Li *Journal of Inequalities and Applications* 2013, 2013:460
http://www.journalofinequalitiesandapplications.com/content/2013/1/460

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

On improvements of Fischer's inequality and Hadamard's inequality for K_0-matrices

Yanxi Li*

*Correspondence: wfulyx@126.com
School of Mathematics and Information Sciences, Weifang University, Weifang, 261061, People's Republic of China

Abstract

In this paper, the class of K_0-matrices, which includes positive definite matrices, totally positive matrices, M-matrices and inverse M-matrices, is first introduced and the refinements of Fischer's inequality and Hadamard's inequality for K_0-matrices are obtained. Some previous well-known results for totally nonnegative matrices can be regarded as the special case of this paper.

MSC: 15A15

Keywords: totally nonnegative matrix; K_0-matrix; Fischer's inequality; Hadamard's inequality

1 Introduction

All matrices considered in this paper are real. For an $n \times n$ matrix $A, \alpha, \beta \subseteq \langle n \rangle = \{1, 2, \ldots, n\}$, the submatrix of A lying in rows indexed by α and the columns indexed by β will be denoted by $A[\alpha, \beta]$. If $\alpha = \beta$, then the principal submatrix $A[\alpha, \alpha]$ is abbreviated to $A[\alpha]$. For any $\alpha \subseteq \langle n \rangle$, let α^c denote the complement of α relative to $\langle n \rangle$, and let $|\alpha|$ denote the cardinality of α. If $\alpha = \emptyset$, we define $\det A[\emptyset] = 1$. We use S_n for the symmetric group on $\langle n \rangle$.

An $n \times n$ matrix A is called a P_0-matrix (P-matrix) if all the principal minors of A are nonnegative (positive). A P-matrix A is called 1-minor symmetric if $\det A[\alpha, \beta] \cdot \det A[\beta, \alpha] \geq 0$, whenever $|\alpha| = |\beta| = 1 + |\alpha \cap \beta|$. Of course, each of the P-matrices, such as positive definite matrices (PD), totally positive matrices (TP), M-matrices (M) and inverse M-matrices (M^{-1}), is 1-minor symmetric, see [1]. We have known that the following multiplicative principal minor inequalities are classical for PD, M and M^{-1} matrices:

Hadamard: $\det A \leq \prod_{i=1}^{n} a_{ii}$;

Fischer: $\det A \leq \det A[S] \cdot \det A[S^c]$, for $\forall S \subseteq \langle n \rangle$;

Koteljanskii: $\det A[S \cup T] \cdot \det A[S \cap T] \leq \det A[S] \cdot \det A[T]$, for $\forall S, T \subseteq \langle n \rangle$.

These inequalities also hold for totally nonnegative matrices (TN), see [1–6].

The study of multiplicative principal minor inequalities has been actively going on for many years, many authors have done various wonderful works on this topic, see [7–11]. In [11], Zhang and Yang improved Hadamard's inequality for totally nonnegative matrices as follows:

If $A = [a_{ij}]$ is an $n \times n$ totally nonnegative matrix with $\prod_{i=1}^{n} a_{ii} > 0$, then

$$\det A \le \min\left\{\prod_{i=1}^{n} a_{ii} - \max_{1\neq\sigma\in S_n}\left(\prod_{i=1}^{n} a_{i\sigma(i)}a_{\sigma(i)i}\right)^{1/2}, \min_{k\in\langle n\rangle} a_{kk}\prod_{\substack{i=1\\ i\neq k}}^{n}\left(a_{ii} - \frac{a_{ik}a_{ki}}{a_{kk}}\right)\right\}. \tag{1.1}$$

Hadamard's inequality for some subclasses of P_0-matrices is an important inequality in matrix analysis, inequality (1.1) is the generalization of Hadamard's inequality for totally nonnegative matrices. It is a noticeable problem to generalize inequality (1.1) for totally nonnegative matrices to other classes of matrices. In this paper we give some new upper bounds of Fischer's inequality and Hadamard's inequality for a subclass of P_0-matrices and extend the corresponding results due to Zhang and Yang (see [11]).

2 Some lemmas

To avoid triviality, we always assume $n > 1$. We will need important Sylvester's identity for determinants (see [12]).

Lemma 2.1 [12] *Let A be an $n \times n$ matrix, $\alpha \subseteq \langle n\rangle$, and suppose $|\alpha| = k$ $(1 \le k \le n-1)$. Define the $(n-k)\times(n-k)$ matrix $B = (b_{ij})$, with $i,j \in \alpha^c$, by setting $b_{ij} = \det A[\alpha\cup\{i\}, \alpha\cup\{j\}]$ for every $i,j \in \alpha^c$. B is called the Sylvester matrix of A associated with $A[\alpha]$. Then Sylvester's identity states that for each $\delta,\gamma \subseteq \alpha^c$, with $|\delta| = |\gamma| = l$,*

$$\det B[\delta,\gamma] = \left(\det A[\alpha]\right)^{l-1}\det A[\alpha\cup\delta, \alpha\cup\gamma]. \tag{2.1}$$

For convenience, we introduce the following definition.

Definition 2.1 A P_0-matrix (P-matrix) A is called a K_0-matrix (K-matrix) if every principal submatrix of A satisfies Koteljanskii's inequality.

Obviously, each principal submatrix of a K_0-matrix (K-matrix) is a K_0-matrix (K-matrix). Of course, each of the matrices *PD*, *TP*, M and M^{-1} is a K-matrix, the totally nonnegative matrices are K_0-matrices. In fact, an evident necessary and sufficient condition for a K-matrix was given in [1].

Lemma 2.2 [1] *A P-matrix satisfies Koteljanskii's inequality if and only if it is* 1*-minor symmetric.*

There are K-matrices that lie in none of the classes *PD*, *TP*, M and M^{-1}, *e.g.*,

$$\begin{pmatrix} 4 & 2 & -3 \\ 1 & 3 & 1 \\ -1 & 1 & 2 \end{pmatrix}.$$

Lemma 2.3 *If an $n\times n$ matrix $A = [a_{ij}]$ is a K_0-matrix, then $a_{ij}a_{ji} \ge 0$ for $\forall i,j \in \langle n\rangle$.*

Proof For $\forall i,j \in \langle n\rangle$, $i < j$, we consider the submatrix $A[\{i,j\}]$ of the K_0-matrix A. Since $A[\{i,j\}]$ is a K_0-matrix, by the Hadamard's inequality, we have

$$\det A\big[\{i,j\}\big] = a_{ii}a_{jj} - a_{ij}a_{ji} \le a_{ii}a_{jj},$$

therefore

$$a_{ij}a_{ji} \geq 0 \quad \text{for } \forall i,j \in \langle n \rangle.$$

This completes the proof. □

Lemma 2.4 *Let A be a K-matrix, B be the Sylvester matrix of A associated with $A[\alpha]$, then B is a K-matrix.*

Proof For $\forall \delta \subseteq \alpha^c$, with $|\delta| = l$, by Sylvester's identity (2.1), we have

$$\det B[\delta,\delta] = \left(\det A[\alpha]\right)^{l-1} \det A[\alpha \cup \delta, \alpha \cup \delta] > 0,$$

this means that B is a P-matrix.

For $\forall \delta', \gamma \subseteq \alpha^c$, with $|\delta'| = |\gamma| = 1 + |\delta' \cap \gamma| = k_1$, by Sylvester's identity (2.1), we have

$$\begin{aligned} &\det B\left[\delta',\gamma\right] \det B\left[\gamma,\delta'\right] \\ &\quad = \left(\det A[\alpha]\right)^{2k_1-2} \det A\left[\alpha \cup \delta', \alpha \cup \gamma\right] \det A\left[\alpha \cup \gamma, \alpha \cup \delta'\right] > 0. \end{aligned} \tag{2.2}$$

Obviously,

$$\begin{aligned} \left|\alpha \cup \delta'\right| &= |\alpha| + \left|\delta'\right| = 1 + |\alpha| + \left|\delta' \cap \gamma\right| \\ &= 1 + \left|\left(\alpha \cup \delta'\right) \cap (\alpha \cup \gamma)\right|, \\ |\alpha \cup \gamma| &= |\alpha| + |\gamma| = 1 + |\alpha| + \left|\delta' \cap \gamma\right| \\ &= 1 + \left|\left(\alpha \cup \delta'\right) \cap (\alpha \cup \gamma)\right|. \end{aligned}$$

That is,

$$\left|\alpha \cup \delta'\right| = |\alpha \cup \gamma| = 1 + \left|\left(\alpha \cup \delta'\right) \cap (\alpha \cup \gamma)\right|.$$

By Lemma 2.2 and (2.2), we conclude that B is 1-minor symmetric, therefore B is a K-matrix. □

Lemma 2.5 *If $A = [a_{ij}]$ is an $n \times n$ K_0-matrix, then*

$$\prod_{i=1}^{n} a_{ii} \geq \left[\prod_{i=1}^{n} a_{i\sigma(i)}a_{\sigma(i)i}\right]^{\frac{1}{2}} \quad \textit{for each } \sigma \in S_n. \tag{2.3}$$

Proof For each $\sigma \in S_n$, by Lemma 2.3, we obtain

$$a_{i\sigma(i)}a_{\sigma(i)i} \geq 0 \quad \text{for } \forall i = 1,2,\ldots,n.$$

Since each 2×2 principal minor of A is nonnegative, we have

$$a_{ii}a_{\sigma(i)i\sigma(i)} \geq a_{i\sigma(i)}a_{\sigma(i)i} \geq 0 \quad \text{for } \forall i = 1,2,\ldots,n,$$

thus

$$\left(\prod_{i=1}^{n} a_{ii}\right)^2 \geq \prod_{i=1}^{n} a_{i\sigma(i)} a_{\sigma(i)i},$$

hence inequality (2.3) follows. □

3 Main results

In this section, we give some new upper bounds for Fischer's inequality and Hadamard's inequality, and extend the corresponding results due to Zhang and Yang (see [11]).

Theorem 3.1 *If $A = [a_{ij}]$ is an $n \times n$ K_0-matrix, with $\prod_{i=1}^{n} a_{ii} > 0$, then*

$$\det A \leq \det A[\alpha] \cdot \det A\big[\alpha^c\big]\left(1 - \max_{\substack{i \in \alpha \\ j \in \alpha^c}} \frac{a_{ij}a_{ji}}{a_{ii}a_{jj}}\right). \tag{3.1}$$

Proof If A is singular, (3.1) is valid obviously. If A is a nonsingular K_0-matrix, by Fisher's inequality, we know that A is a P-matrix, therefore A is a K-matrix. Now we prove (3.1) for the K-matrix by induction on n. When $n = 2$, it is very easy to see that the result is valid. We suppose that the result is valid for all $m \times m$ ($m < n$ and $n \geq 3$) K-matrices, let k be the cardinality of α. Note that α and α^c are symmetric in inequality (3.1). Without loss of generality, we assume $k \geq 2$. Let

$$\max_{\substack{i \in \alpha \\ j \in \alpha^c}} \frac{a_{ij}a_{ji}}{a_{ii}a_{jj}} = \frac{a_{i_1 j_1} a_{j_1 i_1}}{a_{i_1 i_1} a_{j_1 j_1}},$$

where $i_1 \in \alpha$ and $j_1 \in \alpha^c$.

For any $r \in \alpha$ and $r \neq i_1$, let $\omega = \alpha - \{r\}$, and $S = (S_{ij})$ be the $(n-k+1) \times (n-k+1)$ Sylvester matrix of A associated with $A[\omega]$. By Lemma 2.4, S is a K-matrix. Clearly, $S_{rr} = \det A[\alpha]$, then, by Fisher's inequality, we have

$$\det S \leq S_{rr} \cdot \det S\big[\alpha^c\big] = \det A[\alpha] \cdot \det S\big[\alpha^c\big], \tag{3.2}$$

where $\alpha^c = \langle n \rangle - \alpha$. It follows from Sylvester's identity (2.1) and (3.2) that

$$\det A = \frac{\det S}{(\det A[\omega])^{n-k}} \leq \frac{\det A[\alpha] \cdot \det S[\alpha^c]}{(\det A[\omega])^{n-k}}. \tag{3.3}$$

Let $\beta = \langle n \rangle - \{r\}$, from Sylvester's identity (2.1), we obtain

$$\det S\big[\alpha^c\big] = \big(\det A[\omega]\big)^{n-k-1} \cdot \det A[\beta]. \tag{3.4}$$

Thus, by the inductive hypothesis, we have

$$\det A[\beta] \leq \det A[\omega] \cdot \det A\big[\alpha^c\big]\left(1 - \max_{\substack{i \in \omega \\ j \in \alpha^c}} \frac{a_{ij}a_{ji}}{a_{ii}a_{jj}}\right). \tag{3.5}$$

From (3.3), (3.4) and (3.5), it follows that

$$\det A \le \det A[\alpha] \cdot \det A\bigl[\alpha^c\bigr]\Bigl(1 - \max_{\substack{i\in\omega\\ j\in\alpha^c}} \frac{a_{ij}a_{ji}}{a_{ii}a_{jj}}\Bigr). \tag{3.6}$$

Clearly,

$$\max_{\substack{i\in\omega\\ j\in\alpha^c}} \frac{a_{ij}a_{ji}}{a_{ii}a_{jj}} = \max_{\substack{i\in\alpha\\ j\in\alpha^c}} \frac{a_{ij}a_{ji}}{a_{ii}a_{jj}}. \tag{3.7}$$

Combining (3.6) and (3.7), we obtain inequality (3.1). □

Example 3.1 Now, we consider a 4×4 K-matrix that lies in none of the classes *PD*, *TP*, *M* and M^{-1}. Let

$$A = \begin{pmatrix} 15 & 5 & 0 & -3 \\ 1 & 3 & 6 & 9 \\ 0 & 3 & 7 & 11 \\ -1 & 5 & 15 & 30 \end{pmatrix}.$$

Then A is 1-minor symmetric and A is a P-matrix. By Lemma 2.2, we know that A is a K-matrix. Let $\alpha = \{1,2\}$, then $\alpha^c = \{3,4\}$. By calculating, we have

$$\det A = 36 < \det A\bigl[\{1,2\}\bigr] \cdot \det A\bigl[\{3,4\}\bigr]\Bigl(1 - \max_{\substack{i\in\{1,2\}\\ j\in\{3,4\}}} \frac{a_{ij}a_{ji}}{a_{ii}a_{jj}}\Bigr) = \frac{1{,}800}{7},$$

therefore inequality (3.1) holds.

Let $A[\alpha] = A[\{k\}]$ $(k \in \langle n\rangle)$, by Theorem 3.1 and the induction, we can obtain the following conclusion.

Corollary 3.2 *If* $A = [a_{ij}]$ *is an* $n \times n$ K_0*-matrix, with* $\prod_{i=1}^n a_{ii} > 0$, *then*

$$\det A \le \min_{k\in\langle n\rangle} a_{kk} \prod_{\substack{i=1\\ i\ne k}}^{n} \Bigl(a_{ii} - \frac{a_{ik}a_{ki}}{a_{kk}}\Bigr).$$

If $A = [a_{ij}]$ is a totally nonnegative matrix with $\prod_{i=1}^n a_{ii} > 0$, Corollary 3.2 is certainly valid, so Corollary 3.2 is the generalization of Theorem 3 in [11].

Theorem 3.3 *If* $A = [a_{ij}]$ *is an* $n \times n$ K_0*-matrix, then*

$$\det A \le \prod_{i=1}^{n} a_{ii} - \max_{1\ne\sigma\in S_n} \Bigl(\prod_{i=1}^{n} a_{i\sigma(i)}a_{\sigma(i)i}\Bigr)^{1/2}. \tag{3.8}$$

Proof If A is singular, by Lemma 2.3 and Lemma 2.5, we know (3.8) is valid. If A is a nonsingular K_0-matrix, then A is a K-matrix. We suppose that A is a K-matrix in the following.

If $n = 2$, then equality in (3.8) holds. If $n > 2$, then first we prove the following conclusion:

For any $1 \neq \sigma \in S_n$, there exists $k_0 \neq \sigma(k_0)$ such that

$$\left[\prod_{i=1}^{n} a_{i\sigma(i)}a_{\sigma(i)i}\right]^{\frac{1}{2}} \leq a_{k_0\sigma(k_0)}a_{\sigma(k_0)k_0} \prod_{i=1, i\neq k_0,\sigma(k_0)}^{n} a_{ii}. \tag{3.9}$$

Case 1. If $\prod_{i=1}^{n} a_{i\sigma(i)}a_{\sigma(i)i} = 0$. For any $1 \neq \sigma \in S_n$, by Lemma 2.3, we know that (3.9) certainly holds.

Case 2. If $\prod_{i=1}^{n} a_{i\sigma(i)}a_{\sigma(i)i} > 0$. Let $T = \{k|\sigma(k) \neq k\}$, then $|T| \geq 2$, it is easy to see that (3.9) is true even with the equality sign for $|T| = 2$. Now we assume that $|T| > 2$.

Suppose that there exists $1 \neq \sigma \in S_n$, for any $k \neq \sigma(k)$, such that

$$\left[\prod_{i=1}^{n} a_{i\sigma(i)}a_{\sigma(i)i}\right]^{\frac{1}{2}} > a_{k\sigma(k)}a_{\sigma(k)k} \prod_{i=1, i\neq k,\sigma(k)}^{n} a_{ii}. \tag{3.10}$$

Then multiplying all the possible inequalities in (3.10) yields

$$\left[\prod_{i=1}^{n} a_{i\sigma(i)}a_{\sigma(i)i}\right]^{\frac{|T|}{2}} > \prod_{k\in T}\left[a_{k\sigma(k)}a_{\sigma(k)k} \prod_{i=1, i\neq k,\sigma(k)}^{n} a_{ii}\right]. \tag{3.11}$$

Let $S = \langle n\rangle - T$. If $S \neq \emptyset$, by (3.11) we have

$$\left[\prod_{i\in T} a_{i\sigma(i)}a_{\sigma(i)i}\right]^{\frac{|T|}{2}} \left[\prod_{i\in S} a_{ii}\right]^{|T|} > \left[\prod_{i\in T} a_{i\sigma(i)}a_{\sigma(i)i}\right] \cdot \left[\prod_{i\in S} a_{ii}\right]^{|T|} \prod_{i\in T} a_{ii}^{|T|-2}. \tag{3.12}$$

If $S = \emptyset$, by (3.11) we have

$$\left[\prod_{i\in T} a_{i\sigma(i)}a_{\sigma(i)i}\right]^{\frac{|T|}{2}} > \left[\prod_{i\in T} a_{i\sigma(i)}a_{\sigma(i)i}\right] \cdot \prod_{i\in T} a_{ii}^{|T|-2}. \tag{3.13}$$

By (3.12) and (3.13), we have

$$\left[\prod_{i\in T} a_{i\sigma(i)}a_{\sigma(i)i}\right]^{\frac{1}{2}} > \prod_{i\in T} a_{ii}. \tag{3.14}$$

On the other hand, since the principal submatrix $A[T]$ of A is a K-matrix, applying Lemma 2.5 to $A[T]$ yields

$$\left[\prod_{i\in T} a_{i\sigma(i)}a_{\sigma(i)i}\right]^{\frac{1}{2}} \leq \prod_{i\in T} a_{ii}, \tag{3.15}$$

which is a contradiction to (3.14), therefore (3.9) holds.

By Fisher's inequality (3.1) and (3.9), we have

$$\begin{aligned}\det A &\le \det A\big[\{k_0,\sigma(k_0)\}\big]\cdot \det A\big[\{k_0,\sigma(k_0)\}^c\big]\\ &\le [a_{k_0k_0}a_{\sigma(k_0)\sigma(k_0)} - a_{\sigma(k_0)k_0}a_{k_0\sigma(k_0)}]\cdot \prod_{i=1,i\neq k_0,\sigma(k_0)} a_{ii}\\ &\le \prod_{i=1}^n a_{ii} - \left[\prod_{i=1}^n a_{i\sigma(i)}a_{\sigma(i)i}\right]^{\frac{1}{2}},\end{aligned}$$

therefore (3.8) holds. □

If $A = [a_{ij}]$ is a totally nonnegative matrix, Theorem 3.3 is certainly valid too, so Theorem 3.3 is the generalization of Theorem 4 in [11].

Example 3.2 Now we consider the previous K-matrix. Let

$$A = \begin{pmatrix} 4 & 2 & -3\\ 1 & 3 & 1\\ -1 & 1 & 2\end{pmatrix}.$$

By calculating we have

$$\det A = 2 \le \prod_{i=1}^3 a_{ii} - \max_{1\neq\sigma\in S_3}\left[\prod_{i=1}^3 a_{i\sigma(i)}a_{\sigma(i)i}\right]^{\frac{1}{2}} = 24 - \max\{4, 9, \sqrt{6}, 4, \sqrt{6}\} = 15,$$

therefore inequality (3.8) holds.

By Corollary 3.2 and Theorem 3.3, we get the following result.

Corollary 3.4 *If $A = [a_{ij}]$ is an $n \times n$ K_0-matrix, with $\prod_{i=1}^n a_{ii} > 0$, then*

$$\det A \le \min\left\{\prod_{i=1}^n a_{ii} - \max_{1\neq\sigma\in S_n}\left(\prod_{i=1}^n a_{i\sigma(i)}a_{\sigma(i)i}\right)^{1/2}, \min_{k\in\langle n\rangle} a_{kk}\prod_{\substack{i=1\\ i\neq k}}^n\left(a_{ii} - \frac{a_{ik}a_{ki}}{a_{kk}}\right)\right\}.$$

If A is a totally nonnegative matrix, Corollary 3.4 is valid, so we obtain the result in [11].

Corollary 3.5 [11] *If $A = [a_{ij}]$ is an $n \times n$ totally nonnegative matrix, with $\prod_{i=1}^n a_{ii} > 0$, then*

$$\det A \le \min\left\{\prod_{i=1}^n a_{ii} - \max_{1\neq\sigma\in S_n}\left(\prod_{i=1}^n a_{i\sigma(i)}a_{\sigma(i)i}\right)^{1/2}, \min_{k\in\langle n\rangle} a_{kk}\prod_{\substack{i=1\\ i\neq k}}^n\left(a_{ii} - \frac{a_{ik}a_{ki}}{a_{kk}}\right)\right\}.$$

Competing interests
The author declares that he has no competing interests.

Authors' contributions
The studies and manuscript of this paper was written by YL independently.

Acknowledgements
The author expresses his deep gratitude to the referees for their many very valuable suggestions and comments. The research of this paper was supported by the Natural Science Foundation of Shandong Province of China (ZR2010AL017).

Received: 16 May 2013 Accepted: 9 September 2013 Published: 07 Nov 2013

References

1. Fallat, SM, Johnson, CR: Determinantal inequalities: ancient history and recent advances. Contemp. Math. **259**, 199-212 (2000)
2. Ando, T: Totally positive matrices. Linear Algebra Appl. **90**, 165-219 (1987)
3. Carlson, D: Weakly sign-symmetric matrices and some determinantal inequalities. Colloq. Math. **17**, 123-129 (1967)
4. Gantmacher, FR, Krein, MG: Oszillationsmatrizen, Oszillationskerne und kleine Schwingungen Mechanischer Systeme. Akademie Verlag, Berlin (1960)
5. Johnson, CR, Smith, RL: Almost principal minors of inverse *M*-matrices. Linear Algebra Appl. **337**, 253-265 (2001)
6. Koteljanskii, DM: A property of sign-symmetric matrices. Usp. Mat. Nauk **8**, 163-167 (1953) (in Russian). English transl.: Am. Math. Soc. Transl. **27**, 19-24 (1963)
7. Chen, S: Inequalities for *M*-matrices and inverse *M*-matrices. Linear Algebra Appl. **426**, 610-618 (2007)
8. Elhashash, A, Szyld, DB: Generalizations of *M*-matrices which may not have a nonnegative inverse. Linear Algebra Appl. **49**, 2435-2450 (2008)
9. Fallat, SM, Gekhtman, MI: Multiplicative principal-minor inequalities for totally nonnegative matrices. Adv. Appl. Math. **30**, 442-470 (2003)
10. Fallat, SM, Johnson, CR: Hadamard powers and totally positive matrices. Linear Algebra Appl. **423**, 420-427 (2007)
11. Zhang, X, Yang, S: An improvement of Hadamard's inequality for totally nonnegative matrices. SIAM J. Matrix Anal. Appl. **14**, 705-711 (1993)
12. Horn, RA, Johnson, CR: Matrix Analysis. Cambridge University Press, New York (1985)

10.1186/1029-242X-2013-460
Cite this article as: Li: On improvements of Fischer's inequality and Hadamard's inequality for K_0-matrices. *Journal of Inequalities and Applications* 2013, 2013:460