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# Redefined intuitionistic fuzzy bi-ideals of ordered semigroups

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## Abstract

In this article, we try to obtain a more general form than  $(\epsilon, \epsilon \vee \mathbf{q})$ -intuitionistic fuzzy bi-ideals in ordered semigroups. The notion of an  $(\epsilon, \epsilon \vee \mathbf{q}_k)$ -intuitionistic fuzzy bi-ideal is introduced, and several properties are investigated. Characterizations of an  $(\epsilon, \epsilon \vee \mathbf{q}_k)$ -intuitionistic fuzzy bi-ideal are established. A condition for an  $(\epsilon, \epsilon \vee \mathbf{q}_k)$ -intuitionistic fuzzy bi-ideal to be an intuitionistic fuzzy bi-ideal is provided. It is shown that every  $(\epsilon, \epsilon)$ -intuitionistic fuzzy bi-ideal is an  $(\epsilon, \epsilon \vee \mathbf{q})$ -intuitionistic fuzzy bi-ideal, and every  $(\epsilon, \epsilon \vee \mathbf{q})$ -intuitionistic fuzzy bi-ideal is an  $(\epsilon, \epsilon \vee \mathbf{q}_k)$ -intuitionistic fuzzy bi-ideal but the converse is not true. The important achievement of the study with an  $(\epsilon, \epsilon \vee \mathbf{q}_k)$ -intuitionistic fuzzy bi-ideal is that the notion of an  $(\epsilon, \epsilon \vee \mathbf{q})$ -intuitionistic fuzzy bi-ideal is a special case of an  $(\epsilon, \epsilon \vee \mathbf{q}_k)$ -intuitionistic fuzzy bi-ideal, and, thus, several results in the paper (Jun *et al.* in Bi-ideals of ordered semigroups based on the intuitionistic fuzzy points (submitted)) are the corollaries of our results obtained in this paper.

**Keywords:** intuitionistic fuzzy bi-ideal;  $(\epsilon, \epsilon \vee \mathbf{q})$ -intuitionistic fuzzy bi-ideal;  $(\epsilon, \epsilon \vee \mathbf{q}_k)$ -intuitionistic fuzzy bi-ideal

## 1 Introduction

In mathematics, an ordered semigroup is a semigroup together with a partial order that is compatible with the semigroup operation. Ordered semigroups have many applications in the theory of sequential machines, formal languages, computer arithmetics, design of fast adders and error-correcting codes. The concept of a fuzzy filter in ordered semigroups was first introduced by Kehayopulu and Tsingelis in [1], where some basic properties of fuzzy filters and prime fuzzy ideals were discussed. A theory of fuzzy generalized sets on ordered semigroups can be developed. Mordeson *et al.* in [2] presented an up-to-date account of fuzzy sub-semigroups and fuzzy ideals of a semigroup. Murali [3] proposed the definition of a fuzzy point belonging to a fuzzy subset under a natural equivalence on fuzzy subset. The idea of quasi-coincidence of a fuzzy point with a fuzzy set played a vital role in generating different types of fuzzy subgroups. Bhakat and Das [4, 5] gave the concepts of  $(\alpha, \beta)$ -fuzzy subgroups by using the 'belong to' ( $\in$ ) relation and 'quasi-coincident with' ( $\mathbf{q}$ ) relation between a fuzzy point and a fuzzy subgroup, and introduced the concept of  $(\epsilon, \epsilon \vee \mathbf{q})$ -fuzzy subgroup. In [6], Davvaz started the generalized fuzzification in algebra. In [7], Jun *et al.* initiated the study of  $(\alpha, \beta)$ -fuzzy bi-ideals of an ordered semigroup. In [8], Davvaz and Khan studied  $(\epsilon, \epsilon \vee \mathbf{q})$ -fuzzy generalized bi-ideals of an ordered semigroup. Shabir *et al.* [9] studied characterization of regular semigroups by  $(\alpha, \beta)$ -fuzzy ideals. Jun

*et al.* [10] discussed a generalization of  $(\in, \in \vee q)$ -fuzzy ideals of a *BCK/BCI*-algebra. Using the idea of a quasi-coincidence of a fuzzy point with a fuzzy set, Jun *et al.* [11] introduced the concept of  $(\alpha, \beta)$ -intuitionistic fuzzy bi-ideals in an ordered semigroup. They introduced a new sort of intuitionistic fuzzy bi-ideals, called  $(\alpha, \beta)$ -intuitionistic fuzzy bi-ideals, and studied  $(\in, \in \vee q)$ -intuitionistic fuzzy bi-ideals.

In this paper, we try to have more general form of an  $(\in, \in \vee q)$ -intuitionistic bi-ideal of an ordered semigroup. We introduce the notion of an  $(\in, \in \vee q_k)$ -intuitionistic bi-ideal of an ordered semigroup, and give examples which are  $(\in, \in \vee q_k)$ -intuitionistic fuzzy bi-ideals but not  $(\in, \in \vee q)$ -intuitionistic fuzzy bi-ideals. We discuss characterizations of  $(\in, \in \vee q_k)$ -intuitionistic fuzzy bi-ideals in ordered semigroups. We provide a condition for an  $(\in, \in \vee q_k)$ -intuitionistic fuzzy bi-ideal to be an intuitionistic fuzzy bi-ideal. The important achievement of the study with an  $(\in, \in \vee q_k)$ -intuitionistic fuzzy bi-ideal is that the notion of an  $(\in, \in \vee q)$ -intuitionistic fuzzy bi-ideal is a special case of an  $(\in, \in \vee q_k)$ -intuitionistic fuzzy bi-ideal, and, thus, several results in the paper [11] are the corollaries of our results obtained in this paper.

## 2 Basic definitions and preliminary results

By an *ordered semigroup* (or *po-semigroup*) we mean a structure  $(S, \cdot, \leq)$ , in which the following are satisfied:

- (OS1)  $(S, \cdot)$  is a semigroup,
- (OS2)  $(S, \leq)$  is a poset,
- (OS3)  $(\forall x, a, b \in S) (a \leq b \Rightarrow a \cdot x \leq b \cdot x, x \cdot a \leq x \cdot b)$ .

In what follows,  $x \cdot y$  is simply denoted by  $xy$  for all  $x, y \in S$ .

A nonempty subset  $A$  of an ordered semigroup  $S$  is called a *subsemigroup* of  $S$  if  $A^2 \subseteq A$ . A non-empty subset  $A$  of an ordered semigroup  $S$  is called a *bi-ideal* of  $S$  if it satisfies

- (b1)  $(\forall b \in S) (\forall b \in A) (a \leq b \Rightarrow b \in A)$ ,
- (b2)  $(\forall a, b \in S) (a, b \in A \Rightarrow ab \in A)$ ,
- (b3)  $ASA \subseteq A$ .

An intuitionistic fuzzy set (briefly IFS)  $A$  in a non-empty set  $X$  is an object having the form  $A = \{ \langle x, \mu_A(x), \gamma_A(x) \rangle \mid x \in X \}$ , where the function  $\mu_A : X \rightarrow [0, 1]$  and  $\gamma_A : X \rightarrow [0, 1]$  denote the degree of membership (namely,  $\mu_A(x)$ ) and the degree of non-membership (namely,  $\gamma_A(x)$ ) for each element  $x \in X$  to the set  $A$ , respectively, and  $0 \leq \mu_A(x) + \gamma_A(x) \leq 1$  for all  $x \in X$ . For the sake of simplicity, we shall use the symbol  $A = \langle x, \mu_A, \gamma_A \rangle$  for the intuitionistic fuzzy set  $A = \{ \langle x, \mu_A(x), \gamma_A(x) \rangle \mid x \in X \}$ .

Let  $(S, \cdot, \leq)$  be an ordered semigroup and  $A = \langle x, \mu_A, \gamma_A \rangle$  be an IFS of  $S$ . Then  $A = \langle x, \mu_A, \gamma_A \rangle$  is called an *intuitionistic fuzzy subsemigroup* of  $S$  [11] if

$$(\forall x, y \in S) \quad (\mu_A(xy) \geq \min\{\mu_A(x), \mu_A(y)\} \text{ and } \gamma_A(xy) \leq \max\{\gamma_A(x), \gamma_A(y)\}).$$

Let  $(S, \cdot, \leq)$  be an ordered semigroup and  $A = \langle x, \mu_A, \gamma_A \rangle$  be an intuitionistic fuzzy subsemigroup of  $S$ . Then  $A = \langle x, \mu_A, \gamma_A \rangle$  is called an *intuitionistic fuzzy bi-ideal* of  $S$  [11] if

- (b4)  $(\forall x, y \in S) (x \leq y \Rightarrow \mu_A(x) \geq \mu_A(y) \text{ and } \gamma_A(x) \leq \gamma_A(y))$ ,
- (b5)  $(\forall x, y \in S) (\mu_A(xy) \geq \min\{\mu_A(x), \mu_A(y)\} \text{ and } \gamma_A(xy) \leq \max\{\gamma_A(x), \gamma_A(y)\})$ ,
- (b6)  $(\forall x, y, z \in S) (\mu_A(xyz) \geq \min\{\mu_A(x), \mu_A(z)\} \text{ and } \gamma_A(xyz) \leq \max\{\gamma_A(x), \gamma_A(z)\})$ .

Let  $A = \langle x, \mu_A, \gamma_A \rangle$  be an IFS of  $S$  and  $\alpha \in (0, 1]$  and  $\beta \in [0, 1)$ . Then the sets

$$U(\mu_A; \alpha) = \{ x \in S \mid \mu_A(x) \geq \alpha \} \quad \text{and} \quad L(\gamma_A; \beta) = \{ x \in S \mid \gamma_A(x) \leq \beta \}$$

are called  $\mu_A$ -level and  $\gamma_A$ -level cuts of the intuitionistic fuzzy set  $A = \langle x, \mu_A, \gamma_A \rangle$ , respectively. For an IFS  $A = \langle x, \mu_A, \gamma_A \rangle$  and  $\alpha \in (0, 1]$ ,  $\beta \in [0, 1)$ , we define the  $(\mu_A, \gamma_A)$ -level cut as follows

$$C_{(\alpha, \beta)}(A) = \{x \in S \mid \mu_A(x) \geq \alpha \text{ and } \gamma_A(x) \leq \beta\}.$$

Clearly,  $C_{(\alpha, \beta)}(A) = U(\mu_A; \alpha) \cap L(\gamma_A; \beta)$ .

Let  $x$  be a point of a non-empty set  $X$ . If  $\alpha \in (0, 1]$  and  $\beta \in [0, 1)$  are two real numbers such that  $0 \leq \alpha + \beta \leq 1$ , then the IFS of the form

$$\langle x; (\alpha, \beta) \rangle = \langle x; x_\alpha, 1 - x_{1-\beta} \rangle$$

is called an *intuitionistic fuzzy point* (IFP for short) in  $X$ , where  $\alpha$  (resp.  $\beta$ ) is the degree of membership (resp. non-membership) of  $\langle x; (\alpha, \beta) \rangle$  and  $x \in X$  is the support of  $\langle x; (\alpha, \beta) \rangle$ .

Consider an IFP  $\langle x; (\alpha, \beta) \rangle$  in  $S$ , an IFS  $A = \langle x, \mu_A, \gamma_A \rangle$  and  $\alpha \in \{\in, q, \in \vee q\}$ , we define  $\langle x; (\alpha, \beta) \rangle \alpha A$  as follows

(b7)  $\langle x; (\alpha, \beta) \rangle \in A$  (resp.  $\langle x; (\alpha, \beta) \rangle qA$ ) means that  $\mu_A(x) \geq \alpha$  and  $\gamma_A(x) \leq \beta$  (resp.  $\mu_A(x) + \alpha > 1$  and  $\gamma_A(x) + \beta < 1$ ), and in this case, we say that  $\langle x; (\alpha, \beta) \rangle$  belongs to (resp. *quasi-coincident with*) an IFS  $A = \langle x, \mu_A, \gamma_A \rangle$ .

(b8)  $\langle x; (\alpha, \beta) \rangle \in \vee qA$  (resp.  $\langle x; (\alpha, \beta) \rangle \in \wedge qA$ ) means that  $\langle x; (\alpha, \beta) \rangle \in A$  or  $\langle x; (\alpha, \beta) \rangle qA$  (resp.  $\langle x; (\alpha, \beta) \rangle \in A$  and  $\langle x; (\alpha, \beta) \rangle qA$ ).

By  $\langle x; (\alpha, \beta) \rangle \bar{\alpha}A$ , we mean that  $\langle x; (\alpha, \beta) \rangle \alpha A$  does not hold.

### 3 $(\in, \in \vee q_k)$ -Intuitionistic fuzzy bi-ideals

Let  $k$  denote an arbitrary element of  $[0, 1)$  unless specified otherwise. For an IFP  $\langle x; (\alpha, \beta) \rangle$  and an IFS  $A = \langle x, \mu_A, \gamma_A \rangle$  of  $X$ , we say that

- (c1)  $\langle x; (\alpha, \beta) \rangle q_k A$  if  $\mu_A(x) + k + \alpha > 1$  and  $\gamma_A(x) + k + \beta < 1$ .
- (c2)  $\langle x; (\alpha, \beta) \rangle \in \vee q_k A$  if  $\langle x; (\alpha, \beta) \rangle \in A$  or  $\langle x; (\alpha, \beta) \rangle q_k A$ .
- (c3)  $\langle x; (\alpha, \beta) \rangle \bar{\alpha}A$  if  $\langle x; (\alpha, \beta) \rangle \alpha A$  does not hold for  $\alpha \in \{q_k, \in \vee q_k\}$ .

**Theorem 3.1** *Let  $A = \langle x, \mu_A, \gamma_A \rangle$  be an IFS of an ordered semigroup  $S$ . Then the following are equivalent*

- (1)  $(\forall \alpha \in (\frac{1-k}{2}, 1]) (\forall \beta \in [0, \frac{1-k}{2})) (C_{(\alpha, \beta)}(A) \neq \emptyset \implies C_{(\alpha, \beta)}(A) \text{ is a bi-ideal of } S)$ .
- (2)  $A = \langle x, \mu_A, \gamma_A \rangle$  satisfies the following assertions

$$(2.1) \quad \left( \begin{array}{l} x \leq y \implies \mu_A(y) \leq \max\{\mu_A(x), \frac{1-k}{2}\} \\ \text{and } \gamma_A(y) \geq \min\{\gamma_A(x), \frac{1-k}{2}\} \end{array} \right),$$

$$(2.2) \quad \left( \begin{array}{l} \min\{\mu_A(x), \mu_A(y)\} \leq \max\{\mu_A(xy), \frac{1-k}{2}\} \\ \text{and } \max\{\gamma_A(x), \gamma_A(y)\} \geq \min\{\gamma_A(xy), \frac{1-k}{2}\} \end{array} \right),$$

$$(2.3) \quad \left( \begin{array}{l} \min\{\mu_A(x), \mu_A(z)\} \leq \max\{\mu_A(xyz), \frac{1-k}{2}\} \text{ and } \\ \max\{\gamma_A(x), \gamma_A(z)\} \geq \min\{\gamma_A(xyz), \frac{1-k}{2}\} \end{array} \right),$$

for all  $x, y, z \in S$ .

*Proof* Assume that  $C_{(\alpha, \beta)}(A)$  is a bi-ideal of  $S$  for all  $\alpha \in (\frac{1-k}{2}, 1]$  and  $\beta \in [0, \frac{1-k}{2})$  with  $C_{(\alpha, \beta)}(A) \neq \emptyset$ . If there exist  $a, b \in S$  such that condition (2.1) is not valid, that is, there exist  $a, b \in S$  with  $a \leq b$  and  $\mu_A(b) > \max\{\mu_A(a), \frac{1-k}{2}\}$ ,  $\gamma_A(b) < \min\{\gamma_A(a), \frac{1-k}{2}\}$ . Then  $\mu_A(b) \in$

$(\frac{1-k}{2}, 1]$ ,  $\gamma_A(b) \in [0, \frac{1-k}{2})$  and  $b \in C_{(\mu_A(b), \gamma_A(b))}(A)$ . But  $\mu_A(a) < \mu_A(b)$  and  $\gamma_A(a) > \gamma_A(b)$  imply that  $a \notin C_{(\mu_A(b), \gamma_A(b))}(A)$ . This is not possible. Hence (2.1) is valid. Suppose that (2.2) is false, that is,

$$s := \min\{\mu_A(a), \mu_A(c)\} > \max\left\{\mu_A(ac), \frac{1-k}{2}\right\}$$

and

$$t := \max\{\gamma_A(a), \gamma_A(c)\} < \min\left\{\gamma_A(ac), \frac{1-k}{2}\right\}$$

for some  $a, c \in S$ . Then  $s \in (\frac{1-k}{2}, 1]$ ,  $t \in [0, \frac{1-k}{2})$  and  $a, c \in C_{(s,t)}(A)$ . But  $ac \notin C_{(s,t)}(A)$ , since  $\mu_A(ac) < s$  and  $\gamma_A(ac) > t$ . This is not possible, and so (2.2) is valid. If there exist  $a, b, c \in S$  such that (2.3) is not valid, that is,

$$s_0 := \min\{\mu_A(a), \mu_A(b)\} > \max\left\{\mu_A(acb), \frac{1-k}{2}\right\}$$

and

$$t_0 := \max\{\gamma_A(a), \gamma_A(b)\} < \min\left\{\gamma_A(acb), \frac{1-k}{2}\right\}.$$

Then  $s_0 \in (\frac{1-k}{2}, 1]$ ,  $t_0 \in [0, \frac{1-k}{2})$  and  $a, b \in C_{(s_0, t_0)}(A)$ . But  $acb \notin C_{(s_0, t_0)}(A)$ , since  $\mu_A(acb) < s_0$  and  $\gamma_A(acb) > t_0$ . This is a contradiction, and hence (2.3) is valid.

Conversely, assume that  $A = \langle x, \mu_A, \gamma_A \rangle$  satisfies the three conditions (2.1), (2.2) and (2.3). Suppose that  $C_{(\alpha, \beta)}(A) \neq \emptyset$  for all  $\alpha \in (\frac{1-k}{2}, 1]$ , and  $\beta \in [0, \frac{1-k}{2})$ . Let  $x, y \in S$  be such that  $x \leq y$  and  $y \in C_{(\alpha, \beta)}(A)$ . Then  $\mu_A(y) \geq \alpha$  and  $\gamma_A(y) \leq \beta$ . Using (2.1), we have  $\max\{\mu_A(x), \frac{1-k}{2}\} \geq \mu_A(y) \geq \alpha > \frac{1-k}{2}$  and  $\min\{\gamma_A(x), \frac{1-k}{2}\} \leq \gamma_A(y) \leq \beta < \frac{1-k}{2}$ . Hence  $\mu_A(x) \geq \alpha$  and  $\gamma_A(x) \leq \beta$ , i.e.,  $x \in C_{(\alpha, \beta)}(A)$ . If  $x, y \in C_{(\alpha, \beta)}(A)$ , then  $\mu_A(x) \geq \alpha$ ,  $\gamma_A(x) \leq \beta$  and  $\mu_A(y) \geq \alpha$ ,  $\gamma_A(y) \leq \beta$ . By using (2.2), we have

$$\max\left\{\mu_A(xy), \frac{1-k}{2}\right\} \geq \min\{\mu_A(x), \mu_A(y)\} \geq \alpha > \frac{1-k}{2}$$

and

$$\min\left\{\gamma_A(xy), \frac{1-k}{2}\right\} \leq \max\{\gamma_A(x), \gamma_A(y)\} \leq \beta < \frac{1-k}{2},$$

so that  $\mu_A(xy) \geq \alpha$  and  $\gamma_A(xy) \leq \beta$ , i.e.,  $xy \in C_{(\alpha, \beta)}(A)$ . Finally, if  $x, z \in C_{(\alpha, \beta)}(A)$  and  $y \in S$ , then  $\mu_A(x) \geq \alpha$ ,  $\gamma_A(x) \leq \beta$  and  $\mu_A(z) \geq \alpha$ ,  $\gamma_A(z) \leq \beta$ . By using (2.3), we have

$$\max\left\{\mu_A(xyz), \frac{1-k}{2}\right\} \geq \min\{\mu_A(x), \mu_A(z)\} \geq \alpha > \frac{1-k}{2}$$

and

$$\min\left\{\gamma_A(xyz), \frac{1-k}{2}\right\} \leq \max\{\gamma_A(x), \gamma_A(z)\} \leq \beta < \frac{1-k}{2},$$

**Table 1** \* -multiplication table for S

*	a	b	c	d	e
a	a	d	a	d	d
b	a	b	a	d	d
c	a	d	c	d	e
d	a	d	a	d	d
e	a	d	c	d	e

so that  $\mu_A(xyz) \geq \alpha$  and  $\gamma_A(xyz) \leq \beta$ , i.e.,  $xyz \in C_{(\alpha,\beta)}(A)$ . Therefore,  $C_{(\alpha,\beta)}(A)$  is a bi-ideal of S. □

If we take  $k = 0$  in Theorem 3.1, then we have the following corollary.

**Corollary 3.2** [11, Theorem 3.1] *Let  $A = \langle x, \mu_A, \gamma_A \rangle$  be an IFS of S. Then the following assertions are equivalent*

- (1)  $(\forall \alpha \in (0.5, 1]) (\forall \beta \in [0, 0.5]) (C_{(\alpha,\beta)}(A) \neq \emptyset \implies C_{(\alpha,\beta)}(A) \text{ is a bi-ideal of } S)$ .
- (2)  $A = \langle x, \mu_A, \gamma_A \rangle$  satisfies the following conditions

$$(2.1) \quad \left( \begin{array}{l} x \leq y \implies \mu_A(y) \leq \max\{\mu_A(x), 0.5\} \\ \text{and } \gamma_A(y) \geq \min\{\gamma_A(x), 0.5\} \end{array} \right),$$

$$(2.2) \quad \left( \begin{array}{l} \min\{\mu_A(x), \mu_A(y)\} \leq \max\{\mu_A(xy), 0.5\} \\ \text{and } \max\{\gamma_A(x), \gamma_A(y)\} \geq \min\{\gamma_A(xy), 0.5\} \end{array} \right),$$

$$(2.3) \quad \left( \begin{array}{l} \min\{\mu_A(x), \mu_A(z)\} \leq \max\{\mu_A(xyz), 0.5\} \\ \text{and } \max\{\gamma_A(x), \gamma_A(z)\} \geq \min\{\gamma_A(xyz), 0.5\} \end{array} \right),$$

for all  $x, y, z \in S$ .

**Definition 3.3** An IFS  $A = \langle x, \mu_A, \gamma_A \rangle$  in S is called an  $(\in, \in \vee q_k)$ -intuitionistic fuzzy bi-ideal of S if for all  $x, y, z \in S, t, t_1, t_2 \in (0, 1]$  and  $s, s_1, s_2 \in [0, 1]$  it satisfies the following conditions

- (q1)  $(x \leq y, \langle y; (t, s) \rangle \in A \implies \langle x; (t, s) \rangle \in \vee q_k A)$ ,
- (q2)  $(\langle x; (t_1, s_1) \rangle \in A \text{ and } \langle y; (t_2, s_2) \rangle \in A \implies \langle xy; \min\{t_1, t_2\}, \max\{s_1, s_2\} \rangle \in \vee q_k A)$ ,
- (q3)  $(\langle x; (t_1, s_1) \rangle \in A \text{ and } \langle z; (t_2, s_2) \rangle \in A \implies \langle xyz; \min\{t_1, t_2\}, \max\{s_1, s_2\} \rangle \in \vee q_k A)$ .

An  $(\in, \in \vee q_k)$ -intuitionistic fuzzy bi-ideal of S with  $k = 0$  is an  $(\in, \in \vee q)$ -intuitionistic fuzzy bi-ideal of S.

**Example 3.4** Consider the set  $S = \{a, b, c, d, e\}$  with the order relation  $a \leq c \leq e, a \leq d \leq e, b \leq d$  and  $b \leq e$  and \* -multiplication table (see Table 1 above).

- (1) Define an IFS  $A = \langle x, \mu_A, \gamma_A \rangle$  by

$$\mu_A : S \rightarrow [0, 1] \mid \mu_A(x) = \begin{cases} 0.40 & \text{if } x = a, \\ 0.35 & \text{if } x = b, \\ 0.30 & \text{if } x = c, \\ 0.50 & \text{if } x = d, \\ 0.20 & \text{if } x = e \end{cases}$$

and

$$\gamma_A : S \rightarrow [0, 1] \mid \gamma_A(x) = \begin{cases} 0.40 & \text{if } x = a, \\ 0.30 & \text{if } x = b, \\ 0.50 & \text{if } x = c, \\ 0.40 & \text{if } x = d, \\ 0.80 & \text{if } x = e. \end{cases}$$

Then  $A = \langle x, \mu_A, \gamma_A \rangle$  is an  $(\in, \in \vee q_{0.4})$ -intuitionistic fuzzy bi-ideal of  $S$ .

(2) Let  $A = \langle x, \mu_A, \gamma_A \rangle$  be an intuitionistic fuzzy set given by

$$\mu_A : S \rightarrow [0, 1] \mid \mu_A(x) = \begin{cases} 0.80 & \text{if } x = a, \\ 0.60 & \text{if } x = b, e, \\ 0.30 & \text{if } x = c, \\ 0.50 & \text{if } x = d \end{cases}$$

and

$$\gamma_A : S \rightarrow [0, 1] \mid \gamma_A(x) = \begin{cases} 0.20 & \text{if } x = a, \\ 0.30 & \text{if } x = b, e, \\ 0.60 & \text{if } x = c, \\ 0.50 & \text{if } x = d. \end{cases}$$

Then  $A = \langle x, \mu_A, \gamma_A \rangle$  is an  $(\in, \in \vee q_{0.04})$ -intuitionistic fuzzy bi-ideal of  $S$ .

**Theorem 3.5** An IFS  $A = \langle x, \mu_A, \gamma_A \rangle$  of  $S$  is an  $(\in, \in \vee q_k)$ -intuitionistic fuzzy bi-ideal of  $S$  if and only if it satisfies the following conditions

- (1)  $\left( \begin{array}{l} x \leq y \implies \mu_A(x) \geq \min\{\mu_A(y), \frac{1-k}{2}\} \\ \text{and } \gamma_A(x) \leq \max\{\gamma_A(y), \frac{1-k}{2}\} \end{array} \right),$
- (2)  $\left( \begin{array}{l} \mu_A(xy) \geq \min\{\mu_A(x), \mu_A(y), \frac{1-k}{2}\} \\ \text{and } \gamma_A(xy) \leq \max\{\gamma_A(x), \gamma_A(y), \frac{1-k}{2}\} \end{array} \right),$
- (3)  $\left( \begin{array}{l} \mu_A(xyz) \geq \min\{\mu_A(x), \mu_A(z), \frac{1-k}{2}\} \\ \text{and } \gamma_A(xyz) \leq \max\{\gamma_A(x), \gamma_A(z), \frac{1-k}{2}\} \end{array} \right).$

*Proof* Suppose that  $A = \langle x, \mu_A, \gamma_A \rangle$  is an  $(\in, \in \vee q_k)$ -intuitionistic fuzzy bi-ideal of  $S$ . Let  $x, y \in S$  be such that  $x \leq y$ . Assume that  $\mu_A(y) \leq \frac{1-k}{2}$  and  $\gamma_A(y) \geq \frac{1-k}{2}$ . If  $\mu_A(x) < \mu_A(y)$  and  $\gamma_A(x) > \gamma_A(y)$ , then  $\mu_A(x) < t \leq \mu_A(y)$  and  $\gamma_A(x) > s \geq \gamma_A(y)$  for some  $t \in (0, \frac{1-k}{2})$  and  $s \in (\frac{1-k}{2}, 1)$ . It follows  $\langle y; (t, s) \rangle \in A$ , but  $\langle x; (t, s) \rangle \notin A$ . Since  $\mu_A(x) + t < 2t < 1 - k$  and  $\gamma_A(x) + s > 2s > 1 - k$ , we get  $\langle x; (t, s) \rangle \overline{q_k} A$ . Therefore,  $\langle x; (t, s) \rangle \in \vee q_k A$ , which is a contradiction. Hence  $\mu_A(x) \geq \mu_A(y)$  and  $\gamma_A(x) \leq \gamma_A(y)$ . Now, if  $\mu_A(y) \geq \frac{1-k}{2}$  and  $\gamma_A(x) \leq \frac{1-k}{2}$ , then  $\langle y; (\frac{1-k}{2}, \frac{1-k}{2}) \rangle \in A$ , and so,  $\langle x; (\frac{1-k}{2}, \frac{1-k}{2}) \rangle \in \vee q_k A$ , which implies that  $\langle x; (\frac{1-k}{2}, \frac{1-k}{2}) \rangle \in A$  or  $\langle x; (\frac{1-k}{2}, \frac{1-k}{2}) \rangle q_k A$ , that is,  $\mu_A(x) \geq \frac{1-k}{2}$  and  $\gamma_A(x) \leq \frac{1-k}{2}$  or  $\mu_A(x) + \frac{1-k}{2} > 1$  and  $\gamma_A(x) + \frac{1-k}{2} < 1$ .

Hence  $\mu_A(x) \geq \frac{1-k}{2}$  and  $\gamma_A(x) \leq \frac{1-k}{2}$ . Otherwise,  $\mu_A(x) + \frac{1-k}{2} < \frac{1-k}{2} + \frac{1-k}{2} = 1$  and  $\gamma_A(x) + \frac{1-k}{2} > \frac{1-k}{2} + \frac{1-k}{2} = 1$ , a contradiction. Consequently,

$$\mu_A(x) \geq \min\left\{\mu_A(y), \frac{1-k}{2}\right\}$$

and

$$\gamma_A(x) \leq \max\left\{\gamma_A(y), \frac{1-k}{2}\right\}$$

for all  $x, y \in S$  with  $x \leq y$ . Let  $x, y \in S$  be such that  $\min\{\mu_A(x), \mu_A(y)\} < \frac{1-k}{2}$  and  $\max\{\gamma_A(x), \gamma_A(y)\} > \frac{1-k}{2}$ . We claim that  $\mu_A(xy) \geq \min\{\mu_A(x), \mu_A(y)\}$  and  $\gamma_A(xy) \leq \max\{\gamma_A(x), \gamma_A(y)\}$ . If not, then  $\mu_A(xy) < t \leq \min\{\mu_A(x), \mu_A(y)\}$  and  $\gamma_A(xy) > s \geq \max\{\gamma_A(x), \gamma_A(y)\}$  for some  $t \in (0, \frac{1-k}{2})$  and  $s \in (\frac{1-k}{2}, 1)$ . It follows that  $\langle x; (t, s) \rangle \in A$  and  $\langle y; (t, s) \rangle \in A$ , but  $\langle xy; (t, s) \rangle \notin A$  and  $\mu_A(xy) + t < 2t < 1 - k$  and  $\gamma_A(xy) + s > 2s > 1 - k$ , i.e.,  $\langle xy; (t, s) \rangle \notin \overline{q_k}A$ . This is a contradiction. Thus,  $\mu_A(xy) \geq \min\{\mu_A(x), \mu_A(y)\}$  and  $\gamma_A(xy) \leq \max\{\gamma_A(x), \gamma_A(y)\}$  for all  $x, y \in S$  with  $\min\{\mu_A(x), \mu_A(y)\} < \frac{1-k}{2}$  and  $\max\{\gamma_A(x), \gamma_A(y)\} > \frac{1-k}{2}$ . If  $\min\{\mu_A(x), \mu_A(y)\} \geq \frac{1-k}{2}$  and  $\max\{\gamma_A(x), \gamma_A(y)\} \leq \frac{1-k}{2}$ , then  $\langle x; (\frac{1-k}{2}, \frac{1-k}{2}) \rangle \in A$  and  $\langle y; (\frac{1-k}{2}, \frac{1-k}{2}) \rangle \in A$ . Using (q2), we have

$$\begin{aligned} & \left\langle xy; \left(\frac{1-k}{2}, \frac{1-k}{2}\right) \right\rangle \\ &= \left\langle xy; \min\left\{\frac{1-k}{2}, \frac{1-k}{2}\right\}, \max\left\{\frac{1-k}{2}, \frac{1-k}{2}\right\} \right\rangle \in \vee q_k A, \end{aligned}$$

and so,  $\mu_A(xy) \geq \frac{1-k}{2}$  and  $\gamma_A(xy) \leq \frac{1-k}{2}$  or  $\mu_A(xy) + \frac{1-k}{2} > 1$  and  $\gamma_A(xy) + \frac{1-k}{2} < 1$ . If  $\mu_A(xy) < \frac{1-k}{2}$  and  $\gamma_A(xy) > \frac{1-k}{2}$ , then

$$\mu_A(xy) + \frac{1-k}{2} < \frac{1-k}{2} + \frac{1-k}{2} = 1$$

and

$$\gamma_A(xy) + \frac{1-k}{2} > \frac{1-k}{2} + \frac{1-k}{2} = 1,$$

which is impossible. Consequently,  $\mu_A(xy) \geq \min\{\mu_A(x), \mu_A(y), \frac{1-k}{2}\}$  and  $\gamma_A(xy) \leq \max\{\gamma_A(x), \gamma_A(y), \frac{1-k}{2}\}$  for all  $x, y \in S$ . Let  $a, b, c \in S$  be such that  $\min\{\mu_A(a), \mu_A(c)\} < \frac{1-k}{2}$  and  $\max\{\gamma_A(a), \gamma_A(c)\} > \frac{1-k}{2}$ . We claim that

$$\mu_A(abc) \geq \min\{\mu_A(a), \mu_A(c)\} \quad \text{and} \quad \gamma_A(abc) \leq \max\{\gamma_A(a), \gamma_A(c)\}.$$

If not, then  $\mu_A(abc) < t_0 \leq \min\{\mu_A(a), \mu_A(c)\}$  and  $\gamma_A(abc) > s_0 \geq \max\{\gamma_A(a), \gamma_A(c)\}$  for some  $t_0 \in (0, \frac{1-k}{2})$  and  $s_0 \in (\frac{1-k}{2}, 1)$ . It follows that  $\langle a; (t_0, s_0) \rangle \in A$  and  $\langle c; (t_0, s_0) \rangle \in A$ , but  $\langle abc; (t_0, s_0) \rangle \notin A$  and  $\mu_A(abc) + t_0 < 2t_0 < 1 - k$  and  $\gamma_A(abc) + s_0 > 2s_0 > 1 - k$ , i.e.,  $\langle abc; (t_0, s_0) \rangle \notin \overline{q_k}A$ . This is a contradiction. Thus,  $\mu_A(abc) \geq \min\{\mu_A(a), \mu_A(c)\}$  and  $\gamma_A(abc) \leq \max\{\gamma_A(a), \gamma_A(c)\}$  for all  $a, b, c \in S$  with  $\min\{\mu_A(a), \mu_A(c)\} < \frac{1-k}{2}$  and

$\max\{\gamma_A(a), \gamma_A(c)\} > \frac{1-k}{2}$ . If  $\min\{\mu_A(a), \mu_A(c)\} \geq \frac{1-k}{2}$  and  $\max\{\gamma_A(a), \gamma_A(c)\} \leq \frac{1-k}{2}$ , then  $\langle a; (\frac{1-k}{2}, \frac{1-k}{2}) \rangle \in A$  and  $\langle c; (\frac{1-k}{2}, \frac{1-k}{2}) \rangle \in A$ . Using (q3), we have

$$\begin{aligned} & \left\langle abc; \left( \frac{1-k}{2}, \frac{1-k}{2} \right) \right\rangle \\ &= \left\langle abc; \min \left\{ \frac{1-k}{2}, \frac{1-k}{2} \right\}, \max \left\{ \frac{1-k}{2}, \frac{1-k}{2} \right\} \right\rangle \in \vee q A, \end{aligned}$$

and so  $\mu_A(abc) \geq \frac{1-k}{2}$  and  $\gamma_A(abc) \leq \frac{1-k}{2}$  or  $\mu_A(abc) + \frac{1-k}{2} > 1$  and  $\gamma_A(abc) + \frac{1-k}{2} < 1$ . If  $\mu_A(abc) < \frac{1-k}{2}$  and  $\gamma_A(abc) > \frac{1-k}{2}$ , then

$$\mu_A(abc) + \frac{1-k}{2} < \frac{1-k}{2} + \frac{1-k}{2} = 1$$

and

$$\gamma_A(abc) + \frac{1-k}{2} > \frac{1-k}{2} + \frac{1-k}{2} = 1,$$

which is impossible. Therefore,  $\mu_A(xyz) \geq \min\{\mu_A(x), \mu_A(z), \frac{1-k}{2}\}$  and  $\gamma_A(xyz) \leq \max\{\gamma_A(x), \gamma_A(z), \frac{1-k}{2}\}$  for all  $x, y, z \in S$ .

Conversely, let  $A = \langle x, \mu_A, \gamma_A \rangle$  be an IFS of  $S$  that satisfies the three conditions (1), (2) and (3). Let  $x, y \in S$ ,  $t \in (0, 1]$  and  $s \in [0, 1)$  be such that  $x \leq y$  and  $\langle y; (t, s) \rangle \in A$ . Then  $\mu_A(y) \geq t$  and  $\gamma_A(y) \leq s$ , and so,

$$\begin{aligned} \mu_A(x) &\geq \min \left\{ \mu_A(y), \frac{1-k}{2} \right\} \geq \min \left\{ t, \frac{1-k}{2} \right\} \\ &= \begin{cases} t & \text{if } t \leq \frac{1-k}{2}, \\ \frac{1-k}{2} & \text{if } t > \frac{1-k}{2} \end{cases} \end{aligned}$$

and

$$\begin{aligned} \gamma_A(x) &\leq \max \left\{ \gamma_A(y), \frac{1-k}{2} \right\} \leq \max \left\{ s, \frac{1-k}{2} \right\} \\ &= \begin{cases} s & \text{if } s \geq \frac{1-k}{2}, \\ \frac{1-k}{2} & \text{if } s < \frac{1-k}{2}. \end{cases} \end{aligned}$$

It follows that  $\mu_A(x) \geq t$  and  $\gamma_A(x) \leq s$  or  $\mu_A(x) + t \geq \frac{1-k}{2} + t > 1-k$  and  $\gamma_A(x) + s \leq \frac{1-k}{2} + s < 1-k$ , i.e.,  $\langle x; (t, s) \rangle \in A$  or  $\langle x; (t, s) \rangle \in \vee q_k A$ . Hence,  $\langle x; (t, s) \rangle \in \vee q_k A$ . Let  $x, y \in S$ ,  $t_1, t_2 \in (0, 1]$  and  $s_1, s_2 \in [0, 1)$  be such that  $\langle x; (t_1, s_1) \rangle \in A$  and  $\langle y; (t_2, s_2) \rangle \in A$ . Then  $\mu_A(x) \geq t_1$ ,  $\gamma_A(x) \leq s_1$  and  $\mu_A(y) \geq t_2$ ,  $\gamma_A(y) \leq s_2$ . It follows from (2) that

$$\begin{aligned} \mu_A(xy) &\geq \min \left\{ \mu_A(x), \mu_A(y), \frac{1-k}{2} \right\} \geq \min \left\{ t_1, t_2, \frac{1-k}{2} \right\} \\ &= \begin{cases} \min\{t_1, t_2\} & \text{if } \min\{t_1, t_2\} \leq \frac{1-k}{2}, \\ \frac{1-k}{2} & \text{if } \min\{t_1, t_2\} > \frac{1-k}{2} \end{cases} \end{aligned}$$



and

$$\begin{aligned} \gamma_A(xy) &\leq \max\left\{\gamma_A(x), \gamma_A(y), \frac{1-k}{2}\right\} \leq \max\left\{s_1, s_2, \frac{1-k}{2}\right\} \\ &= \begin{cases} \max\{s_1, s_2\} & \text{if } \max\{s_1, s_2\} \geq \frac{1-k}{2}, \\ \frac{1-k}{2} & \text{if } \max\{s_1, s_2\} < \frac{1-k}{2}. \end{cases} \end{aligned}$$

It follows that  $\langle xy; \min\{t_1, t_2\}, \max\{s_1, s_2\} \rangle \in A$  or  $\mu_A(xy) + \min\{t_1, t_2\} \geq \frac{1-k}{2} + \min\{t_1, t_2\} > \frac{1-k}{2} + \frac{1-k}{2} = 1-k$  and  $\gamma_A(xy) + \max\{s_1, s_2\} \leq \frac{1-k}{2} + \max\{s_1, s_2\} < \frac{1-k}{2} + \frac{1-k}{2} = 1-k$ , i.e.,  $\langle xy; \min\{t_1, t_2\}, \max\{s_1, s_2\} \rangle \in \text{q}_k A$ . Therefore,  $\langle xy; \min\{t_1, t_2\}, \max\{s_1, s_2\} \rangle \in \vee \text{q}_k A$ . Let  $x, y, z \in S$ ,  $t_1, t_2 \in (0, 1]$  and  $s_1, s_2 \in [0, 1)$  be such that  $\langle x; (t_1, s_1) \rangle \in A$  and  $\langle z; (t_2, s_2) \rangle \in A$ . Then  $\mu_A(x) \geq t_1$ ,  $\gamma_A(x) \leq s_1$  and  $\mu_A(z) \geq t_2$ ,  $\gamma_A(z) \leq s_2$ . It follows from (3) that

$$\begin{aligned} \mu_A(xyz) &\geq \min\left\{\mu_A(x), \mu_A(z), \frac{1-k}{2}\right\} \geq \min\left\{t_1, t_2, \frac{1-k}{2}\right\} \\ &= \begin{cases} \min\{t_1, t_2\} & \text{if } \min\{t_1, t_2\} \leq \frac{1-k}{2}, \\ \frac{1-k}{2} & \text{if } \min\{t_1, t_2\} > \frac{1-k}{2}. \end{cases} \end{aligned}$$

and

$$\begin{aligned} \gamma_A(xyz) &\leq \max\left\{\gamma_A(x), \gamma_A(z), \frac{1-k}{2}\right\} \leq \max\left\{s_1, s_2, \frac{1-k}{2}\right\} \\ &= \begin{cases} \max\{s_1, s_2\} & \text{if } \max\{s_1, s_2\} \geq \frac{1-k}{2}, \\ \frac{1-k}{2} & \text{if } \max\{s_1, s_2\} < \frac{1-k}{2}. \end{cases} \end{aligned}$$

Thus, we have  $\langle xyz; \min\{t_1, t_2\}, \max\{s_1, s_2\} \rangle \in A$  or  $\mu_A(xyz) + \min\{t_1, t_2\} \geq \frac{1-k}{2} + \min\{t_1, t_2\} > \frac{1-k}{2} + \frac{1-k}{2} = 1-k$  and  $\gamma_A(xyz) + \max\{s_1, s_2\} \leq \frac{1-k}{2} + \max\{s_1, s_2\} < \frac{1-k}{2} + \frac{1-k}{2} = 1-k$ , i.e.,  $\langle xyz; \min\{t_1, t_2\}, \max\{s_1, s_2\} \rangle \in \text{q}_k A$ . Therefore,  $\langle xyz; \min\{t_1, t_2\}, \max\{s_1, s_2\} \rangle \in \vee \text{q}_k A$ . Thus,  $A = \langle x, \mu_A, \gamma_A \rangle$  is an  $(\in, \in \vee \text{q}_k)$ -intuitionistic fuzzy bi-ideal of  $S$ .  $\square$

If we take  $k = 0$  in Theorem 3.5, then we have the following corollary.

**Corollary 3.6** [11, Theorem 3.5] *An IFS  $A = \langle x, \mu_A, \gamma_A \rangle$  of  $S$  is an  $(\in, \in \vee \text{q}_k)$ -intuitionistic fuzzy bi-ideal of  $S$  if and only if it satisfies the conditions*

- (1)  $\left( \begin{array}{l} x \leq y \implies \mu_A(x) \geq \min\{\mu_A(y), 0.5\} \\ \text{and } \gamma_A(x) \leq \max\{\gamma_A(y), 0.5\} \end{array} \right),$
- (2)  $\left( \begin{array}{l} \mu_A(xy) \geq \min\{\mu_A(x), \mu_A(y), 0.5\} \\ \text{and } \gamma_A(xy) \leq \max\{\gamma_A(x), \gamma_A(y), 0.5\} \end{array} \right),$
- (3)  $\left( \begin{array}{l} \mu_A(xyz) \geq \min\{\mu_A(x), \mu_A(z), 0.5\} \\ \text{and } \gamma_A(xyz) \leq \max\{\gamma_A(x), \gamma_A(z), 0.5\} \end{array} \right).$

Obviously, every intuitionistic fuzzy bi-ideal is an  $(\in, \in)$ -intuitionistic fuzzy bi-ideal, and we know that every  $(\in, \in)$ -intuitionistic fuzzy bi-ideal of  $S$  is an  $(\in, \in \vee \text{q})$ -intuitionistic

fuzzy bi-ideal of  $S$ , and every  $(\in, \in \vee q)$ -intuitionistic fuzzy bi-ideal is an  $(\in, \in \vee q_k)$ -intuitionistic fuzzy bi-ideal of  $S$ . But the converse may not be true. The following example shows that every  $(\in, \in \vee q_k)$ -intuitionistic fuzzy bi-ideal of  $S$  may not be an  $(\in, \in \vee q)$ -intuitionistic fuzzy bi-ideal nor an intuitionistic fuzzy bi-ideal of  $S$ .

**Example 3.7** Consider the ordered semigroup of Example 3.4 and define an IFS  $A = \langle x, \mu_A, \gamma_A \rangle$  by

$$\mu_A : S \rightarrow [0, 1] \mid \mu_A(x) = \begin{cases} 0.80 & \text{if } x = a, \\ 0.60 & \text{if } x = b, \\ 0.40 & \text{if } x = c, \\ 0.30 & \text{if } x = d, e \end{cases}$$

and

$$\gamma_A : S \rightarrow [0, 1] \mid \gamma_A(x) = \begin{cases} 0.20 & \text{if } x = a, \\ 0.30 & \text{if } x = b, \\ 0.40 & \text{if } x = c, \\ 0.70 & \text{if } x = d, e. \end{cases}$$

Then  $A = \langle x, \mu_A, \gamma_A \rangle$  is an  $(\in, \in \vee q_{0.4})$ -intuitionistic fuzzy bi-ideal of  $S$ . But

- (1)  $A = \langle x, \mu_A, \gamma_A \rangle$  is not an  $(\in, \in \vee q)$ -intuitionistic fuzzy bi-ideal of  $S$ . Since  $\langle a; (0.8, 0.2) \rangle \in A$  and  $\langle b; (0.6, 0.3) \rangle \in A$  but  $\langle ab; (0.6, 0.3) \rangle \notin \nabla \bar{q} A$ .
- (2)  $A = \langle x, \mu_A, \gamma_A \rangle$  is not an intuitionistic fuzzy bi-ideal of  $S$ . Since

$$\mu_A(ab) = \mu_A(d) = 0.30 < m\{\mu_A(a) = 0.80, \mu_A(b) = 0.60\}$$

and

$$\gamma_A(ab) = \gamma_A(d) = 0.70 > M\{\gamma_A(a) = 0.20, \mu_A(b) = 0.30\}.$$

In the following, we give a condition for an  $(\in, \in \vee q_k)$ -intuitionistic fuzzy bi-ideal of  $S$  to be an ordinary intuitionistic fuzzy bi-ideal of  $S$ .

**Theorem 3.8** *Let  $A = \langle x, \mu_A, \gamma_A \rangle$  be an  $(\in, \in \vee q_k)$ -intuitionistic fuzzy bi-ideal of  $S$ . If  $\mu_A(x) \geq \frac{1-k}{2}$  and  $\gamma_A(x) \leq \frac{1+k}{2}$  for all  $x \in S$ , then  $A = \langle x, \mu_A, \gamma_A \rangle$  is an  $(\in, \in)$ -intuitionistic fuzzy bi-ideal of  $S$ .*

*Proof* The proof is straightforward by Theorem 3.5. □

**Corollary 3.9** [11, Theorem 3.8] *Let  $A = \langle x, \mu_A, \gamma_A \rangle$  be an  $(\in, \in \vee q)$ -intuitionistic fuzzy bi-ideal of  $S$ . If  $\mu_A(x) \geq 0.5$  and  $\gamma_A(x) \leq 0.5$  for all  $x \in S$ , then  $A = \langle x, \mu_A, \gamma_A \rangle$  is an  $(\in, \in)$ -intuitionistic fuzzy bi-ideal of  $S$ .*

*Proof* The proof follows from Theorem 3.8, by taking  $k = 0$ . □

**Theorem 3.10** For an IFS  $A = \langle x, \mu_A, \gamma_A \rangle$  of  $S$ , the following are equivalent:

- (1)  $A = \langle x, \mu_A, \gamma_A \rangle$  is an  $(\in, \in \vee q_k)$ -intuitionistic fuzzy bi-ideal of  $S$ .
- (2)  $(\forall t \in (0, \frac{1-k}{2}]) (\forall s \in [\frac{1-k}{2}, 1)) (C_{(\alpha, \beta)}(A) \neq \emptyset \implies C_{(\alpha, \beta)}(A) \text{ is a bi-ideal of } S)$ .

*Proof* Assume that  $A = \langle x, \mu_A, \gamma_A \rangle$  is an  $(\in, \in \vee q_k)$ -intuitionistic fuzzy bi-ideal of  $S$ , let  $t \in (0, \frac{1-k}{2}]$  and  $s \in [\frac{1-k}{2}, 1)$  be such that  $C_{(\alpha, \beta)}(A) \neq \emptyset$ . Using Theorem 3.5(1), we have

$$\mu_A(x) \geq \min \left\{ \mu_A(y), \frac{1-k}{2} \right\} \quad \text{and} \quad \gamma_A(x) \leq \max \left\{ \gamma_A(y), \frac{1-k}{2} \right\}$$

for any  $x, y \in S$  with  $x \leq y$  and  $x \in C_{(t,s)}(A)$ . It follows that  $\mu_A(x) \geq \min\{t, 0.5\} = t$  and  $\gamma_A(x) \leq \max\{s, 0.5\} = s$ , so that  $y \in C_{(t,s)}(A)$ . Let  $x, y \in C_{(\alpha, \beta)}(A)$ . Then  $\mu_A(x) \geq t$ ,  $\gamma_A(x) \leq s$  and  $\mu_A(y) \geq t$ ,  $\gamma_A(y) \leq s$ . Theorem 3.5(2) implies that

$$\mu_A(xy) \geq \min \left\{ \mu_A(x), \mu_A(y), \frac{1-k}{2} \right\} \geq \min \left\{ t, \frac{1-k}{2} \right\} = t$$

and

$$\gamma_A(xy) \leq \max \left\{ \gamma_A(x), \gamma_A(y), \frac{1-k}{2} \right\} \leq \max \left\{ s, \frac{1-k}{2} \right\} = s.$$

Thus,  $xy \in C_{(\alpha, \beta)}(A)$ . Now let  $x, z \in C_{(\alpha, \beta)}(A)$ . Then  $\mu_A(x) \geq t$ ,  $\gamma_A(x) \leq s$  and  $\mu_A(z) \geq t$ ,  $\gamma_A(z) \leq s$ . Theorem 3.5(3) induces that

$$\mu_A(xyz) \geq \min \left\{ \mu_A(x), \mu_A(z), \frac{1-k}{2} \right\} \geq \min \left\{ t, \frac{1-k}{2} \right\} = t$$

and

$$\gamma_A(xyz) \leq \max \left\{ \gamma_A(x), \gamma_A(z), \frac{1-k}{2} \right\} \leq \max \left\{ s, \frac{1-k}{2} \right\} = s.$$

Thus,  $xyz \in C_{(\alpha, \beta)}(A)$ , therefore,  $C_{(\alpha, \beta)}(A)$  is a bi-ideal of  $S$ .

Conversely, let  $A = \langle x, \mu_A, \gamma_A \rangle$  be an IFS of  $S$  such that  $C_{(\alpha, \beta)}(A)$  is non-empty and is a bi-ideal of  $S$  for all  $t \in (0, \frac{1-k}{2}]$  and  $s \in [\frac{1-k}{2}, 1)$ . If there exist  $a, b \in S$  with  $a \leq b$  and  $b \in C_{(\alpha, \beta)}(A)$  such that  $\mu_A(a) < \min\{\mu_A(b), \frac{1-k}{2}\}$  and  $\gamma_A(a) > \max\{\gamma_A(b), \frac{1-k}{2}\}$ , then  $\mu_A(a) < t_a \leq \min\{\mu_A(b), \frac{1-k}{2}\}$  and  $\gamma_A(a) > s_a \geq \max\{\gamma_A(b), \frac{1-k}{2}\}$  for some  $t_a \in (0, \frac{1-k}{2}]$  and  $s_a \in [\frac{1-k}{2}, 1)$ . Then  $a \notin C_{(t_a, s_a)}(A)$ , a contradiction. Therefore,  $\mu_A(x) \geq \min\{\mu_A(y), \frac{1-k}{2}\}$  and  $\gamma_A(x) > \max\{\gamma_A(y), \frac{1-k}{2}\}$  for all  $x, y \in S$  with  $x \leq y$ . Assume that there exist  $a, b \in S$  such that

$$\mu_A(ab) < \min \left\{ \mu_A(a), \mu_A(b), \frac{1-k}{2} \right\}$$

and  $\gamma_A(ab) > \max\{\gamma_A(a), \gamma_A(b), \frac{1-k}{2}\}$ . Then

$$\mu_A(ab) < t_0 \leq \min \left\{ \mu_A(a), \mu_A(b), \frac{1-k}{2} \right\}$$

and

$$\gamma_A(ab) > s_0 \geq \max \left\{ \gamma_A(a), \gamma_A(b), \frac{1-k}{2} \right\}$$

for some  $t_0 \in (0, \frac{1-k}{2}]$  and  $s_0 \in [\frac{1-k}{2}, 1)$ . It follows that  $a \in C_{(t_0, s_0)}(A)$  and  $b \in C_{(t_0, s_0)}(A)$ , but  $ab \notin C_{(t_0, s_0)}(A)$ . This is a contradiction. Hence

$$\mu_A(xy) \geq \min \left\{ \mu_A(x), \mu_A(y), \frac{1-k}{2} \right\}$$

and

$$\gamma_A(xy) \leq \max \left\{ \gamma_A(x), \gamma_A(y), \frac{1-k}{2} \right\}$$

for all  $x, y \in S$ . Suppose that

$$\mu_A(abc) < \min \left\{ \mu_A(a), \mu_A(c), \frac{1-k}{2} \right\}$$

and

$$\gamma_A(abc) > \max \left\{ \gamma_A(a), \gamma_A(c), \frac{1-k}{2} \right\}$$

for some  $a, b, c \in S$ . Then there exist  $t_1 \in (0, \frac{1-k}{2}]$  and  $s_1 \in [\frac{1-k}{2}, 1)$  such that

$$\mu_A(abc) < t_1 \leq \min \left\{ \mu_A(a), \mu_A(c), \frac{1-k}{2} \right\}$$

and

$$\gamma_A(abc) > s_1 \geq \max \left\{ \gamma_A(a), \gamma_A(c), \frac{1-k}{2} \right\}.$$

Then  $a \in C_{(t_1, s_1)}(A)$  and  $c \in C_{(t_1, s_1)}(A)$ , but  $abc \notin C_{(t_1, s_1)}(A)$ . This is impossible, and hence  $\mu_A(xyz) < \min \{ \mu_A(x), \mu_A(z), \frac{1-k}{2} \}$  and

$$\gamma_A(xyz) > \max \left\{ \gamma_A(x), \gamma_A(z), \frac{1-k}{2} \right\}$$

for all  $x, y, z \in S$ . Therefore,  $A = \langle x, \mu_A, \gamma_A \rangle$  is an  $(\in, \in \vee q_k)$ -intuitionistic fuzzy bi-ideal of  $S$ . □

By taking  $k = 0$  in Theorem 3.10, we get the following corollary.

**Corollary 3.11** [11, Theorem 3.10] *For an IFS  $A = \langle x, \mu_A, \gamma_A \rangle$  of an ordered semigroup  $(S, \cdot, \leq)$ , the following are equivalent*

- (1)  $A = \langle x, \mu_A, \gamma_A \rangle$  is an  $(\in, \in \vee q)$ -intuitionistic fuzzy bi-ideal of  $S$ .
- (2)  $(\forall t \in (0, 0.5]) (\forall s \in [0.5, 1)) (C_{(\alpha, \beta)}(A) \neq \emptyset \implies C_{(\alpha, \beta)}(A) \text{ is a bi-ideal of } S)$ .

For an IFP  $\langle x; (\alpha, \beta) \rangle$  of  $S$  and an IFS  $A = \langle x, \mu_A, \gamma_A \rangle$  of  $S$ , we say that

- (c4)  $\langle x; (\alpha, \beta) \rangle \underline{q}A$  if  $\mu_A(x) + \alpha \geq 1$  and  $\gamma_A(x) + \beta \leq 1$ ,
- (c5)  $\langle x; (\alpha, \beta) \rangle \underline{q}_kA$  if  $\mu_A(x) + \alpha + k \geq 1$  and  $\gamma_A(x) + \beta + k \leq 1$ .

We denote by  $Q_{(\alpha,\beta)}^k(A)$  (resp.  $\underline{Q}_{(\alpha,\beta)}^k(A)$ ) the set  $\{x \in S \mid (x; (\alpha, \beta)) \mathbf{q}_k A\}$  (resp.  $\{x \in S \mid (x; (\alpha, \beta)) \underline{\mathbf{q}}_k A\}$ ), and  $[A]_{(t,s)}^k := \{x \in S \mid [x; (t, s)] \in \vee \mathbf{q}_k A\}$ . It is obvious that  $[A]_{(t,s)}^k = C_{(t,s)}(A) \cup Q_{(t,s)}^k(A)$ .

**Proposition 3.12** *If  $A = \langle x, \mu_A, \gamma_A \rangle$  is an  $(\in, \in \vee \mathbf{q}_k)$ -intuitionistic fuzzy bi-ideal of  $S$ , then*

$$\left( \forall t \in \left( \frac{1-k}{2}, 1 \right] \right) \left( \forall s \in \left[ 0, \frac{1-k}{2} \right) \right) \left( Q_{(t,s)}^k(A) \neq \emptyset \implies Q_{(t,s)}^k(A) \text{ is a bi-ideal of } S \right).$$

*Proof* Assume that  $A = \langle x, \mu_A, \gamma_A \rangle$  is an  $(\in, \in \vee \mathbf{q}_k)$ -intuitionistic fuzzy bi-ideal of  $S$ . Let  $t \in [\frac{1-k}{2}, 1]$  and  $s \in [0, \frac{1-k}{2}]$  be such that  $Q_{(t,s)}^k(A) \neq \emptyset$ . Let  $y \in Q_{(t,s)}^k(A)$  and  $x \in S$  be such that  $x \leq y$ . Then  $\mu_A(y) + t + k > 1$  and  $\gamma_A(y) + s + k < 1$ . By means of Theorem 3.5(1), we have

$$\begin{aligned} \mu_A(x) &\geq \min \left\{ \mu_A(y), \frac{1-k}{2} \right\} \\ &= \begin{cases} \frac{1-k}{2} & \text{if } \mu_A(y) \geq \frac{1-k}{2}, \\ \mu_A(y) & \text{if } \mu_A(y) < \frac{1-k}{2} \end{cases} \\ &> 1 - t - k \end{aligned}$$

and

$$\begin{aligned} \gamma_A(x) &\leq \max \left\{ \gamma_A(y), \frac{1-k}{2} \right\} \\ &= \begin{cases} \frac{1-k}{2} & \text{if } \gamma_A(y) < \frac{1-k}{2}, \\ \gamma_A(y) & \text{if } \gamma_A(y) \geq \frac{1-k}{2} \end{cases} \\ &< 1 - s - k. \end{aligned}$$

It follows that  $x \in Q_{(t,s)}(A)$ . Let  $x, y \in Q_{(t,s)}(A)$ . Then  $\mu_A(x) + t > 1 - k$  and  $\gamma_A(x) + s < 1 - k$ ,  $\mu_A(y) + t > 1 - k$  and  $\gamma_A(y) + s < 1 - k$ . Using (2) of Theorem 3.5, we have that

$$\begin{aligned} \mu_A(xy) &\geq \min \left\{ \mu_A(x), \mu_A(y), \frac{1-k}{2} \right\} \\ &= \begin{cases} \frac{1-k}{2} & \text{if } \min\{\mu_A(x), \mu_A(y)\} \geq \frac{1-k}{2}, \\ \min\{\mu_A(x), \mu_A(y)\} & \text{if } \min\{\mu_A(x), \mu_A(y)\} < \frac{1-k}{2} \end{cases} \\ &> 1 - t - k, \end{aligned}$$

and

$$\begin{aligned} \gamma_A(xy) &\leq \max \left\{ \gamma_A(x), \gamma_A(y), \frac{1-k}{2} \right\} \\ &= \begin{cases} \frac{1-k}{2} & \text{if } \max\{\gamma_A(x), \gamma_A(y)\} < \frac{1-k}{2}, \\ \max\{\gamma_A(x), \gamma_A(y)\} & \text{if } \max\{\gamma_A(x), \gamma_A(y)\} \geq \frac{1-k}{2} \end{cases} \\ &< 1 - s - k. \end{aligned}$$

Thus,  $xy \in Q_{(t,s)}(A)$ . Let  $x, z \in Q_{(t,s)}(A)$  and  $y \in S$ . Then  $\mu_A(x) + t > 1 - k$  and  $\gamma_A(x) + s < 1 - k$ ,  $\mu_A(z) + t > 1 - k$  and  $\gamma_A(z) + s < 1 - k$ . Using (3) of Theorem 3.5, we have that

$$\begin{aligned} \mu_A(xyz) &\geq \min \left\{ \mu_A(x), \mu_A(z), \frac{1-k}{2} \right\} \\ &= \begin{cases} \frac{1-k}{2} & \text{if } \min\{\mu_A(x), \mu_A(z)\} \geq \frac{1-k}{2}, \\ \min\{\mu_A(x), \mu_A(z)\} & \text{if } \min\{\mu_A(x), \mu_A(z)\} < \frac{1-k}{2} \end{cases} \\ &> 1 - t - k \end{aligned}$$

and

$$\begin{aligned} \gamma_A(xyz) &\leq \max \left\{ \gamma_A(x), \gamma_A(z), \frac{1-k}{2} \right\} \\ &= \begin{cases} \frac{1-k}{2} & \text{if } \max\{\gamma_A(x), \gamma_A(z)\} < \frac{1-k}{2}, \\ \max\{\gamma_A(x), \gamma_A(z)\} & \text{if } \max\{\gamma_A(x), \gamma_A(z)\} \geq \frac{1-k}{2} \end{cases} \\ &< 1 - s. \end{aligned}$$

Hence  $xyz \in Q_{(t,s)}^k(A)$ . Therefore,  $Q_{(t,s)}^k(A)$  is a bi-ideal of  $S$ . □

**Theorem 3.13** For any IFS  $A = \langle x, \mu_A, \gamma_A \rangle$  of  $S$ , the following are equivalent

- (1)  $A = \langle x, \mu_A, \gamma_A \rangle$  is an  $(\in, \in \vee q_k)$ -intuitionistic fuzzy bi-ideal of  $S$ .
- (2)  $(\forall t \in (0, 1]) (\forall s \in [0, 1)) ([A]_{(t,s)}^k \neq \emptyset \implies [A]_{(t,s)}^k \text{ is a bi-ideal of } S)$ .

We call  $[A]_{(t,s)}^k$  an  $(\in \vee q)$ -level bi-ideal of  $A = \langle x, \mu_A, \gamma_A \rangle$ .

*Proof* Assume that  $A = \langle x, \mu_A, \gamma_A \rangle$  is an  $(\in, \in \vee q_k)$ -intuitionistic fuzzy bi-ideal of  $S$ , and let  $t \in (0, 1]$  and  $s \in [0, 1)$  be such that  $[A]_{(t,s)}^k \neq \emptyset$ . Let  $y \in [A]_{(t,s)}^k$  and  $x \in S$  be such that  $x \leq y$ . Then  $y \in C_{(t,s)}(A)$  or  $y \in Q_{(t,s)}^k(A)$ , i.e.,  $\mu_A(y) \geq t$  and  $\gamma_A(y) \leq s$  or  $\mu_A(y) + t > 1 - k$  and  $\gamma_A(y) + s < 1 - k$ . Using Theorem 3.5(1), we get

$$\mu_A(x) \geq \min \left\{ \mu_A(y), \frac{1-k}{2} \right\} \quad \text{and} \quad \gamma_A(x) \leq \max \left\{ \gamma_A(y), \frac{1-k}{2} \right\}. \quad (3.1)$$

We consider two cases  $\mu_A(y) \leq \frac{1-k}{2}$ ,  $\gamma_A(y) \geq \frac{1-k}{2}$  and  $\mu_A(y) > \frac{1-k}{2}$ ,  $\gamma_A(y) < \frac{1-k}{2}$ . The first case implies from (3.1) that  $\mu_A(x) \geq \mu_A(y)$  and  $\gamma_A(x) \leq \gamma_A(y)$ . Thus, if  $\mu_A(y) \geq t$  and  $\gamma_A(y) \leq s$ , then  $\mu_A(x) \geq t$  and  $\gamma_A(x) \leq s$ , and so,  $x \in C_{(t,s)}(A) \subseteq [A]_{(t,s)}^k$ . If  $\mu_A(y) + t > 1 - k$  and  $\gamma_A(y) + s < 1 - k$ , then  $\mu_A(x) + t \geq \mu_A(y) + t > 1 - k$  and  $\gamma_A(x) + s \leq \gamma_A(y) + s < 1 - k$ , which implies that  $x \in C_{(t,s)}(A) \subseteq [A]_{(t,s)}^k$ , i.e.,  $x \in Q_{(t,s)}^k(A) \subseteq [A]_{(t,s)}^k$ . Combining the second case and (3.1), we have  $\mu_A(x) \geq \frac{1-k}{2}$  and  $\gamma_A(x) \leq \frac{1-k}{2}$ . If  $t \leq \frac{1-k}{2}$  and  $s \geq \frac{1-k}{2}$ , then  $\mu_A(x) \geq t$  and  $\gamma_A(x) \leq s$ , and hence  $x \in C_{(t,s)}(A) \subseteq [A]_{(t,s)}^k$ . If  $t > \frac{1-k}{2}$  and  $s < \frac{1-k}{2}$ , then  $\mu_A(x) + t > \frac{1-k}{2} + \frac{1-k}{2} = 1 - k$  and  $\gamma_A(x) + s < \frac{1-k}{2} + \frac{1-k}{2} = 1 - k$ , which implies that  $x \in Q_{(t,s)}^k(A) \subseteq [A]_{(t,s)}^k$ . Therefore,  $[A]_{(t,s)}^k$  satisfies the condition (b1). Let  $x, y \in [A]_{(t,s)}^k$ . Then  $x \in C_{(t,s)}(A)$  or  $(x; (t, s))q_k A$  and  $y \in C_{(t,s)}(A)$  or  $(y; (t, s))q_k A$ , that is,  $\mu_A(x) \geq t$ ,  $\gamma_A(x) \leq s$  or  $\mu_A(x) + t + k > 1$ ,  $\gamma_A(x) + s + k < 1$  and  $\mu_A(y) \geq t$ ,  $\gamma_A(y) \leq s$  or  $\mu_A(y) + t + k > 1$ ,  $\gamma_A(y) + s + k < 1$ . We consider the following four cases

- (i)  $\mu_A(x) \geq t$ ,  $\gamma_A(x) \leq s$  and  $\mu_A(y) \geq t$ ,  $\gamma_A(y) \leq s$ ,
- (ii)  $\mu_A(x) \geq t$ ,  $\gamma_A(x) \leq s$  and  $\mu_A(y) + t + k > 1$ ,  $\gamma_A(y) + s + k < 1$ ,

- (iii)  $\mu_A(x) + t + k > 1$ ,  $\gamma_A(x) + s + k < 1$  and  $\mu_A(y) \geq t$ ,  $\gamma_A(y) \leq s$ ,
- (iv)  $\mu_A(x) + t + k > 1$ ,  $\gamma_A(x) + s + k < 1$  and  $\mu_A(y) + t + k > 1$ ,  $\gamma_A(y) + s + k < 1$ .

For the case (i), Theorem 3.5(2) implies that

$$\begin{aligned} \mu_A(xy) &\geq \min\left\{\mu_A(x), \mu_A(y), \frac{1-k}{2}\right\} \geq \min\left\{t, \frac{1-k}{2}\right\} \\ &= \begin{cases} \frac{1-k}{2} & \text{if } t > \frac{1-k}{2}, \\ t & \text{if } t \leq \frac{1-k}{2} \end{cases} \end{aligned}$$

and

$$\begin{aligned} \gamma_A(xy) &\leq \max\left\{\gamma_A(x), \gamma_A(y), \frac{1-k}{2}\right\} \leq \max\left\{s, \frac{1-k}{2}\right\} \\ &= \begin{cases} \frac{1-k}{2} & \text{if } s < \frac{1-k}{2}, \\ s & \text{if } s \geq \frac{1-k}{2}. \end{cases} \end{aligned}$$

Then  $xy \in C_{(t,s)}(A)$  or  $\mu_A(xy) + t + k > \frac{1-k}{2} + \frac{1-k}{2} + k = 1$  and  $\gamma_A(xy) + s + k < \frac{1-k}{2} + \frac{1-k}{2} + k = 1$ , that is,  $xy \in Q_{(t,s)}^k(A)$ . Hence  $xy \in C_{(t,s)}(A) \cup Q_{(t,s)}^k(A) = [A]_{(t,s)}^k$ . For the second case, assume that  $t > \frac{1-k}{2}$  and  $s < \frac{1-k}{2}$ , then  $1 - t - k \leq 1 - t < \frac{1-k}{2}$  and  $1 - s - k \geq 1 - s \geq \frac{1-k}{2}$ . Hence

$$\begin{aligned} \mu_A(xy) &\geq \min\left\{\mu_A(x), \mu_A(y), \frac{1-k}{2}\right\} \\ &= \begin{cases} \min\{\mu_A(y), \frac{1-k}{2}\} > 1 - t - k & \text{if } \min\{\mu_A(y), \frac{1-k}{2}\} \leq \mu_A(x), \\ \mu_A(x) \geq t & \text{if } \min\{\mu_A(y), \frac{1-k}{2}\} > \mu_A(x) \end{cases} \end{aligned}$$

and

$$\begin{aligned} \gamma_A(xy) &\leq \max\left\{\gamma_A(x), \gamma_A(y), \frac{1-k}{2}\right\} \\ &= \begin{cases} \max\{\gamma_A(y), \frac{1-k}{2}\} < 1 - s - k & \text{if } \max\{\gamma_A(y), \frac{1-k}{2}\} \geq \gamma_A(x), \\ \gamma_A(x) \leq s & \text{if } \max\{\gamma_A(y), \frac{1-k}{2}\} < \gamma_A(x). \end{cases} \end{aligned}$$

Thus  $xy \in C_{(t,s)}(A) \cup Q_{(t,s)}^k(A) = [A]_{(t,s)}^k$ . Suppose that  $t \leq \frac{1-k}{2}$  and  $s \geq \frac{1-k}{2}$ . Then

$$\begin{aligned} \mu_A(xy) &\geq \min\left\{\mu_A(x), \mu_A(y), \frac{1-k}{2}\right\} \\ &= \begin{cases} \min\{\mu_A(x), \frac{1-k}{2}\} \geq t & \text{if } \min\{\mu_A(x), \frac{1-k}{2}\} \leq \mu_A(y), \\ \mu_A(y) > 1 - t - k & \text{if } \min\{\mu_A(x), \frac{1-k}{2}\} > \mu_A(y), \end{cases} \end{aligned}$$

and

$$\begin{aligned} \gamma_A(xy) &\leq \max\left\{\gamma_A(x), \gamma_A(y), \frac{1-k}{2}\right\} \\ &= \begin{cases} \max\{\gamma_A(x), \frac{1-k}{2}\} \leq s & \text{if } \max\{\gamma_A(x), \frac{1-k}{2}\} \geq \gamma_A(y), \\ \gamma_A(y) < 1 - s - k & \text{if } \max\{\gamma_A(x), \frac{1-k}{2}\} < \gamma_A(y). \end{cases} \end{aligned}$$

Thus  $xy \in C_{(t,s)}(A) \cup Q_{(t,s)}^k(A) = [A]_{(t,s)}^k$ . We have a similar result for the case (iii). For the final case, if  $t > \frac{1-k}{2}$  and  $s < \frac{1-k}{2}$ , then  $1-t-k < \frac{1-k}{2}$  and  $1-s-k > \frac{1-k}{2}$ . Hence

$$\begin{aligned} \mu_A(xy) &\geq \min\left\{\mu_A(x), \mu_A(y), \frac{1-k}{2}\right\} \\ &= \frac{1-k}{2} > 1-t-k \quad \text{whenever } \min\{\mu_A(x), \mu_A(y)\} \geq \frac{1-k}{2} \end{aligned}$$

and

$$\begin{aligned} \mu_A(xy) &\leq \min\left\{\mu_A(x), \mu_A(y), \frac{1-k}{2}\right\} \\ &= \min\{\mu_A(x), \mu_A(y)\} \\ &> 1-s-k \quad \text{whenever } \min\{\mu_A(x), \mu_A(y)\} \geq \frac{1-k}{2} \end{aligned}$$

and

$$\begin{aligned} \gamma_A(xy) &\leq \max\left\{\gamma_A(x), \gamma_A(y), \frac{1-k}{2}\right\} \\ &= \frac{1-k}{2} < 1-s-k \quad \text{whenever } \max\{\gamma_A(x), \gamma_A(y)\} \leq \frac{1-k}{2} \end{aligned}$$

and

$$\begin{aligned} \gamma_A(xy) &\leq \max\left\{\gamma_A(x), \gamma_A(y), \frac{1-k}{2}\right\} \\ &= \max\{\gamma_A(x), \gamma_A(y)\} \\ &< 1-s-k, \quad \text{whenever } \max\{\gamma_A(x), \gamma_A(y)\} \leq \frac{1-k}{2}. \end{aligned}$$

Thus,  $xy \in Q_{(t,s)}^k(A) \subseteq [A]_{(t,s)}^k$ . If  $t \leq \frac{1-k}{2}$  and  $s \geq \frac{1-k}{2}$ , then

$$\begin{aligned} \mu_A(xy) &\geq \min\left\{\mu_A(x), \mu_A(y), \frac{1-k}{2}\right\} \\ &= \begin{cases} \frac{1-k}{2} \geq t & \text{if } \min\{\mu_A(x), \mu_A(y)\} \geq \frac{1-k}{2}, \\ \min\{\mu_A(x), \mu_A(y)\} > 1-t-k & \text{if } \min\{\mu_A(x), \mu_A(y)\} < \frac{1-k}{2}, \end{cases} \end{aligned}$$

and

$$\begin{aligned} \gamma_A(xy) &\leq \max\left\{\gamma_A(x), \gamma_A(y), \frac{1-k}{2}\right\} \\ &= \begin{cases} \frac{1-k}{2} \leq s & \text{if } \max\{\gamma_A(x), \gamma_A(y)\} \leq \frac{1-k}{2}, \\ \max\{\gamma_A(x), \gamma_A(y)\} < 1-s-k & \text{if } \max\{\gamma_A(x), \gamma_A(y)\} > \frac{1-k}{2}, \end{cases} \end{aligned}$$

which implies that  $xy \in Q_{(t,s)}^k(A) \subseteq [A]_{(t,s)}^k$ . Let  $x, z \in [A]_{(t,s)}^k$ . Then  $x \in C_{(t,s)}(A)$  or  $\langle x; (t, s) \rangle_{q_k A}$  and  $z \in C_{(t,s)}(A)$  or  $\langle z; (t, s) \rangle_{q_k A}$ , that is,  $\mu_A(x) \geq t$ ,  $\gamma_A(x) \leq s$  or  $\mu_A(x) + t + k > 1$ ,



$\gamma_A(x) + s + k < 1$  and  $\mu_A(z) \geq t, \gamma_A(z) \leq s$  or  $\mu_A(z) + t + k > 1, \gamma_A(z) + s + k < 1$ . We consider the following four cases

- (i)  $\mu_A(x) \geq t, \gamma_A(x) \leq s$  and  $\mu_A(z) \geq t, \gamma_A(z) \leq s$ ,
- (ii)  $\mu_A(x) \geq t, \gamma_A(x) \leq s$  and  $\mu_A(z) + t > 1 - k, \gamma_A(z) + s < 1 - k$ ,
- (iii)  $\mu_A(x) + t > 1 - k, \gamma_A(x) + s < 1 - k$  and  $\mu_A(z) \geq t, \gamma_A(z) \leq s$ ,
- (iv)  $\mu_A(x) + t > 1 - k, \gamma_A(x) + s < 1 - k$  and  $\mu_A(z) + t > 1 - k, \gamma_A(z) + s < 1 - k$ .

For the case (i), Theorem 3.5(3) implies that

$$\mu_A(xyz) \geq \min \left\{ \mu_A(x), \mu_A(z), \frac{1-k}{2} \right\} \geq \min \left\{ t, \frac{1-k}{2} \right\} = \begin{cases} \frac{1-k}{2} & \text{if } t > \frac{1-k}{2}, \\ t & \text{if } t \leq \frac{1-k}{2} \end{cases}$$

and

$$\gamma_A(xyz) \leq \max \left\{ \gamma_A(x), \gamma_A(z), \frac{1-k}{2} \right\} \leq \max \left\{ s, \frac{1-k}{2} \right\} = \begin{cases} \frac{1-k}{2} & \text{if } s < \frac{1-k}{2}, \\ s & \text{if } s \geq \frac{1-k}{2}. \end{cases}$$

Then  $xyz \in C_{(t,s)}(A)$  or  $\mu_A(xyz) + t + k > \frac{1-k}{2} + \frac{1-k}{2} + k = 1$  and  $\gamma_A(xyz) + s + k < \frac{1-k}{2} + \frac{1-k}{2} + k = 1$ , that is,  $xyz \in Q_{(t,s)}^k(A)$ . Hence  $xyz \in C_{(t,s)}(A) \cup Q_{(t,s)}^k(A) = [A]_{(t,s)}^k$ . For the second case, assume that  $t > \frac{1-k}{2}$  and  $s < \frac{1-k}{2}$ , then  $1 - t - k \leq 1 - t < \frac{1-k}{2}$  and  $1 - s - k \geq 1 - s \geq \frac{1-k}{2}$ . Hence

$$\begin{aligned} \mu_A(xyz) &\geq \min \left\{ \mu_A(x), \mu_A(z), \frac{1-k}{2} \right\} \\ &= \begin{cases} \min \{ \mu_A(x), \frac{1-k}{2} \} > 1 - t - k & \text{if } \min \{ \mu_A(z), \frac{1-k}{2} \} \leq \mu_A(x), \\ \mu_A(x) \geq t & \text{if } \min \{ \mu_A(z), \frac{1-k}{2} \} > \mu_A(x) \end{cases} \end{aligned}$$

and

$$\begin{aligned} \gamma_A(xyz) &\leq \max \left\{ \gamma_A(x), \gamma_A(z), \frac{1-k}{2} \right\} \\ &= \begin{cases} \max \{ \gamma_A(z), \frac{1-k}{2} \} < 1 - s - k & \text{if } \max \{ \gamma_A(z), \frac{1-k}{2} \} \geq \gamma_A(x), \\ \gamma_A(x) \leq s & \text{if } \max \{ \gamma_A(z), \frac{1-k}{2} \} < \gamma_A(x). \end{cases} \end{aligned}$$

Thus,  $xyz \in C_{(t,s)}(A) \cup Q_{(t,s)}^k(A) = [A]_{(t,s)}^k$ . Suppose that  $t \leq \frac{1-k}{2}$  and  $s \geq \frac{1-k}{2}$ . Then

$$\begin{aligned} \mu_A(xyz) &\geq \min \left\{ \mu_A(x), \mu_A(z), \frac{1-k}{2} \right\} \\ &= \begin{cases} \min \{ \mu_A(x), \frac{1-k}{2} \} \geq t & \text{if } \min \{ \mu_A(x), \frac{1-k}{2} \} \leq \mu_A(z), \\ \mu_A(z) > 1 - t - k & \text{if } \min \{ \mu_A(x), \frac{1-k}{2} \} > \mu_A(z), \end{cases} \end{aligned}$$

and

$$\begin{aligned} \gamma_A(xyz) &\leq \max \left\{ \gamma_A(x), \gamma_A(z), \frac{1-k}{2} \right\} \\ &= \begin{cases} \max \{ \gamma_A(x), \frac{1-k}{2} \} \leq s & \text{if } \max \{ \gamma_A(x), \frac{1-k}{2} \} \geq \gamma_A(z), \\ \gamma_A(y) < 1 - s - k & \text{if } \max \{ \gamma_A(x), \frac{1-k}{2} \} < \gamma_A(z). \end{cases} \end{aligned}$$

Thus,  $xyz \in C_{(t,s)}(A) \cup Q_{(t,s)}^k(A) = [A]_{(t,s)}^k$ . We have a similar result for the case (iii). For the final case, if  $t > \frac{1-k}{2}$  and  $s < \frac{1-k}{2}$ , then  $1-t-k < \frac{1-k}{2}$  and  $1-s-k > \frac{1-k}{2}$ . Hence

$$\begin{aligned} \mu_A(xyz) &\geq \min\left\{\mu_A(x), \mu_A(z), \frac{1-k}{2}\right\} \\ &= \frac{1-k}{2} > 1-t-k \quad \text{whenever } \min\{\mu_A(x), \mu_A(z)\} \geq \frac{1-k}{2} \end{aligned}$$

and

$$\begin{aligned} \mu_A(xyz) &\leq \min\left\{\mu_A(x), \mu_A(z), \frac{1-k}{2}\right\} \\ &= \min\{\mu_A(x), \mu_A(z)\} \\ &> 1-s-k \quad \text{whenever } \min\{\mu_A(x), \mu_A(z)\} \geq \frac{1-k}{2} \end{aligned}$$

and

$$\begin{aligned} \gamma_A(xyz) &\leq \max\left\{\gamma_A(x), \gamma_A(z), \frac{1-k}{2}\right\} \\ &= \frac{1-k}{2} < 1-s-k \quad \text{whenever } \max\{\gamma_A(x), \gamma_A(z)\} \leq \frac{1-k}{2} \end{aligned}$$

and

$$\begin{aligned} \gamma_A(xyz) &\leq \max\left\{\gamma_A(x), \gamma_A(z), \frac{1-k}{2}\right\} \\ &= \max\{\gamma_A(x), \gamma_A(z)\} \\ &< 1-s-k \quad \text{whenever } \max\{\gamma_A(x), \gamma_A(z)\} \leq \frac{1-k}{2}. \end{aligned}$$

Thus,  $xyz \in Q_{(t,s)}^k(A) \subseteq [A]_{(t,s)}^k$ . If  $t \leq \frac{1-k}{2}$  and  $s \geq \frac{1-k}{2}$ , then

$$\begin{aligned} \mu_A(xyz) &\geq \min\left\{\mu_A(x), \mu_A(z), \frac{1-k}{2}\right\} \\ &= \begin{cases} \frac{1-k}{2} \geq t & \text{if } \min\{\mu_A(x), \mu_A(z)\} \geq \frac{1-k}{2}, \\ \min\{\mu_A(x), \mu_A(z)\} > 1-t-k & \text{if } \min\{\mu_A(x), \mu_A(z)\} < \frac{1-k}{2}, \end{cases} \end{aligned}$$

and

$$\begin{aligned} \gamma_A(xyz) &\leq \max\left\{\gamma_A(x), \gamma_A(z), \frac{1-k}{2}\right\} \\ &= \begin{cases} \frac{1-k}{2} \leq s & \text{if } \max\{\gamma_A(x), \gamma_A(z)\} \leq \frac{1-k}{2}, \\ \max\{\gamma_A(x), \gamma_A(z)\} < 1-s-k & \text{if } \max\{\gamma_A(x), \gamma_A(z)\} > \frac{1-k}{2}, \end{cases} \end{aligned}$$

which implies that  $xyz \in Q_{(t,s)}^k(A) \subseteq [A]_{(t,s)}^k$ . Therefore,  $[A]_{(t,s)}^k$  is a bi-ideal of  $S$ .

Conversely, suppose that (2) is valid. If there exist  $a, b \in S$  such that  $a \leq b$  and

$$\mu_A(a) < \min\left\{\mu_A(b), \frac{1-k}{2}\right\} \quad \text{and} \quad \gamma_A(a) > \max\left\{\gamma_A(b), \frac{1-k}{2}\right\}.$$

Then  $\mu_A(a) < t_a \leq \min\{\mu_A(b), \frac{1-k}{2}\}$  and  $\gamma_A(a) > s_a \geq \max\{\gamma_A(b), \frac{1-k}{2}\}$  for some  $t_a \in (0, 1]$  and  $s_a \in [0, 1)$ . It follows that  $b \in C_{(t_a, s_a)}(A) \subseteq [A]_{(t_a, s_a)}^k$  but  $a \notin C_{(t_a, s_a)}(A)$ . Also we have  $\mu_A(a) + t_a < 2t_a \leq 1 - k$  and  $\gamma_A(a) + s_a > 2s_a \geq 1 - k$  and so  $\langle a; (t_a, s_a) \rangle \notin \overline{q_k}A$ , i.e.,  $b \notin Q_{(t_a, s_a)}^k(A)$ . Therefore,  $a \notin [A]_{(t_a, s_a)}$ , a contradiction. Hence  $\mu_A(x) \geq \min\{\mu_A(y), \frac{1-k}{2}\}$  and  $\gamma_A(x) \leq \max\{\gamma_A(y), \frac{1-k}{2}\}$  for all  $x, y \in S$  with  $x \leq y$ . Suppose that there exist  $a, b \in S$  such that

$$\mu_A(ab) < \min\left\{\mu_A(a), \mu_A(b), \frac{1-k}{2}\right\}$$

and

$$\gamma_A(ab) > \max\left\{\gamma_A(a), \gamma_A(b), \frac{1-k}{2}\right\}.$$

Then

$$\mu_A(ab) < t \leq \min\left\{\mu_A(a), \mu_A(b), \frac{1-k}{2}\right\}$$

and

$$\gamma_A(ab) > s \geq \max\left\{\gamma_A(a), \gamma_A(b), \frac{1-k}{2}\right\}$$

for  $t \in (0, 1]$  and  $s \in [0, 1)$ . It follows that  $a \in C_{(t, s)}(A) \subseteq [A]_{(t, s)}^k$  and  $b \in C_{(t, s)}(A) \subseteq [A]_{(t, s)}^k$ , so from (b2)  $ab \in [A]_{(t, s)}^k$ . Thus,  $\mu_A(ab) \geq t$ ,  $\gamma_A(ab) \leq s$  or  $\mu_A(ab) + t + k > 1$ ,  $\gamma_A(ab) + s + k < 1$ , a contradiction. Therefore,  $\mu_A(xy) \geq \min\{\mu_A(x), \mu_A(y), \frac{1-k}{2}\}$  and  $\gamma_A(xy) \leq \max\{\gamma_A(x), \gamma_A(y), \frac{1-k}{2}\}$  for all  $x, y \in S$ . Assume that there exist  $a, b, c \in S$  such that

$$\mu_A(abc) < \min\left\{\mu_A(a), \mu_A(c), \frac{1-k}{2}\right\}$$

and

$$\gamma_A(abc) > \max\left\{\gamma_A(a), \gamma_A(c), \frac{1-k}{2}\right\}.$$

Then  $\mu_A(abc) < t_0 \leq \min\{\mu_A(a), \mu_A(c), \frac{1-k}{2}\}$  and  $\gamma_A(abc) > s_0 \geq \max\{\gamma_A(a), \gamma_A(c), \frac{1-k}{2}\}$  for  $t_0 \in (0, 1]$  and  $s_0 \in [0, 1)$ . It follows that  $a \in C_{(t_0, s_0)}(A) \subseteq [A]_{(t_0, s_0)}^k$  and  $c \in C_{(t_0, s_0)}(A) \subseteq [A]_{(t_0, s_0)}^k$  so from (b2)  $abc \in [A]_{(t_0, s_0)}$ . Thus,  $\mu_A(abc) \geq t_0$ ,  $\gamma_A(abc) \leq s_0$  or  $\mu_A(abc) + t_0 + k > 1$ ,  $\gamma_A(abc) + s_0 + k < 1$ , a contradiction. Therefore,  $\mu_A(xyz) \geq \min\{\mu_A(x), \mu_A(z), \frac{1-k}{2}\}$  and  $\gamma_A(xyz) \leq \max\{\gamma_A(x), \gamma_A(z), \frac{1-k}{2}\}$  for all  $x, y, z \in S$ . Thus,  $A = \langle x, \mu_A, \gamma_A \rangle$  is an  $(\in, \in \vee q_k)$ -intuitionistic fuzzy bi-ideal of  $S$ .  $\square$

**Theorem 3.14** *Let  $\{A_i | i \in \Lambda\}$  be a family of  $(\in, \in \vee q_k)$ -intuitionistic fuzzy bi-ideals of  $S$ . Then  $A = \bigcap_{i \in \Lambda} A_i$  is an  $(\in, \in \vee q_k)$ -intuitionistic fuzzy bi-ideal of  $S$ , where  $\bigcap_{i \in \Lambda} A_i = \langle x, \bigwedge_{i \in \Lambda} \mu_{A_i}, \bigvee_{i \in \Lambda} \gamma_{A_i} \rangle$  and*

$$\bigwedge_{i \in \Lambda} \mu_{A_i}(x) = \inf\{\mu_{A_i}(x) | i \in \Lambda \text{ and } x \in S\},$$

$$\bigvee_{i \in \Lambda} \gamma_{A_i}(x) = \sup\{\gamma_{A_i}(x) | i \in \Lambda \text{ and } x \in S\}.$$

*Proof* Let  $x, y \in S$  with  $x \leq y$ ,  $t \in (0, 1]$  and  $s \in [0, 1)$  be such that  $\langle y; (t, s) \rangle \in A$ . Assume that  $\langle x; (t, s) \rangle \notin \overline{\vee q_k} A$ . Then  $\mu_A(x) < t$ ,  $\gamma_A(x) > s$  and  $\mu_A(x) + t + k \leq 1$ ,  $\gamma_A(x) + s + k \geq 1$ , which imply that

$$\mu_A(x) < \frac{1-k}{2} \quad \text{and} \quad \gamma_A(x) > \frac{1-k}{2}.$$

Let  $\Omega_1 := \{i \in \Lambda | \mu_{A_i}(x) \geq t, \gamma_{A_i}(x) \leq s\}$  and  $\Omega_2 := \{i \in \Lambda | \langle x; (t, s) \rangle q_k A_i \text{ and } \mu_{A_i}(x) < t, \gamma_{A_i}(x) > s\}$ .

Then  $\Lambda = \Omega_1 \cup \Omega_2$  and  $\Omega_1 \cap \Omega_2 = \emptyset$ . If  $\Omega_2 = \emptyset$ , then  $\mu_{A_i}(x) \geq t$ ,  $\gamma_{A_i}(x) \leq s$  for all  $i \in \Lambda$ , and so,  $\mu_A(x) \geq t$ ,  $\gamma_A(x) \leq s$ , which is a contradiction. Hence  $\Omega_2 \neq \emptyset$ , and so,  $\mu_{A_i}(x) + t + k > 1$ ,  $\gamma_{A_i}(x) + s + k < 1$  and  $\mu_{A_i}(x) < t$ ,  $\gamma_{A_i}(x) > s$  for every  $i \in \Lambda$ . It follows that  $t > \frac{1-k}{2}$  and  $s < \frac{1-k}{2}$ , so that  $\mu_{A_i}(x) \geq \mu_A(x) \geq t > \frac{1-k}{2}$  and  $\gamma_{A_i}(x) \leq \gamma_A(x) \leq s < \frac{1-k}{2}$  for all  $i \in \Lambda$ . Now, suppose that  $t_x := \mu_{A_i}(x) < \frac{1-k}{2}$  and  $s_x := \gamma_{A_i}(x) > \frac{1-k}{2}$  for some  $i \in \Lambda$ . Let  $t'_x \in (0, \frac{1-k}{2})$  and  $s'_x \in (\frac{1-k}{2}, 1)$  be such that  $t_x < t'_x$  and  $s'_x < s_x$ . Then  $\mu_{A_i}(y) > \frac{1-k}{2} > t'_x$  and  $\gamma_{A_i}(y) < \frac{1-k}{2} < s'_x$ , i.e.,  $\langle y; (t'_x, s'_x) \rangle \in A_i$ . But  $\mu_{A_i}(x) = t_x < t'_x$ ,  $\gamma_{A_i}(x) = s_x > s'_x$  and  $\mu_{A_i}(x) + t'_x + k < 1$ ,  $\gamma_{A_i}(x) + s'_x + k > 1$ , that is,  $\langle x; (t'_x, s'_x) \rangle \notin \overline{\vee q_k} A_i$ . This is a contradiction, and so,  $\mu_{A_i}(x) \geq \frac{1-k}{2}$  and  $\gamma_{A_i}(x) \leq \frac{1-k}{2}$  for all  $i \in \Lambda$ . Thus,  $\mu_A(x) \geq \frac{1-k}{2}$  and  $\gamma_A(x) \leq \frac{1-k}{2}$ , which is impossible. Therefore,  $\langle y; (t, s) \rangle \in \overline{\vee q_k} A$ .

Let  $x, y \in S$ ,  $t_1, t_2 \in (0, 1]$  and  $s_1, s_2 \in [0, 1)$  be such that  $\langle x; (t_1, s_1) \rangle \in A$  and  $\langle y; (t_2, s_2) \rangle \in A$ . Assume that  $\langle xy; \min\{t_1, t_2\}, \max\{s_1, s_2\} \rangle \notin \overline{\vee q_k} A$ . Then

$$\mu_A(xy) < \min\{t_1, t_2\}, \quad \gamma_A(xy) > \max\{s_1, s_2\}$$

and

$$\mu_A(xy) + \min\{t_1, t_2\} < 1 - k, \quad \gamma_A(xy) + \max\{s_1, s_2\} > 1 - k.$$

It follows that  $\mu_A(xy) < \frac{1-k}{2}$  and  $\gamma_A(xy) > \frac{1-k}{2}$ . Let  $\Omega_3 := \{i \in \Lambda | \mu_{A_i}(xy) \geq \min\{t_1, t_2\} \text{ and } \gamma_{A_i}(xy) \leq \max\{s_1, s_2\}\}$  and  $\Omega_4 := \{i \in \Lambda | \langle xy; \min\{t_1, t_2\}, \max\{s_1, s_2\} \rangle q_k A_i \text{ and } \mu_{A_i}(xy) < \min\{t_1, t_2\} \text{ and } \gamma_{A_i}(xy) > \max\{s_1, s_2\}\}$ . Then  $\Omega_3 \cup \Omega_4 = \Lambda$  and  $\Omega_3 \cap \Omega_4 = \emptyset$ . If  $\Omega_4 = \emptyset$ , then  $\mu_{A_i}(xy) \geq \min\{t_1, t_2\}$  and  $\gamma_{A_i}(xy) \leq \max\{s_1, s_2\}$  for all  $i \in \Lambda$ , and so,  $\mu_A(xy) \geq \min\{t_1, t_2\}$  and  $\gamma_A(xy) \leq \max\{s_1, s_2\}$ , which is a contradiction. Hence  $\Omega_4 \neq \emptyset$  and  $\langle xy; \min\{t_1, t_2\}, \max\{s_1, s_2\} \rangle q_k A_i$ , i.e.,  $\mu_{A_i}(xy) + \min\{t_1, t_2\} > 1 - k$ ,  $\gamma_{A_i}(xy) + \max\{s_1, s_2\} < 1 - k$ . It follows that  $\min\{t_1, t_2\} > \frac{1-k}{2}$  and  $\max\{s_1, s_2\} < \frac{1-k}{2}$ , so that  $\mu_{A_i}(x) \geq \mu_A(x) \geq t_1 \geq \min\{t_1, t_2\} > \frac{1-k}{2}$  and  $\gamma_{A_i}(x) \leq \gamma_A(x) \leq s_1 \leq \max\{s_1, s_2\}$  for all  $i \in \Lambda$ . By a similar way, we have  $\mu_{A_i}(y) \geq \mu_A(y) \geq t_2 \geq \min\{t_1, t_2\} > \frac{1-k}{2}$  and  $\gamma_{A_i}(y) \leq \gamma_A(y) \leq s_2 \leq \max\{s_1, s_2\}$  for all  $i \in \Lambda$ . Now, suppose that  $t := \mu_{A_i}(xy) < \frac{1-k}{2}$  and  $s := \gamma_{A_i}(xy) > \frac{1-k}{2}$  for some  $i \in \Lambda$ . Let  $t' \in (0, \frac{1-k}{2})$  and  $s' \in (\frac{1-k}{2}, 1)$  be such that  $t < t'$  and  $s > s'$ . Then  $\mu_{A_i}(x) > \frac{1-k}{2} > t'$ ,  $\gamma_{A_i}(x) < \frac{1-k}{2} < s'$  and  $\mu_{A_i}(y) > \frac{1-k}{2} > t'$ ,

$\gamma_{A_i}(y) < \frac{1-k}{2} < s'$ , i.e.,  $\langle x; (t', s') \rangle \in A$  and  $\langle y; (t', s') \rangle \in A$ . But  $\mu_{A_i}(xy) = t < t'$ ,  $\gamma_{A_i}(xy) = s > s'$  and  $\mu_{A_i}(xy) + t' + k < 1$ ,  $\gamma_{A_i}(xy) + s' + k > 1$ , that is,  $\langle xy; (t', s') \rangle \notin \bigvee \overline{q_k} A_i$ . This is a contradiction. Thus,  $\mu_{A_i}(xy) \geq \frac{1-k}{2}$  and  $\gamma_{A_i}(xy) \leq \frac{1-k}{2}$  for all  $i \in \Lambda$ . Therefore,  $\mu_A(xy) \geq \frac{1-k}{2}$  and  $\gamma_A(xy) \leq \frac{1-k}{2}$ , which is invalid. Consequently,  $\langle xy; \min\{t_1, t_2\}, \max\{s_1, s_2\} \rangle \in \bigvee q_k A$ . Finally, suppose that  $x, y, z \in S$ ,  $t_1, t_2 \in (0, 1]$  and  $s_1, s_2 \in [0, 1)$  be such that  $\langle x; (t_1, s_1) \rangle \in A$  and  $\langle z; (t_2, s_2) \rangle \in A$ . Assume that  $\langle xyz; \min\{t_1, t_2\}, \max\{s_1, s_2\} \rangle \in \bigvee q_k A$ . Then

$$\mu_A(xyz) < \min\{t_1, t_2\}, \quad \gamma_A(xyz) > \max\{s_1, s_2\}$$

and

$$\mu_A(xyz) + \min\{t_1, t_2\} < 1 - k, \quad \gamma_A(xyz) + \max\{s_1, s_2\} > 1 - k.$$

It follows that  $\mu_A(xyz) < \frac{1-k}{2}$  and  $\gamma_A(xyz) > \frac{1-k}{2}$ . Let

$$\Omega_5 := \left\{ i \in \Lambda \mid \mu_{A_i}(xyz) \geq \min\{t_1, t_2\} \text{ and } \gamma_{A_i}(xyz) \leq \max\{s_1, s_2\} \right\}$$

and

$$\begin{aligned} \Omega_6 := & \left\{ i \in \Lambda \mid \langle xyz; \min\{t_1, t_2\}, \max\{s_1, s_2\} \rangle q_k A_i \right. \\ & \left. \text{and } \mu_{A_i}(xyz) < \min\{t_1, t_2\} \text{ and } \gamma_{A_i}(xyz) > \max\{s_1, s_2\} \right\}. \end{aligned}$$

Then  $\Omega_5 \cup \Omega_6 = \Lambda$  and  $\Omega_5 \cap \Omega_6 = \emptyset$ . If  $\Omega_6 = \emptyset$ , then  $\mu_{A_i}(xyz) \geq \min\{t_1, t_2\}$  and  $\gamma_{A_i}(xyz) \leq \max\{s_1, s_2\}$  for all  $i \in \Lambda$ , and so  $\mu_A(xyz) \geq \min\{t_1, t_2\}$  and  $\gamma_A(xyz) \leq \max\{s_1, s_2\}$  which is a contradiction. Hence  $\Omega_6 \neq \emptyset$  and

$$\langle xyz; \min\{t_1, t_2\}, \max\{s_1, s_2\} \rangle q_k A_i,$$

i.e.,

$$\mu_{A_i}(xyz) + \min\{t_1, t_2\} > 1 - k, \quad \gamma_{A_i}(xyz) + \max\{s_1, s_2\} < 1 - k.$$

It follows that  $\min\{t_1, t_2\} > \frac{1-k}{2}$  and  $\max\{s_1, s_2\} < \frac{1-k}{2}$ , so that

$$\mu_{A_i}(x) \geq \mu_A(x) \geq t_1 \geq \min\{t_1, t_2\} > \frac{1-k}{2}$$

and

$$\gamma_{A_i}(x) \leq \gamma_A(x) \leq s_1 \leq \max\{s_1, s_2\}$$

for all  $i \in \Lambda$ . Similarly, we have

$$\mu_{A_i}(z) \geq \mu_A(z) \geq t_2 \geq \min\{t_1, t_2\} > \frac{1-k}{2}$$

and

$$\gamma_{A_i}(z) \leq \gamma_A(z) \leq s_2 \leq \max\{s_1, s_2\}$$

for all  $i \in \Lambda$ . Now, suppose that  $t := \mu_{A_i}(xyz) < \frac{1-k}{2}$  and  $s := \gamma_{A_i}(xyz) > \frac{1-k}{2}$  for some  $i \in \Lambda$ . Let  $t' \in (0, \frac{1-k}{2})$  and  $s' \in (\frac{1-k}{2}, 1)$  be such that  $t < t'$  and  $s > s'$ . Then  $\mu_{A_i}(x) > \frac{1-k}{2} > t'$ ,  $\gamma_{A_i}(x) < \frac{1-k}{2} < s'$  and  $\mu_{A_i}(y) > \frac{1-k}{2} > t'$ ,  $\gamma_{A_i}(y) < \frac{1-k}{2} < s'$ , i.e.,  $\langle x; (t', s') \rangle \in A$  and  $\langle y; (t', s') \rangle \in A$ . But

$$\mu_{A_i}(xyz) = t < t', \quad \gamma_{A_i}(xyz) = s > s'$$

and

$$\mu_{A_i}(xyz) + t' < 1, \quad \gamma_{A_i}(xyz) + s' > 1,$$

that is,  $\langle xyz; (t', s') \rangle \notin \bigvee \overline{q_k} A_i$ . This is a contradiction. Thus,  $\mu_{A_i}(xyz) \geq \frac{1-k}{2}$  and  $\gamma_{A_i}(xyz) \leq \frac{1-k}{2}$  for all  $i \in \Lambda$ . Therefore,  $\mu_A(xyz) \geq \frac{1-k}{2}$  and  $\gamma_A(xyz) \leq \frac{1-k}{2}$ , which is invalid. Thus,  $\langle xyz; \min\{t_1, t_2\}, \max\{s_1, s_2\} \rangle \in \bigvee q_k A$ . Therefore,  $\bigcap_{i \in \Lambda} A_i$  is an  $(\in, \in \bigvee q_k)$ -intuitionistic fuzzy bi-ideal of  $S$ .  $\square$

The following example shows that the union of two  $(\in, \in \bigvee q_k)$ -intuitionistic fuzzy bi-ideals of  $S$  may not be an  $(\in, \in \bigvee q_k)$ -intuitionistic fuzzy bi-ideal of  $S$ .

**Example 3.15** Consider the ordered semigroup of Example 3.4 with the  $*$ -multiplication Table 1 and the IFS of Examples 3.4 and 3.7, then  $\langle a; (0.7, 0.3) \rangle \in A \cup B$  and  $\langle b; (0.5, 0.4) \rangle \in A \cup B$ , but  $\langle ab \min\{0.7, 0.5\}, \max\{0.3, 0.4\} \rangle = \langle d; (0.5, 0.4) \rangle \notin \bigvee q_k A \cup B$ .

**Definition 3.16** An IFS  $A = \langle x, \mu_A, \gamma_A \rangle$  of  $S$  is called an  $(\overline{\in}, \overline{\in} \bigvee \overline{q_k})$ -intuitionistic fuzzy bi-ideal of  $S$  if for all  $x, y, z \in S$ ,  $t, t_1, t_2 \in (0, 1]$  and  $s, s_1, s_2 \in [0, 1)$ , it satisfies the following conditions

- (q4)  $(\langle x; (t, s) \rangle \overline{\in} A \implies \langle y; (t, s) \rangle \overline{\in} \bigvee \overline{q_k} A$  with  $x \leq y$ ),
- (q5)  $(\langle xy; \min\{t_1, t_2\}, \max\{s_1, s_2\} \rangle \overline{\in} A \implies \langle x; (t_1, s_1) \rangle \overline{\in} \bigvee \overline{q_k} A$  or  $\langle y; (t_2, s_2) \rangle \overline{\in} \bigvee \overline{q_k} A)$ ,
- (q6)  $(\langle xyz; \min\{t_1, t_2\}, \max\{s_1, s_2\} \rangle \overline{\in} A \implies \langle x; (t_1, s_1) \rangle \overline{\in} \bigvee \overline{q_k} A$  or  $\langle z; (t_2, s_2) \rangle \overline{\in} \bigvee \overline{q_k} A)$ .

Let  $A = \langle x, \mu_A, \gamma_A \rangle$  be an  $(\overline{\in}, \overline{\in} \bigvee \overline{q_k})$ -intuitionistic fuzzy bi-ideal of an ordered semi-group  $S$ . Suppose that there exist  $a, b \in S$  with  $a \leq b$  such that

$$\mu_A(b) > \max\left\{\mu_A(a), \frac{1-k}{2}\right\} \quad \text{and} \quad \gamma_A(b) < \min\left\{\gamma_A(a), \frac{1-k}{2}\right\}.$$

Then  $\mu_A(b) \geq t > \max\{\mu_A(a), \frac{1-k}{2}\}$  and  $\gamma_A(b) \leq s < \min\{\gamma_A(a), \frac{1-k}{2}\}$  for some  $t \in (\frac{1-k}{2}, 1]$  and  $s \in [0, \frac{1-k}{2})$ . It follows that  $\langle a; (t, s) \rangle \overline{\in} A$ ,  $\langle b; (t, s) \rangle \in A$  and  $\mu_A(b) + t \geq 2t > 1 - k$ ,  $\gamma_A(b) + s \leq 2s < 1 - k$ , i.e.,  $\langle b; (t, s) \rangle q_k A$ . This is a contradiction, and so the following inequalities hold.

$$(e1) \quad \left( x \leq y \implies \mu_A(b) \leq \max\left\{\mu_A(a), \frac{1-k}{2}\right\} \text{ and } \gamma_A(b) \geq \min\left\{\gamma_A(a), \frac{1-k}{2}\right\} \right).$$

Suppose that  $\max\{\mu_A(ab), \frac{1-k}{2}\} < \min\{\mu_A(a), \mu_A(b)\}$  and  $\min\{\gamma_A(ab), \frac{1-k}{2}\} > \max\{\gamma_A(a), \gamma_A(b)\}$  for some  $a, b \in S$ . Then  $\max\{\mu_A(ab), \frac{1-k}{2}\} < t \leq \min\{\mu_A(a), \mu_A(b)\}$  and  $\min\{\gamma_A(ab), \frac{1-k}{2}\} > s \geq \max\{\gamma_A(a), \gamma_A(b)\}$ . Thus,  $\langle a; (t, s) \rangle \overline{\in} A$ ,  $\langle b; (t, s) \rangle \overline{\in} A$ ,  $\mu_A(a) + t \geq 2t > 1 - k$ ,  $\gamma_A(a) + s \leq 2s < 1 - k$ , i.e.,  $\langle a; (t, s) \rangle q_k A$  and  $\mu_A(b) + t \geq 2t > 1 - k$ ,  $\gamma_A(b) + s \leq 2s < 1 - k$ , i.e.,  $\langle b; (t, s) \rangle q_k A$ . This is impossible, and hence  $A = \langle x, \mu_A, \gamma_A \rangle$  satisfies the following assertion

(e2)  $\max\{\mu_A(xy), \frac{1-k}{2}\} < \min\{\mu_A(x), \mu_A(y)\}$  and  $\min\{\gamma_A(xy), \frac{1-k}{2}\} > \max\{\gamma_A(x), \gamma_A(y)\}$  for all  $x, y \in S$ .

Now, assume that

$$\max\left\{\mu_A(abc), \frac{1-k}{2}\right\} < \min\{\mu_A(a), \mu_A(c)\}$$

and

$$\min\left\{\gamma_A(abc), \frac{1-k}{2}\right\} > \max\{\gamma_A(a), \gamma_A(c)\}$$

for some  $a, b, c \in S$ . Then

$$\max\left\{\mu_A(abc), \frac{1-k}{2}\right\} < t \leq \min\{\mu_A(a), \mu_A(c)\}$$

and

$$\min\left\{\gamma_A(abc), \frac{1-k}{2}\right\} > s \geq \max\{\gamma_A(a), \gamma_A(c)\}.$$

Thus,  $\langle a; (t, s) \rangle \bar{\in} A$ ,  $\langle c; (t, s) \rangle \bar{\in} A$ ,  $\mu_A(a) + t \geq 2t > 1 - k$ ,  $\gamma_A(a) + s \leq 2s < 1 - k$ , i.e.,  $\langle a; (t, s) \rangle q_k A$  and  $\mu_A(c) + t \geq 2t > 1 - k$ ,  $\gamma_A(c) + s \leq 2s < 1 - k$ , i.e.,  $\langle c; (t, s) \rangle q_k A$ . This is a contradiction, and hence we have the following assertion

(e3)  $\max\{\mu_A(xyz), \frac{1-k}{2}\} < \min\{\mu_A(x), \mu_A(z)\}$  and  $\min\{\gamma_A(xyz), \frac{1-k}{2}\} > \max\{\gamma_A(x), \gamma_A(z)\}$  for all  $x, y, z \in S$ .

Let  $A = \langle x, \mu_A, \gamma_A \rangle$  be an IFS of  $S$  satisfying the three conditions (e1), (e2) and (e3). Let  $t \in (\frac{1-k}{2}, 1]$  and  $s \in [0, \frac{1-k}{2})$  be such that  $C_{(t,s)}(A) \neq \emptyset$ . Then there exist  $b \in C_{(t,s)}(A)$  and  $S \ni a \leq b$ , by using (e1), we get

$$\frac{1-k}{2} < t \leq \mu_A(b) \leq \max\left\{\mu_A(a), \frac{1-k}{2}\right\} = \mu_A(a)$$

and

$$\frac{1-k}{2} > s \geq \gamma_A(b) \geq \min\left\{\gamma_A(a), \frac{1-k}{2}\right\} = \gamma_A(a).$$

Hence  $a \in C_{(t,s)}(A)$ . Let  $a, b \in S$  be such that  $a \in C_{(t,s)}(A)$  and  $b \in C_{(t,s)}(A)$ . Then  $\mu_A(a) \geq t$ ,  $\gamma_A(a) \leq s$  and  $\mu_A(b) \geq t$ ,  $\gamma_A(b) \leq s$ . Using (e2), we get

$$\max\left\{\mu_A(ab), \frac{1-k}{2}\right\} \geq \min\{\mu_A(a), \mu_A(b)\} \geq t > \frac{1-k}{2}$$

and

$$\min\left\{\gamma_A(ab), \frac{1-k}{2}\right\} \leq \max\{\gamma_A(a), \gamma_A(b)\} \leq s < \frac{1-k}{2},$$

which implies that  $\mu_A(ab) = \max\{\mu_A(ab), \frac{1-k}{2}\} \geq t$  and  $\gamma_A(ab) = \min\{\gamma_A(ab), \frac{1-k}{2}\} \leq s$ . Thus,  $ab \in C_{(t,s)}(A)$ . Now, suppose that  $a, c \in C_{(t,s)}(A)$  and  $b \in S$ . Then  $\mu_A(a) \geq t$ ,  $\gamma_A(a) \leq s$  and

$\mu_A(c) \geq t, \gamma_A(c) \leq s$ . Using (e3), we get

$$\max\left\{\mu_A(abc), \frac{1-k}{2}\right\} \geq \min\{\mu_A(a), \mu_A(c)\} \geq t > \frac{1-k}{2}$$

and

$$\min\left\{\gamma_A(abc), \frac{1-k}{2}\right\} \leq \max\{\gamma_A(a), \gamma_A(c)\} \leq s < \frac{1-k}{2},$$

which implies that  $\mu_A(abc) = \max\{\mu_A(abc), \frac{1-k}{2}\} \geq t$  and  $\gamma_A(abc) = \min\{\gamma_A(abc), \frac{1-k}{2}\} \leq s$ . Thus,  $abc \in C_{(t,s)}(A)$ . Consequently,  $C_{(t,s)}(A)$  is a bi-ideal of  $S$ . Therefore, we conclude that if an IFS  $A = \langle x, \mu_A, \gamma_A \rangle$  of  $S$  satisfies the three conditions (e1), (e2) and (e3), then the following assertion is valid

(e4)  $(\forall t \in (0.5, 1]) (\forall s \in [0, \frac{1-k}{2})) (C_{(t,s)}(A) \neq \emptyset \implies C_{(t,s)}(A)$  is a bi-ideal of  $S$ ).

Now, let  $A = \langle x, \mu_A, \gamma_A \rangle$  be an IFS of  $S$  satisfying (e4). Let  $a, b \in S$  with  $a \leq b$  and  $t \in (0, 1]$  and  $s \in [0, 1]$  be such that  $\langle b; (t, s) \rangle \overline{\in} \vee \overline{q_k} A$ . Then  $\langle b; (t, s) \rangle \in A$  and  $\langle b; (t, s) \rangle q_k A$ . Hence  $b \in C_{(t,s)}(A)$  and  $C_{(t,s)}(A) \neq \emptyset$ . Thus, by (e4),  $a \in C_{(t,s)}(A)$  and so  $\mu_A(a) \geq t, \gamma_A(a) \leq s$ , i.e.,  $\langle a; (t, s) \rangle \in A$ . This shows that (q4) is valid. Let  $a, b \in S, t_1, t_2 \in (0, 1]$  and  $s_1, s_2 \in [0, 1]$  be such that  $\langle a; (t_1, s_1) \rangle \overline{\in} \vee \overline{q_k} A$  and  $\langle b; (t_2, s_2) \rangle \overline{\in} \vee \overline{q_k} A$ . Then  $\langle a; (t_1, s_1) \rangle \in A, \langle a; (t_1, s_1) \rangle q_k A$  and  $\langle b; (t_2, s_2) \rangle \in A, \langle b; (t_2, s_2) \rangle q_k A$ , which implies that  $a \in C_{(t,s)}(A) \subseteq C_{(\min\{t_1, t_2\}, \max\{s_1, s_2\})}(A)$  and  $b \in C_{(t,s)}(A) \subseteq C_{(\min\{t_1, t_2\}, \max\{s_1, s_2\})}(A)$ . Since  $C_{(\min\{t_1, t_2\}, \max\{s_1, s_2\})}(A)$  is bi-ideal of  $S$  by (e4), it follows by (b2) that  $ab \in C_{(\min\{t_1, t_2\}, \max\{s_1, s_2\})}(A)$ , that is,  $\mu_A(ab) \geq \min\{t_1, t_2\}$ , and  $\gamma_A(ab) \leq \max\{s_1, s_2\}$ , so that  $\langle ab; \min\{t_1, t_2\}, \max\{s_1, s_2\} \rangle \in A$ . Hence (q5) is valid. Finally, let  $a, b, c \in S, t_1, t_2 \in (0, 1]$  and  $s_1, s_2 \in [0, 1]$  be such that  $\langle a; (t_1, s_1) \rangle \overline{\in} \vee \overline{q_k} A$  and  $\langle c; (t_2, s_2) \rangle \overline{\in} \vee \overline{q_k} A$ . Then  $\langle a; (t_1, s_1) \rangle \in A, \langle a; (t_1, s_1) \rangle q_k A$  and  $\langle c; (t_2, s_2) \rangle \in A, \langle c; (t_2, s_2) \rangle q_k A$ , which implies that  $a \in C_{(t,s)}(A) \subseteq C_{(\min\{t_1, t_2\}, \max\{s_1, s_2\})}(A)$  and  $c \in C_{(t,s)}(A) \subseteq C_{(\min\{t_1, t_2\}, \max\{s_1, s_2\})}(A)$ . Since  $C_{(\min\{t_1, t_2\}, \max\{s_1, s_2\})}(A)$  is bi-ideal of  $S$  by (e4), it follows by (b3) that  $abc \in C_{(\min\{t_1, t_2\}, \max\{s_1, s_2\})}(A)$ , that is,  $\mu_A(abc) \geq \min\{t_1, t_2\}$ , and  $\gamma_A(abc) \leq \max\{s_1, s_2\}$ , so that  $\langle abc; \min\{t_1, t_2\}, \max\{s_1, s_2\} \rangle \in A$ . Hence (q6) is valid.

Therefore, as a concluding remark, we have the following theorem.

**Theorem 3.17** For an IFS  $A = \langle x, \mu_A, \gamma_A \rangle$  of  $S$ , the following are equivalent

- (1)  $A = \langle x, \mu_A, \gamma_A \rangle$  is an  $(\overline{\in}, \overline{\in} \vee \overline{q_k})$ -intuitionistic fuzzy bi-ideal of  $S$ .
- (2)  $A = \langle x, \mu_A, \gamma_A \rangle$  satisfies the condition (e4).
- (3)  $A = \langle x, \mu_A, \gamma_A \rangle$  satisfies the three conditions (e1), (e2) and (e3).

For an IFS  $A = \langle x; \mu_A, \gamma_A \rangle$  of  $S$ , we consider the following sets

$$\Gamma_1 := \{t \in (0, 1] | U(\mu_A; t) \neq \emptyset \implies U(\mu_A; t) \text{ is a bi-ideal of } S\},$$

$$\Gamma_2 := \{s \in [0, 1] | L(\mu_A; s) \neq \emptyset \implies L(\mu_A; s) \text{ is a bi-ideal of } S\}.$$

Then

- (1) If  $\Gamma_1 = (0, 1]$  and  $\Gamma_2 = [0, 1]$ , then  $A = \langle x; \mu_A, \gamma_A \rangle$  is an intuitionistic fuzzy bi-ideal of  $S$ .
- (2) If  $\Gamma_1 = (0, 0.5]$  and  $\Gamma_2 = [0.5, 1]$ , then  $A = \langle x; \mu_A, \gamma_A \rangle$  is an  $(\in, \in \vee q)$ -intuitionistic fuzzy bi-ideal of  $S$ .



- (3) If  $\Gamma_1 = (0, \frac{1-k}{2})$  and  $\Gamma_2 = [\frac{1-k}{2}, 1)$ , then  $A = \langle x; \mu_A, \gamma_A \rangle$  is an  $(\epsilon, \epsilon \vee q_k)$ -intuitionistic fuzzy bi-ideal of  $S$ .
- (4) If  $\Gamma_1 = (\frac{1-k}{2}, 1]$  and  $\Gamma_2 = [0, \frac{1-k}{2})$ , then  $A = \langle x; \mu_A, \gamma_A \rangle$  is an  $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q}_k)$ -intuitionistic fuzzy bi-ideal of  $S$ .

#### Competing interests

The authors declare that they have no competing interests.

#### Authors' contributions

AK proposed the structure, reviewed the mathematical concepts and coordinated the manuscript. BD participated in its design and helped to check the examples constructed for new concepts in the manuscript. NHS linguistically edited, sequenced and drafted the manuscript. HK introduced new concepts, proofs to mathematical results and typed the manuscript.

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Received: 7 March 2013 Accepted: 31 July 2013 Published: 21 August 2013

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doi:10.1186/1029-242X-2013-397

**Cite this article as:** Khan et al.: Redefined intuitionistic fuzzy bi-ideals of ordered semigroups. *Journal of Inequalities and Applications* 2013 2013:397.

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