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Weyl-type theorems and k-quasi-M-hyponormal operators

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Abstract

In this paper, we show that if E is the Riesz idempotent for a non-zero isolated point λ of the spectrum of a k-quasi-M-hyponormal operator T, then E is self-adjoint, and $R(E) = N(T - \lambda) = N(T - \lambda)^*$. Also, we obtain that Weyl-type theorems hold for algebraically k-quasi-M-hyponormal operators.

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1 Introduction

Let T be a bounded linear operator on a complex Hilbert space H, write it for $T \in B(H)$, take a complex number λ in $\mathbb{C}$, and, henceforth, shorten $T - \lambda I$ to $T - \lambda$. One of recent trends in operator theory is studying natural extensions of normal operators. We introduce some of these operators as follows.

T is said to be a hyponormal operator if $T^*T \geq TT^*$;

T is M-hyponormal [1] if there exists a real positive number M such that

$$M^2(T - \lambda)^*(T - \lambda) \geq (T - \lambda)(T - \lambda)^* \quad \text{for all } \lambda \in \mathbb{C};$$

T is quasi-M-hyponormal [2] if there exits a real positive number M such that

$$T^*\big(M^2(T - \lambda)^*(T - \lambda)\big)T \geq T^*(T - \lambda)(T - \lambda)^*T \quad \text{for all } \lambda \in \mathbb{C};$$

T is k-quasi-M-hyponormal [3] if there exists a real positive number M such that

$$T^{*k}\big(M^2(T - \lambda)^*(T - \lambda)\big)T^k \geq T^{*k}(T - \lambda)(T - \lambda)^*T^k \quad \text{for all } \lambda \in \mathbb{C},$$

where k is a natural number.

It is clear that hyponormal $\Rightarrow$ M-hyponormal $\Rightarrow$ k-quasi-M-hyponormal.

We give the following example to indicate that there exists an M-hyponormal operator, which is not hyponormal.

Example 1.1 Consider the unilateral weighted shift operator as an infinite-dimensional Hilbert space operator. Recall that given a bounded sequence of positive numbers α:

$\alpha_1, \alpha_2, \alpha_3, \dots$ (called weights), the unilateral weighted shift W_α associated with α is the operator on $H = l_2$ defined by $W_\alpha e_n := \alpha_n e_{n+1}$ for all $n \geq 1$, where $\{e_n\}_{n=1}^{\infty}$ is the canonical orthogonal basis for l_2. It is well known that W_α is hyponormal if and only if α is monotonically increasing. Also, W_α is M-hyponormal if and only if α is eventually increasing. Hence, if we take the weights α such that $\alpha_1 = 2$, $\alpha_2 = 1$, $\alpha_3 = 2$, $\alpha_3 = \alpha_4 = \cdots$, then W_α is an M-hyponormal operator, but it is not hyponormal.

Next, we give a 2-quasi-M-hyponormal operator, which is not M-hyponormal.

Example 1.2 Let $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ defined on $\mathbb{C}^2$. Then by simple calculations, we see that T is a 2-quasi-M-hyponormal operator, but is not M-hyponormal.

If $T \in B(H)$, we shall write $N(T)$ and $R(T)$ for the null space and the range space of T. Also, let $\alpha(T) := \dim N(T)$, $\beta(T) := \dim N(T^*)$, $\sigma(T)$ and $\operatorname{iso}\sigma(T)$ for the spectrum and the isolated points of the spectrum of T, respectively.

Let $\lambda \in \operatorname{iso}\sigma(T)$. The Riesz idempotent E of T with respect to λ is defined by $E = \frac{1}{2\pi i}\int_{\partial D}(\mu - T)^{-1}\,d\mu$, where D is a closed disk, centered at λ, which contains no other points of $\sigma(T)$. It is well known that the Riesz idempotent satisfies $E^2 = E$, $ET = TE$, $\sigma(T|_{R(E)}) = \{\lambda\}$, and $N(T-\lambda) \subseteq R(E)$. Stampfli [4] showed that if T satisfies the growth condition G_1, then E is self-adjoint and $R(E) = N(T - \lambda)$. Recently, Chō and Tanahashi [5] obtained an improvement of Stampfli's result to p-hyponormal operators or log-hyponormal operators. Furthermore, Chō and Han extended it to M-hyponormal operators as follows.

Proposition 1.3 [6, Theorem 4] *Let T be an M-hyponormal operator, and let λ be an isolated point of $\sigma(T)$. If E is the Riesz idempotent for λ, then E is self-adjoint, and* $R(E) = N(T - \lambda) = N(T - \lambda)^*$.

2 Isolated point of spectrum of *k*-quasi-*M*-hyponormal operators

Lemma 2.1 *Let T be a k-quasi-M-hyponormal operator. If $0 \neq \lambda \in \mathbb{C}$, and assume that $\sigma(T) = \{\lambda\}$, then $T = \lambda I$.*

Proof If $\lambda \neq 0$ and $\sigma(T) = \{\lambda\}$, then T is invertible, so T is an M-hyponormal operator, and hence, $T = \lambda I$ by [6]. □

Lemma 2.2 *Let T be a k-quasi-M-hyponormal operator and $0 \neq \lambda \in \mathbb{C}$. Then $Tx = \lambda x$ implies that $T^*x = \overline{\lambda}x$.*

Proof Suppose that $Tx = \lambda x$. Since T is a k-quasi-M-hyponormal operator, $M\|(T - \alpha)T^k y\| \geq \|(T-\alpha)^* T^k y\|$ for all vectors $y \in H$ and $\alpha \in \mathbb{C}$. In particular, $M\|(T-\lambda)T^k x\| \geq \|(T-\lambda)^* T^k x\|$. Since $Tx = \lambda x$, $0 = M|\lambda|^k\|(T-\lambda)x\| = M\|(T-\lambda)T^k x\| \geq \|(T-\lambda)^* T^k x\| = |\lambda|^k\|(T-\lambda)^* x\|$. $|\lambda| \neq 0$, therefore $\|(T-\lambda)^* x\| = 0$. □

Theorem 2.3 *Let T be a k-quasi-M-hyponormal operator, and let λ be a non-zero isolated point of $\sigma(T)$. Then the Riesz idempotent E for λ is self-adjoint, and*

$$R(E) = N(T - \lambda) = N(T - \lambda)^*.$$

Proof We can derive the result from Lemma 2.2, [3, Theorem 2.5] and [7, Lemma 5.2]. □

3 Weyl-type theorems of algebraically k-quasi-M-hyponormal operators

We say that T is an algebraically k-quasi-M-hyponormal operator if there exists a nonconstant complex polynomial p such that $p(T)$ is a k-quasi-M-hyponormal operator. From the definition above, T is an algebraically k-quasi-M-hyponormal operator, then so is $T-\lambda$ for each $\lambda \in \mathbb{C}$.

An operator T is called Fredholm if $R(T)$ is closed, and both $N(T)$ and $N(T^*)$ are finite-dimensional. The index of a Fredholm operator T is given by $i(T) = \alpha(T) - \beta(T)$. An operator T is called Weyl if it is Fredholm of index zero. The Weyl spectrum of T [8] is defined by $w(T) := \{\lambda \in \mathbb{C} : T-\lambda \text{ is not Weyl}\}$. Following [9], we say that Weyl's theorem holds for T if $\sigma(T)\backslash w(T) = \pi_{00}(T)$, where $\pi_{00}(T) := \{\lambda \in \operatorname{iso}\sigma(T) : 0 < \alpha(T-\lambda) < \infty\}$.

More generally, Berkani investigated the B-Fredholm theory (see [10–12]). We define $T \in SBF_+^-(H)$ if there exists a positive integer n such that $R(T^n)$ is closed, $T_{[n]} : R(T^n) \ni x \to Tx \in R(T^n)$ is upper semi-Fredholm (*i.e.*, $R(T_{[n]}) = R(T^{n+1})$ is closed, $\dim N(T_{[n]}) = \dim N(T) \cap R(T^n) < \infty$) and $i(T_{[n]}) \leq 0$ [12]. We define $\sigma_{SBF_+^-}(T) = \{\lambda \in \mathbb{C} : T-\lambda \notin SBF_+^-(H)\}$. Let $E^a(T)$ denote the set of all isolated points λ of $\sigma_a(T)$ with $0 < \alpha(T-\lambda)$. We say that generalized a-Weyl's theorem holds for T if $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = E^a(T)$.

We know that Weyl's theorem holds for hermitian operators [13], which have been extended to hyponormal operators [14], algebraically hyponormal operators by [15], algebraically M-hyponormal operators [6] and algebraically quasi-M-hyponormal operators [2], respectively. In this section, we obtain that generalized a-Weyl's theorems hold for algebraically k-quasi-M-hyponormal operators.

Lemma 3.1 [3] *Let $T \in B(H)$ be a k-quasi-M-hyponormal operator, let the range of T^k be not dense and*

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \quad \text{on} \quad H = \overline{R(T^k)} \oplus N(T^{*k}).$$

Then T_1 is M-hyponormal, $T_3^k = 0$ and $\sigma(T) = \sigma(T_1) \cup \{0\}$.

Theorem 3.2 *Let T be a quasinilpotent algebraically k-quasi-M-hyponormal operator. Then T is nilpotent.*

Proof We first assume that T is a k-quasi-M-hyponormal operator. Consider two cases, Case I: If the range of T^k has dense range, then it is an M-hyponormal operator. Hence, by [6, Lemma 8], T is nilpotent. Case II: If T does not have dense range, then by Lemma 3.1, we can represent T as the upper triangular matrix

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \quad \text{on} \quad H = \overline{R(T^k)} \oplus N(T^{*k}),$$

where $T_1 := T|\overline{R(T^k)}$ is an M-hyponormal operator. Since T is quasinilpotent, $\sigma(T) = \{0\}$. But $\sigma(T) = \sigma(T_1) \cup \{0\}$, hence, $\sigma(T_1) = \{0\}$. Since T_1 is an M-hyponormal operator, $T_1 = 0$. Since $T_3^k = 0$, simple computation shows that

$$T^{k+1} = \begin{pmatrix} 0 & T_2 T_3^k \\ 0 & T_3^{k+1} \end{pmatrix} = 0.$$

Now, suppose that T is an algebraically k-quasi-M-hyponormal operator. Then there exists a nonconstant polynomial p such that $p(T)$ is a k-quasi-M-hyponormal operator. If $p(T)$ has dense range, then $p(T)$ is an M-hyponormal operator. Thus T is an algebraically M-hyponormal operator. It follows from [6, Lemma 8] that it is nilpotent. If $(p(T))^k$ does not have a dense range, then by Lemma 3.1, we can represent $p(T)$ as the upper triangular matrix

$$p(T) = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \quad \text{on} \quad H = \overline{R\big(\big(p(T)\big)^k\big)} \oplus N\big(\big(p(T)\big)^{*k}\big),$$

where $A := p(T)|\overline{R((p(T))^k)}$ is an M-hyponormal operator. Since $\sigma(T) = \{0\}$ and $\sigma(p(T)) = p(\sigma(T)) = \{p(0)\}$, the operator $p(T) - p(0)$ is quasinilpotent. But $\sigma(p(T)) = \sigma(A) \cup \{0\}$, thus $\sigma(A) \cup \{0\} = \{p(0)\}$. So $p(0) = 0$, and hence, $p(T)$ is quasinilpotent. Since $p(T)$ is a k-quasi-M-hyponormal operator, by the previous argument $p(T)$ is nilpotent. On the other hand, since $p(0) = 0$, $p(z) = cz^m(z - \lambda_1)(z - \lambda_2)\cdots(z - \lambda_n)$ for some natural number m. $p(T) = cT^m(T - \lambda_1)(T - \lambda_2)\cdots(T - \lambda_n)$. $p(T)$ is nilpotent, therefore, T is nilpotent. □

Recall that an operator T is said to be isoloid if every isolated point of $\sigma(T)$ is an eigenvalue of T and polaroid if every isolated point of $\sigma(T)$ is a pole of the resolvent of T. In general, if T is polaroid, then it is isoloid. However, the converse is not true. In [6], it is showed that every algebraically M-hyponormal operator is isoloid, we can prove more.

Theorem 3.3 *Let T be an algebraically k-quasi-M-hyponormal operator. Then T is polaroid.*

Proof Suppose that T is an algebraically k-quasi-M-hyponormal operator. Then $p(T)$ is a k-quasi-M-hyponormal operator for some nonconstant polynomial p. Let $\lambda \in \operatorname{iso}\sigma(T)$ and E_λ be the Riesz idempotent associated to λ defined by $E_\lambda := \frac{1}{2\pi i}\int_{\partial D}(\mu - T)^{-1}\,d\mu$, where D is a closed disk of center λ, which contains no other point of $\sigma(T)$. We can represent T as the direct sum in the following form:

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix},$$

where $\sigma(T_1) = \{\lambda\}$ and $\sigma(T_2) = \sigma(T)\backslash\{\lambda\}$. Since T_1 is an algebraically k-quasi-M-hyponormal operator, so is $T_1 - \lambda$. But $\sigma(T_1 - \lambda) = \{0\}$, it follows from Theorem 3.2 that $T_1 - \lambda$ is nilpotent, thus $T_1 - \lambda$ has finite ascent and descent. On the other hand, since $T_2 - \lambda$ is invertible, clearly, it has finite ascent and descent. $T - \lambda$ has finite ascent and descent, and hence, λ is a pole of the resolvent of T, therefore, T is polaroid. □

Corollary 3.4 *Let T be an algebraically k-quasi-M-hyponormal operator. Then T is isoloid.*

We say that T has the single valued extension property (abbreviated SVEP) if, for every open set U of $\mathbb{C}$, the only analytic solution $f: U \to H$ of the equation

$$(T - \lambda)f(\lambda) = 0 \quad \text{for all } \lambda \in U$$

is a zero function on U.

Theorem 3.5 *Let T be an algebraically k-quasi-M-hyponormal operator. Then T has SVEP.*

Proof Suppose that T is an algebraically k-quasi-M-hyponormal operator. Then $p(T)$ is a k-quasi-M-hyponormal operator for some nonconstant complex polynomial p, and hence, $p(T)$ has SVEP by [3, Theorem 2.1]. Therefore, T has SVEP by [16, Theorem 3.3.9]. □

In the following theorem, $H(\sigma(T))$ denotes the space of functions analytic in an open neighborhood of $\sigma(T)$.

Theorem 3.6 *Let T or T^* be an algebraically k-quasi-M-hyponormal operator. Then Weyl's theorem holds for $f(T)$ for every $f \in H(\sigma(T))$.*

Proof Firstly, suppose that T is an algebraically k-quasi-M-hyponormal operator. We first show that Weyl's theorem holds for T. Using the fact [17, Theorem 2.2] that if T is polaroid, then Weyl's theorem holds for T if and only if T has SVEP at points of $\lambda \in \sigma(T) \setminus w(T)$. We have that T is polaroid by Theorem 3.3, and T has SVEP by Theorem 3.5. Hence, T satisfies Weyl's theorem.

Next, suppose that T^* is an algebraically k-quasi-M-hyponormal operator. Now we show that Weyl's theorem holds for T. We use the fact [18, Theorem 3.1] that if T or T^* has SVEP, then Weyl's theorem holds for T if and only if $\pi_{00}(T) = p_{00}(T)$. Since T^* has SVEP, it is sufficient to show that $\pi_{00}(T) = p_{00}(T)$. $p_{00}(T) \subseteq \pi_{00}(T)$ is clear, so we only need to prove $\pi_{00}(T) \subseteq p_{00}(T)$. Let $\lambda \in \pi_{00}(T)$. Then λ is an isolated point of $\sigma(T)$. Hence, λ is a pole of the resolvent of T, since T is polaroid by Theorem 3.3, that is, $p(\lambda - T) = q(\lambda - T) < \infty$. By assumption, we have $\alpha(\lambda - T) < \infty$, so $\beta(\lambda - T) < \infty$. Hence, we conclude that $\lambda \in p_{00}(T)$. Therefore, Weyl's theorem holds for T.

Finally, we can derive the result by Theorem 3.5 and [17, Theorem 2.4]. □

Following [19, Theorem 3.12], we obtain the following result.

Theorem 3.7 *Let f be an analytic function on $\sigma(T)$, and f is not constant on each connected component of the open set U containing $\sigma(T)$.*

(i) *If T^* is an algebraically k-quasi-M-hyponormal operator, then $f(T)$ satisfies a generalized a-Weyl's theorem.*
(ii) *If T is an algebraically k-quasi-M-hyponormal operator, then $f(T^*)$ satisfies a generalized a-Weyl's theorem.*

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors have given equal contributions in this paper.

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