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Some new Gronwall-type inequalities arising in the research of fractional differential equations

Qinghua Feng[1,2*] and Fanwei Meng[2]

*Correspondence: fqhua@sina.com
[1]School of Science, Shandong University of Technology, Zibo, Shandong 255049, China
[2]School of Mathematical Sciences, Qufu Normal University, Qufu, Shandong 273165, China

Abstract

In this paper, some new Gronwall-type inequalities, which can be used as a handy tool in the qualitative and quantitative analysis of the solutions to certain fractional differential equations, are presented. The established results are extensions of some existing Gronwall-type inequalities in the literature. Based on the inequalities established, we investigate the boundedness, uniqueness, and continuous dependence on the initial value and parameter for the solution to a certain fractional differential equation.

MSC: 26D10

Keywords: Gronwall-type inequality; fractional differential equation; qualitative analysis; quantitative analysis

1 Introduction

In the research of the theory of differential equations, if their solutions are unknown, then it is important to seek for their qualitative and quantitative properties including boundedness, uniqueness, continuous dependence on initial data and so on. It is known that Gronwall's inequality is very useful in the research of this domain. This inequality reads as follows:

Gronwall's inequality Suppose u, a, b are continuous functions with $b(x) \geq 0$. Then

$$u(x) \leq a(x) + \int_{x_0}^{x} b(t)u(t)\,dt$$

implies

$$u(x) \leq a(x) + \int_{x_0}^{x} a(t)b(t)\exp\left(\int_{t}^{x} b(\tau)\,d\tau\right)dt.$$

Furthermore, if a is nondecreasing, then we have

$$u(x) \leq a(x)\exp\left(\int_{x_0}^{x} b(t)\,dt\right).$$

The inequality above has proved to be very effective in the research of boundedness, uniqueness, and continuous dependence on initial data for the solutions to certain differential equations, as it can provide explicit bounds for the unknown function $u(t)$. In the last few decades, motivated by the analysis of solutions to differential equations with more and more complicated forms, various generalizations of this inequality have been presented (see [1–23] for example). But we notice that most of these developed Gronwall-type inequalities are aimed for the research of differential equations of integer order, while less results are concerned with research of fractional differential equations. In order to obtain the desired analysis of the qualitative and quantitative properties of solutions to certain fractional differential equations, it is necessary to further present some new such inequalities suitable for fractional calculus analysis.

In this paper, we establish some new generalized Gronwall-type inequalities suitable for the qualitative and quantitative analysis of the solutions to fractional differential equations. In Section 2, we present the main results, in which new explicit bounds for unknown functions concerned are established. Then, in Section 3, we investigate a certain fractional differential equation, in which the boundedness, uniqueness, and continuous dependence on initial data for the solution to the fractional differential equation are investigated by use of the generalized Gronwall-type inequalities established.

2 Main results

Lemma 1 [24] *Assume that $a \geq 0$, $p \geq q \geq 0$ with $p \neq 0$. Then, for any $K > 0$, we have*

$$a^{\frac{q}{p}} \leq \frac{q}{p} K^{\frac{q-p}{p}} a + \frac{p-q}{p} K^{\frac{q}{p}}.$$

Theorem 2 *Suppose that $\alpha > 0$, $p \geq 1$ are constants, $L \in C(R_+ \times R_+, R_+)$ with $0 \leq L(t,u) - L(t,v) \leq T(u-v)$ for $u \geq v \geq 0$, where T is the Lipschitz constant, u, a, h are nonnegative functions locally integrable on $[0,X)$ with h nondecreasing and bounded by M, where M is a positive constant. If the following inequality is satisfied*:

$$u^p(x) \leq a(x) + \frac{1}{\Gamma(\alpha)} h(x) \int_0^x (x-t)^{\alpha-1} L\bigl(t, u(t)\bigr)\, dt, \quad 0 \leq x < X, \tag{1}$$

then we have the following explicit estimate for u:

$$u(x) \leq \left\{ \widetilde{a}(x) + \int_0^x \left[\sum_{n=1}^{\infty} \left(\frac{T}{p} K^{\frac{1-p}{p}} \right)^n \frac{h^n(x)}{\Gamma(n\alpha)} (x-t)^{n\alpha-1} \widetilde{a}(t) \right] dt \right\}^{\frac{1}{p}}, \quad 0 \leq x < X, \tag{2}$$

where $\widetilde{a}(x) = a(x) + \frac{1}{\Gamma(\alpha)} h(x) \int_0^x (x-t)^{\alpha-1} L(t, \frac{p-1}{p} K^{\frac{1}{p}})\, dt$, and $K > 0$ is a constant.

Proof Denote the right-hand side of (1) by $v(x)$. Then we have

$$u(x) \leq v^{\frac{1}{p}}(x), \quad 0 \leq x < X. \tag{3}$$

Furthermore,

$$v(x) \leq a(x) + \frac{1}{\Gamma(\alpha)} h(x) \int_0^x (x-t)^{\alpha-1} L\bigl(t, v^{\frac{1}{p}}(t)\bigr)\, dt, \quad 0 \leq x < X. \tag{4}$$

By use of Lemma 1, we obtain that

$$\begin{aligned}
v(x) &\le a(x) + \frac{1}{\Gamma(\alpha)} h(x) \int_0^x (x-t)^{\alpha-1} L\left(t, \frac{1}{p} K^{\frac{1-p}{p}} v(t) + \frac{p-1}{p} K^{\frac{1}{p}}\right) dt \\
&= a(x) + \frac{1}{\Gamma(\alpha)} h(x) \int_0^x (x-t)^{\alpha-1} \left[L\left(t, \frac{1}{p} K^{\frac{1-p}{p}} v(t) + \frac{p-1}{p} K^{\frac{1}{p}}\right) \right. \\
&\quad \left. - L\left(t, \frac{p-1}{p} K^{\frac{1}{p}}\right) + L\left(t, \frac{p-1}{p} K^{\frac{1}{p}}\right) \right] dt \\
&\le a(x) + \frac{1}{\Gamma(\alpha)} h(x) \int_0^x (x-t)^{\alpha-1} \left[\frac{T}{p} K^{\frac{1-p}{p}} v(t) + L\left(t, \frac{p-1}{p} K^{\frac{1}{p}}\right) \right] dt \\
&= a(x) + \frac{1}{\Gamma(\alpha)} h(x) \int_0^x (x-t)^{\alpha-1} L\left(t, \frac{p-1}{p} K^{\frac{1}{p}}\right) dt \\
&\quad + \frac{T}{p} K^{\frac{1-p}{p}} \frac{1}{\Gamma(\alpha)} h(x) \int_0^x (x-t)^{\alpha-1} v(t)\, dt \\
&= \widetilde{a}(x) + \frac{T}{p} K^{\frac{1-p}{p}} \frac{1}{\Gamma(\alpha)} h(x) \int_0^x (x-t)^{\alpha-1} v(t)\, dt. \qquad (5)
\end{aligned}$$

Applying Theorem 1 in [25] to (5), we can get the desired inequality (2). □

Corollary 3 *Under the conditions of Theorem* 2, *furthermore, assume that a is nondecreasing. Then we have the following estimate*:

$$u(x) \le \left\{ \widetilde{a}(x) \sum_{n=0}^{\infty} \left(\frac{T}{p} K^{\frac{1-p}{p}} \right)^n \frac{(h(x)x^{\alpha})^n}{\Gamma(n\alpha+1)} \right\}^{\frac{1}{p}}, \quad 0 \le x < X. \qquad (6)$$

Proof Since a is nondecreasing, then $\widetilde{a}(x)$ is also nondecreasing, and from (2) we obtain

$$\begin{aligned}
u(x) &\le \left\{ \widetilde{a}(x) + \int_0^x \left[\sum_{n=1}^{\infty} \left(\frac{T}{p} K^{\frac{1-p}{p}} \right)^n \frac{h^n(x)}{\Gamma(n\alpha)} (x-t)^{n\alpha-1} \widetilde{a}(t) \right] dt \right\}^{\frac{1}{p}} \\
&\le \widetilde{a}^{\frac{1}{p}}(x) \left\{ 1 + \int_0^x \left[\sum_{n=1}^{\infty} \left(\frac{T}{p} K^{\frac{1-p}{p}} \right)^n \frac{h^n(x)}{\Gamma(n\alpha)} (x-t)^{n\alpha-1} \right] dt \right\}^{\frac{1}{p}} \\
&= \left\{ \widetilde{a}(x) \sum_{n=0}^{\infty} \left(\frac{T}{p} K^{\frac{1-p}{p}} \right)^n \frac{(h(x)x^{\alpha})^n}{\Gamma(n\alpha+1)} \right\}^{\frac{1}{p}},
\end{aligned}$$

which is the desired result. □

Theorem 4 *Suppose that* α, u, a, h *are defined as in Theorem* 2, b *is a nonnegative function locally integrable on* $[0, X)$, *and* p, q *are constants with* $p \ge q \ge 1$. *If* a *is nondecreasing and the following inequality is satisfied*:

$$u^p(x) \le a(x) + \int_0^x b(t) u^q(t)\, dt + \frac{1}{\Gamma(\alpha)} h(x) \int_0^x (x-t)^{\alpha-1} L\big(t, u(t)\big)\, dt, \quad 0 \le x < X, \qquad (7)$$

then we have

$$u(x) \le \exp\left(\frac{q}{p^2}K^{\frac{q-p}{p}}\int_0^x b(t)\,dt\right) \times \left\{\widehat{a}(x) + \int_0^x \left[\sum_{n=1}^{\infty}\left(\frac{T}{p}K^{\frac{1-p}{p}}\right)^n \frac{\widehat{h}^n(x)}{\Gamma(n\alpha)}(x-t)^{n\alpha-1}\widehat{a}(t)\right]dt\right\}^{\frac{1}{p}}, \quad 0 \le x < X, \tag{8}$$

where

$$\widehat{a}(x) = a(x) + \frac{p-q}{p}K^{\frac{q}{p}}\int_0^x b(t)\,dt + \frac{1}{\Gamma(\alpha)}h(x)\int_0^x (x-t)^{\alpha-1}L\left(t, \frac{p-1}{p}K^{\frac{1}{p}}\right)dt,$$
$$\widehat{h}(x) = \exp\left(\frac{q}{p}K^{\frac{q-p}{p}}\int_0^x b(t)\,dt\right)h(x).$$

Proof Denote the right-hand side of (1) by $v(x)$. Then we have

$$u(x) \le v^{\frac{1}{p}}(x), \quad 0 \le x < X. \tag{9}$$

Furthermore, an application of Lemma 1 yields that

$$\begin{aligned} v(x) &\le a(x) + \int_0^x b(t)v^{\frac{q}{p}}(t)\,dt + \frac{1}{\Gamma(\alpha)}h(x)\int_0^x (x-t)^{\alpha-1}L\big(t, v^{\frac{1}{p}}(t)\big)\,dt \\ &\le a(x) + \int_0^x b(t)\left[\frac{q}{p}K^{\frac{q-p}{p}}v(t) + \frac{p-q}{p}K^{\frac{q}{p}}\right]dt \\ &\quad + \frac{1}{\Gamma(\alpha)}h(x)\int_0^x (x-t)^{\alpha-1}\left[L\left(t, \frac{1}{p}K^{\frac{1-p}{p}}v(t) + \frac{p-1}{p}K^{\frac{1}{p}}\right)\right. \\ &\quad \left. - L\left(t, \frac{p-1}{p}K^{\frac{1}{p}}\right) + L\left(t, \frac{p-1}{p}K^{\frac{1}{p}}\right)\right]dt \\ &\le a(x) + \int_0^x b(t)\left[\frac{q}{p}K^{\frac{q-p}{p}}v(t) + \frac{p-q}{p}K^{\frac{q}{p}}\right]dt \\ &\quad + \frac{1}{\Gamma(\alpha)}h(x)\int_0^x (x-t)^{\alpha-1}\left[\frac{T}{p}K^{\frac{1-p}{p}}v(t) + L\left(t, \frac{p-1}{p}K^{\frac{1}{p}}\right)\right]dt, \quad 0 \le x < X. \end{aligned} \tag{10}$$

Let

$$z(x) = a(x) + \frac{p-q}{p}K^{\frac{q}{p}}\int_0^x b(t)\,dt + \frac{1}{\Gamma(\alpha)}h(x)\int_0^x (x-t)^{\alpha-1}L\left(t, \frac{p-1}{p}K^{\frac{1}{p}}\right)dt + \frac{T}{p}K^{\frac{1-p}{p}}\frac{1}{\Gamma(\alpha)}h(x)\int_0^x (x-t)^{\alpha-1}v(t)\,dt.$$

Then we have

$$v(x) \le z(x) + \frac{q}{p}K^{\frac{q-p}{p}}\int_0^x b(t)v(t)\,dt, \quad 0 \le x < X. \tag{11}$$

Since a is nondecreasing, then z is also nondecreasing, and by use of Gronwall's inequality, we get that

$$v(x) \le z(x)\exp\left(\frac{q}{p}K^{\frac{q-p}{p}}\int_0^x b(t)\,dt\right), \quad 0 \le x < X. \tag{12}$$

Moreover,

$$\begin{aligned} z(x) &\le a(x) + \frac{p-q}{p}K^{\frac{q}{p}}\int_0^x b(t)\,dt + \frac{1}{\Gamma(\alpha)}h(x)\int_0^x (x-t)^{\alpha-1}L\left(t, \frac{p-1}{p}K^{\frac{1}{p}}\right)dt \\ &\quad + \frac{T}{p}K^{\frac{1-p}{p}}\frac{1}{\Gamma(\alpha)}h(x)\int_0^x (x-t)^{\alpha-1}\left[z(t)\exp\left(\frac{q}{p}K^{\frac{q-p}{p}}\int_0^t b(\tau)\,d\tau\right)\right]dt \\ &\le a(x) + \frac{p-q}{p}K^{\frac{q}{p}}\int_0^x b(t)\,dt + \frac{1}{\Gamma(\alpha)}h(x)\int_0^x (x-t)^{\alpha-1}L\left(t, \frac{p-1}{p}K^{\frac{1}{p}}\right)dt \\ &\quad + \frac{T}{p}K^{\frac{1-p}{p}}\frac{1}{\Gamma(\alpha)}\exp\left(\frac{q}{p}K^{\frac{q-p}{p}}\int_0^x b(t)\,dt\right)h(x)\int_0^x (x-t)^{\alpha-1}z(t)\,dt \\ &= \widehat{a}(x) + \frac{T}{p}K^{\frac{1-p}{p}}\frac{1}{\Gamma(\alpha)}\widehat{h}(x)\int_0^x (x-t)^{\alpha-1}z(t)\,dt, \quad 0 \le x < X. \end{aligned} \tag{13}$$

Since the structure of (13) is the same as that of (5), following in a similar manner to the proof in Theorem 2, we get that

$$z(x) \le \widehat{a}(x) + \int_0^x \left[\sum_{n=1}^{\infty}\left(\frac{T}{p}K^{\frac{1-p}{p}}\right)^n \frac{\widehat{h}^n(x)}{\Gamma(n\alpha)}(x-t)^{n\alpha-1}\widehat{a}(t)\right]dt, \quad 0 \le x < X. \tag{14}$$

Combining (9), (12) and (14), we get the desired result. □

Corollary 5 *For Theorem* 4, *similar to the proof of Corollary* 3, *we can obtain the following estimate for u*:

$$u(x) \le \exp\left(\frac{q}{p^2}K^{\frac{q-p}{p}}\int_0^x b(t)\,dt\right)\left\{\widehat{a}(x)\sum_{n=0}^{\infty}\left(\frac{T}{p}K^{\frac{1-p}{p}}\right)^n \frac{(\widehat{h}(x)x^{\alpha})^n}{\Gamma(n\alpha+1)}\right\}^{\frac{1}{p}}, \quad 0 \le x < X.$$

Remark In Theorem 2, if we let $p = 1$, $L(t, u(t)) = u(t)$, then Theorem 2 becomes Theorem 1 in [25].

3 Applications

In this section, we show that the inequalities established above are useful in the research of boundedness, uniqueness, continuous dependence on the initial value and parameter for the solutions to fractional differential equations. Consider the following IVP for a certain fractional differential equation:

$$D_x^{\alpha}u^3(x) = f\big(x, u(x)\big), \quad 0 \le x < X, \tag{15}$$

with the initial condition

$$D_x^{\alpha-1}u^3(x)|_{x=0} = \delta, \tag{16}$$

where $0 < \alpha < 1$, $f \in C(R \times R, R)$, D_x^{α} denotes the Riemann-Liouville fractional derivative defined by $D_x^{\alpha}v(x) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_0^x (x-t)^{-\alpha}v(t)\,dt$.

Theorem 6 *For IVP* (15)-(16), *if* $|f(x,u)| \leq L(x,|u|)$, *where L is defined as in Theorem* 2, *then we have the following estimate*:

$$u(x) \leq \sqrt[3]{\frac{x^{\alpha-1}}{\Gamma(\alpha)}|\delta| + \sum_{n=1}^{\infty}\left[\left(\frac{T}{3}K^{-\frac{2}{3}}\right)^n \frac{x^{(n+1)\alpha-1}}{\Gamma((n+1)\alpha)}|\delta|\right]}, \quad 0 \leq x < X, \tag{17}$$

where $K > 0$ *is a constant, and T is defined as in Theorem* 1.

Proof The equivalent integral form of IVP (15)-(16) is denoted as follows:

$$u^3(x) = \frac{\delta}{\Gamma(\alpha)}x^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} f\bigl(t,u(t)\bigr)\,dt.$$

So,

$$\begin{aligned}\bigl|u(x)\bigr|^3 &\leq \left|\frac{\delta}{\Gamma(\alpha)}x^{\alpha-1}\right| + \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}\bigl|f\bigl(t,u(t)\bigr)\bigr|\,dt \\ &\leq \frac{x^{\alpha-1}}{\Gamma(\alpha)}|\delta| + \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} L\bigl(t,\bigl|u(t)\bigr|\bigr)\,dt, \quad 0 \leq x < X.\end{aligned} \tag{18}$$

Then a suitable application of Theorem 2 (with $p = 3$) to (18) yields

$$\begin{aligned}u(x) &\leq \sqrt[3]{\frac{x^{\alpha-1}}{\Gamma(\alpha)}|\delta| + \int_0^x \left[\sum_{n=1}^{\infty}\left(\frac{T}{3}K^{-\frac{2}{3}}\right)^n \frac{1}{\Gamma(n\alpha)}(x-t)^{n\alpha-1}\frac{t^{\alpha-1}}{\Gamma(\alpha)}|\delta|\right] dt} \\ &= \sqrt[3]{\frac{x^{\alpha-1}}{\Gamma(\alpha)}|\delta| + \sum_{n=1}^{\infty}\left[\left(\frac{T}{3}K^{-\frac{2}{3}}\right)^n \frac{x^{(n+1)\alpha-1}B(\alpha,n\alpha)}{\Gamma(n\alpha)\Gamma(\alpha)}|\delta|\right]} \\ &= \sqrt[3]{\frac{x^{\alpha-1}}{\Gamma(\alpha)}|\delta| + \sum_{n=1}^{\infty}\left[\left(\frac{T}{3}K^{-\frac{2}{3}}\right)^n \frac{x^{(n+1)\alpha-1}}{\Gamma((n+1)\alpha)}|\delta|\right]}, \quad 0 \leq x < X,\end{aligned}$$

which is the desired result. □

Theorem 7 *If* $|f(x,u) - f(x,v)| \leq L(x,|u^3 - v^3|)$, *where L is defined as in Theorem* 2, *and* $L(t,0) \equiv 0$, *then IVP* (15)-(16) *has a unique solution*.

Proof Suppose that IVP (15)-(16) has two solutions $u_1(x)$, $u_2(x)$. Then we have

$$u_1(x) = \frac{\delta}{\Gamma(\alpha)}x^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} f\bigl(t,u_1(t)\bigr)\,dt, \tag{19}$$

$$u_2(x) = \frac{\delta}{\Gamma(\alpha)}x^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} f\bigl(t,u_2(t)\bigr)\,dt. \tag{20}$$

Furthermore,

$$u_1^3(x) - u_2^3(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}\bigl[f\bigl(t,u_1(t)\bigr) - f\bigl(t,u_2(t)\bigr)\bigr]\,dt, \tag{21}$$

which implies

$$\begin{aligned} \left|u_1^3(x) - u_2^3(x)\right| &\le \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} \left|f\left(t, u_1(t)\right) - f\left(t, u_2(t)\right)\right| dt \\ &\le \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} L\left(t, \left|u_1^3(t) - u_2^3(t)\right|\right) dt. \end{aligned} \tag{22}$$

Treating $|u_1^3(x) - u_2^3(x)|$ as one independent function, applying Theorem 2 to (22), we obtain $|u_1^3(x) - u_2^3(x)| \le 0$, which implies $u_1(x) \equiv u_2(x)$. So, the proof is complete. □

Now we study the continuous dependence on the initial value and parameter for the solution of IVP (15)-(16).

Theorem 8 *Let u be the solution of IVP* (15)-(16), *and let $\overline{u}(x)$ be the solution of the following IVP*:

$$\begin{cases} D_x^\alpha \overline{u}^3(x) = f(x, \overline{u}(x)), \\ D_x^{\alpha-1} \overline{u}^3(x)|_{x=0} = \overline{\delta}. \end{cases} \tag{23}$$

If $|\delta - \overline{\delta}| < \varepsilon$, where ε is arbitrarily small, and $|f(x,u) - f(x,v)| \le L(x, |u^3 - v^3|)$, where L is defined as in Theorem 2, *and $L(t,0) \equiv 0$, then we have*

$$\left|u^3(x) - \overline{u}^3(x)\right| \le \sqrt[3]{\varepsilon} \sqrt[3]{\frac{x^{\alpha-1}}{\Gamma(\alpha)} + \sum_{n=1}^{\infty} \left[\left(\frac{T}{3} K^{-\frac{2}{3}}\right)^n \frac{x^{(n+1)\alpha-1}}{\Gamma((n+1)\alpha)}\right]}. \tag{24}$$

Proof The equivalent integral form of IVP (23) is denoted as follows:

$$\overline{u}^3(x) = \frac{\overline{\delta}}{\Gamma(\alpha)} x^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f\left(t, \overline{u}(t)\right) dt. \tag{25}$$

So, we have

$$u^3(x) - \overline{u}^3(x) = \frac{\delta - \overline{\delta}}{\Gamma(\alpha)} x^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} \left[f\left(t, u(t)\right) - f\left(t, \overline{u}(t)\right)\right] dt. \tag{26}$$

Furthermore,

$$\begin{aligned} \left|u^3(x) - \overline{u}^3(x)\right| &\le \frac{|\delta - \overline{\delta}|}{\Gamma(\alpha)} x^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} \left|f\left(t, u(t)\right) - f\left(t, \overline{u}(t)\right)\right| dt, \\ &\le \varepsilon \frac{x^{\alpha-1}}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} L\left(t, \left|u_1^3(t) - u_2^3(t)\right|\right) dt. \end{aligned} \tag{27}$$

Applying Theorem 2 to (27), after some basic computation, we can get the desired result. □

4 Conclusions

In this paper, we have established some new generalized Gronwall-type inequalities, which are generalizations of some existing results in the literature. Based on these inequalities, we investigated the boundedness, uniqueness, and continuous dependence on the initial value and parameter for the solution to a certain fractional differential equation. Finally, we note that the presented results in Theorems 2 and 4 can be generalized to Gronwall-type inequalities with more general forms involving arbitrary nonlinear functional terms $\varphi(u(x))$, and also can be generalized to the 2D case.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
QF carried out the main part of this article. All authors read and approved the final manuscript.

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