, we consider the following four cases for u and v.

(I) Decreasing for u in $f(t,u,v)$ (b1) and decreasing for v in $f(t,u,v)$ (b3).
(II) Decreasing for u in $f(t,u,v)$ (b1) and increasing for v in $f(t,u,v)$ (b4).
(III) Increasing for u in $f(t,u,v)$ (b2) and decreasing for v in $f(t,u,v)$ (b3).
(IV) Increasing for u in $f(t,u,v)$ (b2) and increasing for v in $f(t,u,v)$ (b4).

Theorems 2.1, 2.2, 2.3 and 2.4 are the cases of (I), (II), (III) and (IV), respectively.

2 Main results

In this section, we consider the Cauchy problem

$$\begin{cases} u''(t) = f(t,u(t),u'(t)), & \text{for almost every } t \in [0,1], \\ u(0) = 0, \qquad u'(0) = \lambda, \end{cases} \tag{3}$$

where f is a mapping from $[0,1] \times (0,\infty) \times \mathbb{R}$ into $\mathbb{R}$ and $\lambda \in \mathbb{R}$ with $\lambda > 0$.

First, we consider the case of (I).

Theorem 2.1 *Let λ be a real number with $\lambda > 0$. Suppose that a mapping f from $[0,1] \times (0,\infty) \times \mathbb{R}$ into $\mathbb{R}$ satisfies the following:*

(a) *The mapping $t \longmapsto f(t,u,v)$ is measurable for any $(u,v) \in (0,\infty) \times \mathbb{R}$, and the mapping $(u,v) \longmapsto f(t,u,v)$ is continuous for almost every $t \in [0,1]$;*
(b1) *$|f(t,u_1,v)| \geq |f(t,u_2,v)|$ for almost every $t \in [0,1]$, for any $u_1, u_2 \in (0,\infty)$ with $u_1 \leq u_2$ and for any $v \in \mathbb{R}$;*
(b3) *$|f(t,u,v_1)| \geq |f(t,u,v_2)|$ for almost every $t \in [0,1]$, for any $u \in (0,\infty)$ and for any $v_1, v_2 \in \mathbb{R}$ with $v_1 \leq v_2$;*
(c1) *There exist $\alpha_1 \in \mathbb{R}$ with $0 < \alpha_1 < \lambda$ and $\alpha_2 \in \mathbb{R}$ with $\alpha_2 < \lambda$ such that*

$$\int_0^1 \left| f(t,\alpha_1 t,\alpha_2) \right| dt < \infty;$$

(d1) *There exists $\beta_1 \in \mathbb{R}$ with $\beta_1 > 0$ such that*

$$\left|\frac{\partial f}{\partial u}(t,u,v)\right| \le \frac{\beta_1 |f(t,u,v)|}{u}$$

for almost every $t \in [0,1]$, for any $u \in (0,\infty)$ and for any $v \in \mathbb{R}$;

(d2) *There exists $\beta_2 \in \mathbb{R}$ with $\beta_2 > 0$ such that*

$$\left|\frac{\partial f}{\partial v}(t,u,v)\right| \le \beta_2 \left|f(t,u,v)\right|$$

for almost every $t \in [0,1]$, for any $u \in (0,\infty)$ and for any $v \in \mathbb{R}$;

(e) *There exists the limit*

$$\lim_{t\to 0+} \frac{1}{t^2}\int_0^t sf\bigl(s,u(s),u'(s)\bigr)\,ds$$

for any continuously differentiable mapping u from $[0,1]$ into $[0,\infty)$;

(f1) *For α_1 and α_2,*

$$\lim_{t\to 0+} \frac{1}{t^2}\int_0^t s\left|f(s,\alpha_1 s,\alpha_2)\right|\,ds = 0.$$

Then there exists $h \in \mathbb{R}$ with $0 < h \le 1$ such that Cauchy problem (3) *has a unique solution in X, where X is a subset*

$$X = \left\{ u \;\middle|\; \begin{array}{l} u \in C^1[0,h], u(0) = 0, u'(0) = \lambda, \\ \alpha_1 t \le u(t) \textit{ and } \alpha_2 \le u'(t) \textit{ for any } t \in [0,h] \\ \textit{and there exists the limit } \lim_{t\to 0+} \frac{tu'(t)-u(t)}{t^2} \end{array} \right\}$$

of $C^1[0,h]$, which is the class of continuously differentiable mappings from $[0,h]$ into $\mathbb{R}$.

Proof It is noted that $C^1[0,h]$ is a Banach space by the maximum norm

$$\|u\| = \max\bigl\{\max\bigl\{\left|u(t)\right| \mid t \in [0,h]\bigr\}, \max\bigl\{\left|u'(t)\right| \mid t \in [0,h]\bigr\}\bigr\}.$$

Instead of Cauchy problem (3), we consider the integral equation

$$u(t) = \lambda t + \int_0^t (t-s) f\bigl(s,u(s),u'(s)\bigr)\,ds.$$

By condition (c1), there exists $h_1 \in \mathbb{R}$ with $0 < h_1 \le 1$ such that

$$\int_0^{h_1} \left|f(t,\alpha_1 t,\alpha_2)\right|\,dt < \min\left\{\lambda - \alpha_1, \lambda - \alpha_2, \left(\frac{\beta_1}{\alpha_1} + 2\beta_2\right)^{-1}\right\}.$$

By condition (f1), there exists $h \in \mathbb{R}$ with $0 < h \le h_1$ such that

$$\sup_{t\in(0,h]} \frac{1}{t^2}\int_0^t s\left|f(s,\alpha_1 s,\alpha_2)\right|\,ds \le \int_0^{h_1} \left|f(t,\alpha_1 t,\alpha_2)\right|\,dt.$$

Let A be an operator from X into $C^1[0,h]$ defined by

$$Au(t) = \lambda t + \int_0^t (t-s)f\bigl(s,u(s),u'(s)\bigr)\,ds.$$

Since a mapping $t \longmapsto \lambda t$ belongs to X, $X \neq \emptyset$. Moreover, we have $A(X) \subset X$. Indeed, by condition (a), $Au \in C^1[0,h]$, $Au(0) = 0$ and

$$(Au)'(0) = \left[\lambda + \int_0^t f\bigl(s,u(s),u'(s)\bigr)\,ds\right]_{t=0} = \lambda.$$

By conditions (b1) and (b3), we obtain that

$$\begin{aligned} Au(t) &= \lambda t + \int_0^t (t-s)f\bigl(s,u(s),u'(s)\bigr)\,ds \\ &\geq \lambda t - t\int_0^h \bigl|f\bigl(s,u(s),u'(s)\bigr)\bigr|\,ds \\ &\geq \lambda t - t\int_0^h \bigl|f(s,\alpha_1 s,\alpha_2)\bigr|\,ds \\ &\geq \alpha_1 t \end{aligned}$$

and

$$\begin{aligned} (Au)'(t) &= \lambda + \int_0^t f\bigl(s,u(s),u'(s)\bigr)\,ds \\ &\geq \lambda - \int_0^h \bigl|f\bigl(s,u(s),u'(s)\bigr)\bigr|\,ds \\ &\geq \lambda - \int_0^h \bigl|f(s,\alpha_1 s,\alpha_2)\bigr|\,ds \\ &\geq \alpha_2 \end{aligned}$$

for any $t \in [0,h]$. Moreover, by condition (e), there exists the limit

$$\lim_{t\to 0+} \frac{t(Au)'(t) - Au(t)}{t^2} = \lim_{t\to 0+} \frac{1}{t^2}\int_0^t sf\bigl(s,u(s),u'(s)\bigr)\,ds.$$

We will find a fixed point of A. Let φ be an operator from X into $C^1[0,h]$ defined by

$$\varphi[u](t) = \begin{cases} \frac{u(t)}{t} & \text{if } t \in (0,h], \\ \lambda & \text{if } t = 0. \end{cases}$$

Let $\varphi[X]$ be a subset defined by

$$\varphi[X] = \bigl\{\varphi[u] \mid u \in X\bigr\}.$$

Then we have

$$\varphi[X] = \left\{ v \;\middle|\; \begin{array}{l} v \in C^1[0,h], v(0) = \lambda, \\ \alpha_1 \leq v(t) \text{ and } \alpha_2 \leq v(t) + tv'(t) \text{ for any } t \in [0,h] \end{array} \right\}$$

and $\varphi[X]$ is a closed subset of $C^1[0,h]$. Hence it is a complete metric space. Let Φ be an operator from $\varphi[X]$ into $\varphi[X]$ defined by

$$\Phi\varphi[u] = \varphi[Au].$$

By the mean value theorem, for any $u_1, u_2 \in X$, there exist mappings ξ, η such that

$$\begin{aligned}
&f\bigl(t, u_1(t), u_1'(t)\bigr) - f\bigl(t, u_2(t), u_2'(t)\bigr) \\
&\quad = \frac{\partial f}{\partial u}\bigl(t, \xi(t), u_1'(t)\bigr)\bigl(u_1(t) - u_2(t)\bigr) + \frac{\partial f}{\partial v}\bigl(t, u_2(t), \eta(t)\bigr)\bigl(u_1'(t) - u_2'(t)\bigr) \\
&\quad = \left(t\frac{\partial f}{\partial u}\bigl(t, \xi(t), u_1'(t)\bigr) + \frac{\partial f}{\partial v}\bigl(t, u_2(t), \eta(t)\bigr)\right)\bigl(\varphi[u_1](t) - \varphi[u_2](t)\bigr) \\
&\qquad + t\frac{\partial f}{\partial v}\bigl(t, u_2(t), \eta(t)\bigr)\bigl(\varphi[u_1]'(t) - \varphi[u_2]'(t)\bigr), \\
&\min\bigl\{u_1(t), u_2(t)\bigr\} \le \xi(t) \le \max\bigl\{u_1(t), u_2(t)\bigr\}
\end{aligned}$$

and

$$\min\bigl\{u_1'(t), u_2'(t)\bigr\} \le \eta(t) \le \max\bigl\{u_1'(t), u_2'(t)\bigr\}$$

for almost every $t \in [0,h]$. Therefore, by conditions (b1), (b3), (d1) and (d2), we obtain that

$$\begin{aligned}
&\bigl|f\bigl(t, u_1(t), u_1'(t)\bigr) - f\bigl(t, u_2(t), u_2'(t)\bigr)\bigr| \\
&\quad = \left|\left(t\frac{\partial f}{\partial u}\bigl(t, \xi(t), u_1'(t)\bigr) + \frac{\partial f}{\partial v}\bigl(t, u_2(t), \eta(t)\bigr)\right)\bigl(\varphi[u_1](t) - \varphi[u_2](t)\bigr)\right. \\
&\qquad \left. + t\frac{\partial f}{\partial v}\bigl(t, u_2(t), \eta(t)\bigr)\bigl(\varphi[u_1]'(t) - \varphi[u_2]'(t)\bigr)\right| \\
&\quad \le \left(t\left|\frac{\partial f}{\partial u}\bigl(t, \xi(t), u_1'(t)\bigr)\right| + \left|\frac{\partial f}{\partial v}\bigl(t, u_2(t), \eta(t)\bigr)\right|\right)\bigl|\varphi[u_1](t) - \varphi[u_2](t)\bigr| \\
&\qquad + t\left|\frac{\partial f}{\partial v}\bigl(t, u_2(t), \eta(t)\bigr)\right|\bigl|\bigl(\varphi[u_1]'(t) - \varphi[u_2]'(t)\bigr)\bigr| \\
&\quad \le \left(\frac{\beta_1}{\alpha_1} + \beta_2\right)\bigl|f(t, \alpha_1 t, \alpha_2)\bigr|\bigl|\varphi[u_1](t) - \varphi[u_2](t)\bigr| \\
&\qquad + \beta_2 t\bigl|f(t, \alpha_1 t, \alpha_2)\bigr|\bigl|\bigl(\varphi[u_1]'(t) - \varphi[u_2]'(t)\bigr)\bigr|
\end{aligned}$$

for almost every $t \in [0,h]$. Therefore we have

$$\begin{aligned}
&\bigl|\Phi\varphi[u_1](t) - \Phi\varphi[u_2](t)\bigr| \\
&\quad = \left|\frac{1}{t}\int_0^t (t-s)\bigl(f\bigl(s, u_1(s), u_1'(s)\bigr) - f\bigl(s, u_2(s), u_2'(s)\bigr)\bigr)\,ds\right| \\
&\quad \le \int_0^t \bigl|f\bigl(s, u_1(s), u_1'(s)\bigr) - f\bigl(s, u_2(s), u_2'(s)\bigr)\bigr|\,ds \\
&\quad \le \int_0^t \left[\left(\frac{\beta_1}{\alpha_1} + \beta_2\right)\bigl|f(s, \alpha_1 s, \alpha_2)\bigr|\bigl|\varphi[u_1](s) - \varphi[u_2](s)\bigr|\right.
\end{aligned}$$

$$
\begin{aligned}
&\quad + \beta_2 s\big|f(s,\alpha_1 s,\alpha_2)\big|\big|\big(\varphi[u_1]'(s) - \varphi[u_2]'(s)\big)\big| \Bigg] ds \\
&\leq \left(\frac{\beta_1}{\alpha_1} + 2\beta_2\right) \int_0^h \big|f(s,\alpha_1 s,\alpha_2)\big|\, ds \big\|\varphi[u_1] - \varphi[u_2]\big\|
\end{aligned}
$$

for any $t \in [0,h]$. Moreover, we have

$$
\begin{aligned}
&\big|\big(\Phi\varphi[u_1]\big)'(t) - \big(\Phi\varphi[u_2]\big)'(t)\big| \\
&\quad = \left|\frac{1}{t^2}\int_0^t s\big(f\big(s,u_1(s),u_1'(s)\big) - f\big(s,u_2(s),u_2'(s)\big)\big)\, ds\right| \\
&\quad \leq \frac{1}{t^2}\int_0^t s\big|f\big(s,u_1(s),u_1'(s)\big) - f\big(s,u_2(s),u_2'(s)\big)\big|\, ds \\
&\quad \leq \frac{1}{t^2}\int_0^t s\Bigg[\left(\frac{\beta_1}{\alpha_1} + \beta_2\right)\big|f(s,\alpha_1 s,\alpha_2)\big|\big|\varphi[u_1](s) - \varphi[u_2](s)\big| \\
&\qquad + \beta_2 s\big|f(s,\alpha_1 s,\alpha_2)\big|\big|\big(\varphi[u_1]'(s) - \varphi[u_2]'(s)\big)\big| \Bigg] ds \\
&\quad \leq \Bigg[\left(\frac{\beta_1}{\alpha_1} + \beta_2\right)\int_0^{h_1} \big|f(s,\alpha_1 s,\alpha_2)\big|\, ds \\
&\qquad + \beta_2 \int_0^h \big|f(s,\alpha_1 s,\alpha_2)\big|\, ds\Bigg] \big\|\varphi[u_1] - \varphi[u_2]\big\|
\end{aligned}
$$

for any $t \in [0,h]$. Hence we obtain that

$$
\begin{aligned}
&\big\|\Phi\varphi[u_1] - \Phi\varphi[u_2]\big\| \\
&\quad \leq \left(\frac{\beta_1}{\alpha_1} + 2\beta_2\right)\int_0^{h_1} \big|f(s,\alpha_1 s,\alpha_2)\big|\, ds \big\|\varphi[u_1] - \varphi[u_2]\big\|.
\end{aligned}
$$

By the Banach fixed point theorem, there exists a unique mapping $\varphi[u] \in \varphi[X]$ such that $\Phi\varphi[u] = \varphi[u]$. Then $Au = u$. u is a solution of (3). □

Next, we consider the case of (II).

Theorem 2.2 *Let* λ *be a real number with* $\lambda > 0$. *Suppose that a mapping* f *from* $[0,1] \times (0,\infty) \times \mathbb{R}$ *into* $\mathbb{R}$ *satisfies the following*:

(a) *The mapping* $t \longmapsto f(t,u,v)$ *is measurable for any* $(u,v) \in (0,\infty) \times \mathbb{R}$, *and the mapping* $(u,v) \longmapsto f(t,u,v)$ *is continuous for almost every* $t \in [0,1]$;

(b1) $|f(t,u_1,v)| \geq |f(t,u_2,v)|$ *for almost every* $t \in [0,1]$, *for any* $u_1,u_2 \in (0,\infty)$ *with* $u_1 \leq u_2$ *and for any* $v \in \mathbb{R}$;

(b4) $|f(t,u,v_1)| \leq |f(t,u,v_2)|$ *for almost every* $t \in [0,1]$, *for any* $u \in (0,\infty)$ *and for any* $v_1,v_2 \in \mathbb{R}$ *with* $v_1 \leq v_2$;

(c2) *There exist* $\alpha_1 \in \mathbb{R}$ *with* $0 < \alpha_1 < \lambda$ *and* $\alpha_2 \in \mathbb{R}$ *with* $\alpha_2 > \lambda$ *such that*

$$
\int_0^1 \big|f(t,\alpha_1 t,\alpha_2)\big|\, dt < \infty;
$$

(d1) *There exists* $\beta_1 \in \mathbb{R}$ *with* $\beta_1 > 0$ *such that*

$$\left|\frac{\partial f}{\partial u}(t,u,v)\right| \le \frac{\beta_1|f(t,u,v)|}{u}$$

for almost every $t \in [0,1]$, *for any* $u \in (0,\infty)$ *and for any* $v \in \mathbb{R}$;

(d2) *There exists* $\beta_2 \in \mathbb{R}$ *with* $\beta_2 > 0$ *such that*

$$\left|\frac{\partial f}{\partial v}(t,u,v)\right| \le \beta_2\bigl|f(t,u,v)\bigr|$$

for almost every $t \in [0,1]$, *for any* $u \in (0,\infty)$ *and for any* $v \in \mathbb{R}$;

(e) *There exists the limit*

$$\lim_{t\to 0+}\frac{1}{t^2}\int_0^t sf\bigl(s,u(s),u'(s)\bigr)\,ds$$

for any continuously differentiable mapping u from $[0,1]$ *into* $[0,\infty)$;

(f1) *For* α_1 *and* α_2,

$$\lim_{t\to 0+}\frac{1}{t^2}\int_0^t s\bigl|f(s,\alpha_1 s,\alpha_2)\bigr|\,ds = 0.$$

Then there exists $h \in \mathbb{R}$ *with* $0 < h \le 1$ *such that Cauchy problem* (3) *has a unique solution in* X, *where* X *is a subset*

$$X = \left\{ u \;\middle|\; \begin{array}{l} u \in C^1[0,h], u(0) = 0, u'(0) = \lambda, \\ \alpha_1 t \le u(t) \text{ and } u'(t) \le \alpha_2 \text{ for any } t \in [0,h] \\ \text{and there exists the limit } \lim_{t\to 0+}\frac{tu'(t)-u(t)}{t^2} \end{array} \right\}$$

of $C^1[0,h]$.

Proof By condition (c2), there exists $h_1 \in \mathbb{R}$ with $0 < h_1 \le 1$ such that

$$\int_0^{h_1}\bigl|f(t,\alpha_1 t,\alpha_2)\bigr|\,dt < \min\left\{\lambda - \alpha_1, \alpha_2 - \lambda, \left(\frac{\beta_1}{\alpha_1} + 2\beta_2\right)^{-1}\right\}.$$

By condition (f1), there exists $h \in \mathbb{R}$ with $0 < h \le h_1$ such that

$$\sup_{t\in(0,h]}\frac{1}{t^2}\int_0^t s\bigl|f(s,\alpha_1 s,\alpha_2)\bigr|\,ds \le \int_0^{h_1}\bigl|f(t,\alpha_1 t,\alpha_2)\bigr|\,dt.$$

Let A be an operator from X into $C^1[0,h]$ defined by

$$Au(t) = \lambda t + \int_0^t (t-s)f\bigl(s,u(s),u'(s)\bigr)\,ds.$$

Since a mapping $t \longmapsto \lambda t$ belongs to X, $X \neq \emptyset$. Moreover, we have $A(X) \subset X$. Indeed, by condition (a), $Au \in C^1[0,h]$, $Au(0) = 0$ and

$$(Au)'(0) = \left[\lambda + \int_0^t f\bigl(s,u(s),u'(s)\bigr)\,ds\right]_{t=0} = \lambda.$$

By conditions (b1) and (b4), we obtain that

$$\begin{aligned} Au(t) &= \lambda t + \int_0^t (t-s) f\bigl(s, u(s), u'(s)\bigr)\,ds \\ &\ge \lambda t - t\int_0^h \bigl|f\bigl(s, u(s), u'(s)\bigr)\bigr|\,ds \\ &\ge \lambda t - t\int_0^h \bigl|f(s, \alpha_1 s, \alpha_2)\bigr|\,ds \\ &\ge \alpha_1 t \end{aligned}$$

and

$$\begin{aligned} (Au)'(t) &= \lambda + \int_0^t f\bigl(s, u(s), u'(s)\bigr)\,ds \\ &\le \lambda + \int_0^h \bigl|f\bigl(s, u(s), u'(s)\bigr)\bigr|\,ds \\ &\le \lambda + \int_0^h \bigl|f(s, \alpha_1 s, \alpha_2)\bigr|\,ds \\ &\le \alpha_2 \end{aligned}$$

for any $t \in [0,h]$. Moreover, by condition (e), there exists the limit

$$\lim_{t\to 0+} \frac{t(Au)'(t) - Au(t)}{t^2} = \lim_{t\to 0+} \frac{1}{t^2}\int_0^t s f\bigl(s, u(s), u'(s)\bigr)\,ds.$$

We will find a fixed point of A. Let φ be an operator from X into $C^1[0,h]$ defined by

$$\varphi[u](t) = \begin{cases} \frac{u(t)}{t} & \text{if } t \in (0,h], \\ \lambda & \text{if } t = 0, \end{cases}$$

and

$$\begin{aligned} \varphi[X] &= \bigl\{\varphi[u] \mid u \in X\bigr\} \\ &= \left\{ v \;\middle|\; \begin{array}{l} v \in C^1[0,h], v(0) = \lambda, \\ \alpha_1 \le v(t) \text{ and } v(t) + tv'(t) \le \alpha_2 \text{ for any } t \in [0,h] \end{array} \right\}. \end{aligned}$$

Then $\varphi[X]$ is a closed subset of $C^1[0,h]$ and hence it is a complete metric space. Let Φ be an operator from $\varphi[X]$ into $\varphi[X]$ defined by

$$\Phi\varphi[u] = \varphi[Au].$$

Then we can show, just like Theorem 2.1, that by the Banach fixed point theorem there exists a unique mapping $\varphi[u] \in \varphi[X]$ such that $\Phi\varphi[u] = \varphi[u]$ and hence $Au = u$. □

Next, we consider the case of (III).

Theorem 2.3 *Let λ be a real number with $\lambda > 0$. Suppose that a mapping f from $[0,1] \times (0,\infty) \times \mathbb{R}$ into $\mathbb{R}$ satisfies the following*:

(a) *The mapping $t \longmapsto f(t,u,v)$ is measurable for any $(u,v) \in (0,\infty) \times \mathbb{R}$, and the mapping $(u,v) \longmapsto f(t,u,v)$ is continuous for almost every $t \in [0,1]$;*

(b2) *$|f(t,u_1,v)| \leq |f(t,u_2,v)|$ for almost every $t \in [0,1]$, for any $u_1, u_2 \in (0,\infty)$ with $u_1 \leq u_2$ and for any $v \in \mathbb{R}$;*

(b3) *$|f(t,u,v_1)| \geq |f(t,u,v_2)|$ for almost every $t \in [0,1]$, for any $u \in (0,\infty)$ and for any $v_1, v_2 \in \mathbb{R}$ with $v_1 \leq v_2$;*

(c3) *There exist $\alpha_1 \in \mathbb{R}$ with $0 < \alpha_1 < \lambda$ and $\alpha_2 \in \mathbb{R}$ with $\alpha_2 < \lambda$ such that*

$$\int_0^1 \left|f\bigl(t,(2\lambda - \alpha_1)t, \alpha_2\bigr)\right| dt < \infty;$$

(d1) *There exists $\beta_1 \in \mathbb{R}$ with $\beta_1 > 0$ such that*

$$\left|\frac{\partial f}{\partial u}(t,u,v)\right| \leq \frac{\beta_1 |f(t,u,v)|}{u}$$

for almost every $t \in [0,1]$, for any $u \in (0,\infty)$ and for any $v \in \mathbb{R}$;

(d2) *There exists $\beta_2 \in \mathbb{R}$ with $\beta_2 > 0$ such that*

$$\left|\frac{\partial f}{\partial v}(t,u,v)\right| \leq \beta_2 \left|f(t,u,v)\right|$$

for almost every $t \in [0,1]$, for any $u \in (0,\infty)$ and for any $v \in \mathbb{R}$;

(e) *There exists the limit*

$$\lim_{t \to 0+} \frac{1}{t^2} \int_0^t s f\bigl(s, u(s), u'(s)\bigr)\, ds$$

for any continuously differentiable mapping u from $[0,1]$ into $[0,\infty)$;

(f2) *For α_1 and α_2,*

$$\lim_{t \to 0+} \frac{1}{t^2} \int_0^t s \left|f\bigl(s,(2\lambda - \alpha_1)s, \alpha_2\bigr)\right| ds = 0.$$

Then there exists $h \in \mathbb{R}$ with $0 < h \leq 1$ such that Cauchy problem (3) *has a unique solution in X, where X is a subset*

$$X = \left\{ u \;\middle|\; \begin{array}{l} u \in C^1[0,h], u(0) = 0, u'(0) = \lambda, \\ \alpha_1 t \leq u(t) \leq (2\lambda - \alpha_1)t \text{ and } \alpha_2 \leq u'(t) \text{ for any } t \in [0,h] \\ \text{and there exists the limit } \lim_{t \to 0+} \frac{tu'(t) - u(t)}{t^2} \end{array} \right\}$$

of $C^1[0,h]$.

Proof By condition (c3), there exists $h_1 \in \mathbb{R}$ with $0 < h_1 \leq 1$ such that

$$\int_0^{h_1} \left|f\bigl(t,(2\lambda - \alpha_1)t, \alpha_2\bigr)\right| dt < \min\left\{\lambda - \alpha_1, \lambda - \alpha_2, \left(\frac{\beta_1}{\alpha_1} + 2\beta_2\right)^{-1}\right\}.$$

By condition (f2), there exists $h \in \mathbb{R}$ with $0 < h \leq h_1$ such that

$$\sup_{t\in(0,h]} \frac{1}{t^2}\int_0^t s\left|f\bigl(s,(2\lambda-\alpha_1)s,\alpha_2\bigr)\right| ds \leq \int_0^{h_1}\left|f\bigl(t,(2\lambda-\alpha_1)t,\alpha_2\bigr)\right| dt.$$

Let A be an operator from X into $C^1[0,h]$ defined by

$$Au(t) = \lambda t + \int_0^t (t-s)f\bigl(s,u(s),u'(s)\bigr)\,ds.$$

Since a mapping $t \longmapsto \lambda t$ belongs to X, $X \neq \emptyset$. Moreover, $A(X) \subset X$. Indeed, by condition (a), $Au \in C^1[0,h]$, $Au(0) = 0$,

$$(Au)'(0) = \left[\lambda + \int_0^t f\bigl(s,u(s),u'(s)\bigr)\,ds\right]_{t=0} = \lambda,$$

by conditions (b2) and (b3),

$$\begin{aligned}
Au(t) &= \lambda t + \int_0^t (t-s)f\bigl(s,u(s),u'(s)\bigr)\,ds \\
&\geq \lambda t - t\int_0^h \left|f\bigl(s,u(s),u'(s)\bigr)\right| ds \\
&\geq \lambda t - t\int_0^h \left|f\bigl(s,(2\lambda-\alpha_1)s,\alpha_2\bigr)\right| ds \\
&\geq \alpha_1 t, \\
Au(t) &= \lambda t + \int_0^t (t-s)f\bigl(s,u(s),u'(s)\bigr)\,ds \\
&\leq \lambda t + t\int_0^h \left|f\bigl(s,u(s),u'(s)\bigr)\right| ds \\
&\leq \lambda t + t\int_0^h \left|f\bigl(s,(2\lambda-\alpha_1)s,\alpha_2\bigr)\right| ds \\
&\leq (2\lambda-\alpha_1)t, \\
(Au)'(t) &= \lambda + \int_0^t f\bigl(s,u(s),u'(s)\bigr)\,ds \\
&\geq \lambda - \int_0^h \left|f\bigl(s,u(s),u'(s)\bigr)\right| ds \\
&\geq \lambda - \int_0^h \left|f\bigl(s,(2\lambda-\alpha_1)s,\alpha_2\bigr)\right| ds \\
&\geq \alpha_2
\end{aligned}$$

for any $t \in [0,h]$, and by condition (e), there exists the limit

$$\lim_{t\to 0+}\frac{t(Au)'(t) - Au(t)}{t^2} = \lim_{t\to 0+}\frac{1}{t^2}\int_0^t sf\bigl(s,u(s),u'(s)\bigr)\,ds.$$

We will find a fixed point of A. Let φ be an operator from X into $C^1[0,h]$ defined by

$$\varphi[u](t) = \begin{cases} \frac{u(t)}{t} & \text{if } t \in (0,h], \\ \lambda & \text{if } t = 0, \end{cases}$$

and

$$\begin{aligned} \varphi[X] &= \{\varphi[u] \mid u \in X\} \\ &= \left\{ v \;\middle|\; \begin{array}{l} v \in C^1[0,h], v(0) = \lambda, \\ \alpha_1 \le v(t) \le 2\lambda - \alpha_1 \text{ and } \alpha_2 \le v(t) + tv'(t) \text{ for any } t \in [0,h] \end{array} \right\}. \end{aligned}$$

Then $\varphi[X]$ is a closed subset of $C^1[0,h]$, and hence it is a complete metric space. Let Φ be an operator from $\varphi[X]$ into $\varphi[X]$ defined by

$$\Phi\varphi[u] = \varphi[Au].$$

Then we can show, just like Theorem 2.1, that by the Banach fixed point theorem there exists a unique mapping $\varphi[u] \in \varphi[X]$ such that $\Phi\varphi[u] = \varphi[u]$ and hence $Au = u$. □

Finally, we consider the case of (IV).

Theorem 2.4 *Let λ be a real number with $\lambda > 0$. Suppose that a mapping f from $[0,1] \times (0,\infty) \times \mathbb{R}$ into $\mathbb{R}$ satisfies the following:*

(a) *The mapping $t \longmapsto f(t,u,v)$ is measurable for any $(u,v) \in (0,\infty) \times \mathbb{R}$, and the mapping $(u,v) \longmapsto f(t,u,v)$ is continuous for almost every $t \in [0,1]$;*

(b2) *$|f(t,u_1,v)| \le |f(t,u_2,v)|$ for almost every $t \in [0,1]$, for any $u_1, u_2 \in (0,\infty)$ with $u_1 \le u_2$ and for any $v \in \mathbb{R}$;*

(b4) *$|f(t,u,v_1)| \le |f(t,u,v_2)|$ for almost every $t \in [0,1]$, for any $u \in (0,\infty)$ and for any $v_1, v_2 \in \mathbb{R}$ with $v_1 \le v_2$;*

(c4) *There exist $\alpha_1 \in \mathbb{R}$ with $0 < \alpha_1 < \lambda$ and $\alpha_2 \in \mathbb{R}$ with $\alpha_2 > \lambda$ such that*

$$\int_0^1 \left| f\big(t,(2\lambda - \alpha_1)t, \alpha_2\big) \right| dt < \infty;$$

(d1) *There exists $\beta_1 \in \mathbb{R}$ with $\beta_1 > 0$ such that*

$$\left| \frac{\partial f}{\partial u}(t,u,v) \right| \le \frac{\beta_1 |f(t,u,v)|}{u}$$

for almost every $t \in [0,1]$, for any $u \in (0,\infty)$ and for any $v \in \mathbb{R}$;

(d2) *There exists $\beta_2 \in \mathbb{R}$ with $\beta_2 > 0$ such that*

$$\left| \frac{\partial f}{\partial v}(t,u,v) \right| \le \beta_2 \left| f(t,u,v) \right|$$

for almost every $t \in [0,1]$, for any $u \in (0,\infty)$ and for any $v \in \mathbb{R}$;

(e) *There exists the limit*

$$\lim_{t\to 0+} \frac{1}{t^2} \int_0^t sf\big(s, u(s), u'(s)\big)\, ds$$

for any continuously differentiable mapping u from $[0,1]$ *into* $[0,\infty)$;

(f2) *For α_1 and α_2,*

$$\lim_{t\to 0+} \frac{1}{t^2} \int_0^t s\big|f\big(s, (2\lambda-\alpha_1)s, \alpha_2\big)\big|\, ds = 0.$$

Then there exists $h \in \mathbb{R}$ with $0 < h \le 1$ such that Cauchy problem (3) *has a unique solution in X, where X is a subset*

$$X = \left\{ u \;\middle|\; \begin{array}{l} u \in C^1[0,h], u(0) = 0, u'(0) = \lambda, \\ \alpha_1 t \le u(t) \le (2\lambda-\alpha_1)t \text{ and } u'(t) \le \alpha_2 \text{ for any } t \in [0,h] \\ \text{and there exists the limit } \lim_{t\to 0+} \frac{tu'(t)-u(t)}{t^2} \end{array} \right\}$$

of $C^1[0,h]$.

Proof By condition (c4), there exists $h_1 \in \mathbb{R}$ with $0 < h_1 \le 1$ such that

$$\int_0^{h_1} \big|f\big(t, (2\lambda-\alpha_1)t, \alpha_2\big)\big|\, dt < \min\left\{\lambda-\alpha_1, \alpha_2-\lambda, \left(\frac{\beta_1}{\alpha_1} + 2\beta_2\right)^{-1}\right\}.$$

By condition (f2), there exists $h \in \mathbb{R}$ with $0 < h \le h_1$ such that

$$\sup_{t\in(0,h]} \frac{1}{t^2} \int_0^t s\big|f\big(s, (2\lambda-\alpha_1)s, \alpha_2\big)\big|\, ds \le \int_0^{h_1} \big|f\big(t, (2\lambda-\alpha_1)t, \alpha_2\big)\big|\, dt.$$

Let A be an operator from X into $C^1[0,h]$ defined by

$$Au(t) = \lambda t + \int_0^t (t-s) f\big(s, u(s), u'(s)\big)\, ds.$$

Since a mapping $t \longmapsto \lambda t$ belongs to X, $X \neq \emptyset$. Moreover, $A(X) \subset X$. Indeed, by condition (a), $Au \in C^1[0,h]$, $Au(0) = 0$,

$$(Au)'(0) = \left[\lambda + \int_0^t f\big(s, u(s), u'(s)\big)\, ds\right]_{t=0} = \lambda,$$

by conditions (b2) and (b4),

$$\begin{aligned} Au(t) &= \lambda t + \int_0^t (t-s) f\big(s, u(s), u'(s)\big)\, ds \\ &\ge \lambda t - t \int_0^h \big|f\big(s, u(s), u'(s)\big)\big|\, ds \\ &\ge \lambda t - t \int_0^h \big|f\big(s, (2\lambda-\alpha_1)s, \alpha_2\big)\big|\, ds \\ &\ge \alpha_1 t, \end{aligned}$$

$$\begin{aligned}
Au(t) &= \lambda t + \int_0^t (t-s) f\bigl(s, u(s), u'(s)\bigr)\, ds \\
&\le \lambda t + t \int_0^h \bigl|f\bigl(s, u(s), u'(s)\bigr)\bigr|\, ds \\
&\le \lambda t + t \int_0^h \bigl|f\bigl(s, (2\lambda - \alpha_1)s, \alpha_2\bigr)\bigr|\, ds \\
&\le (2\lambda - \alpha_1)t, \\
(Au)'(t) &= \lambda + \int_0^t f\bigl(s, u(s), u'(s)\bigr)\, ds \\
&\le \lambda + \int_0^h \bigl|f\bigl(s, u(s), u'(s)\bigr)\bigr|\, ds \\
&\le \lambda + \int_0^h \bigl|f(s, \alpha_1 s, \alpha_2)\bigr|\, ds \\
&\le \alpha_2
\end{aligned}$$

for any $t \in [0,h]$, and by condition (e), there exists the limit

$$\lim_{t\to 0+} \frac{t(Au)'(t) - Au(t)}{t^2} = \lim_{t\to 0+} \frac{1}{t^2} \int_0^t sf\bigl(s, u(s), u'(s)\bigr)\, ds.$$

We will find a fixed point of A. Let φ be an operator from X into $C^1[0,h]$ defined by

$$\varphi[u](t) = \begin{cases} \frac{u(t)}{t} & \text{if } t \in (0,h], \\ \lambda & \text{if } t = 0, \end{cases}$$

and

$$\begin{aligned}
\varphi[X] &= \bigl\{\varphi[u] \mid u \in X\bigr\} \\
&= \left\{ v \;\middle|\; \begin{array}{l} v \in C^1[0,h], v(0) = \lambda, \\ \alpha_1 \le v(t) \le 2\lambda - \alpha_1 \text{ and } v(t) + tv'(t) \le \alpha_2 \text{ for any } t \in [0,h] \end{array} \right\}.
\end{aligned}$$

Then $\varphi[X]$ is a closed subset of $C^1[0,h]$ and hence it is a complete metric space. Let Φ be an operator from $\varphi[X]$ into $\varphi[X]$ defined by

$$\Phi\varphi[u] = \varphi[Au].$$

Then we can show, just like Theorem 2.1, that by the Banach fixed point theorem there exists a unique mapping $\varphi[u] \in \varphi[X]$ such that $\Phi\varphi[u] = \varphi[u]$ and hence $Au = u$. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
TK wrote first draft. MT wrote final manuscript. All authors read and approved the final manuscript.

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