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On some inequalities for s-convex functions and applications

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Abstract

Some new results related to the left-hand side of the Hermite-Hadamard type inequalities for the class of functions whose second derivatives at certain powers are s-convex functions in the second sense are obtained. Also, some applications to special means of real numbers are provided.

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1 Introduction

The following definition is well known in the literature: a function $f : I \to \mathbb{R}$, $\emptyset \neq I \subset \mathbb{R}$, is said to be convex on I if the inequality

$$f\big(tx + (1-t)y\big) \leq tf(x) + (1-t)f(y)$$

holds for all $x, y \in I$ and $t \in [0,1]$. Geometrically, this means that if P, Q and R are three distinct points on the graph of f with Q between P and R, then Q is on or below the chord PR.

In their paper [1], Hudzik and Maligranda considered, among others, the class of functions which are s-convex in the second sense. This class is defined in the following way: a function $f : [0,\infty) \to \mathbb{R}$ is said to be *s-convex* in the second sense if

$$f\big(tx + (1-t)y\big) \leq t^s f(x) + (1-t)^s f(y)$$

holds for all $x, y \in [0,\infty)$, $t \in [0,1]$ and for some fixed $s \in (0,1]$. The class of s-convex functions in the second sense is usually denoted by K_s^2.

It can be easily seen that for $s = 1$ s-convexity reduces to the ordinary convexity of functions defined on $[0,\infty)$.

In the same paper [1], Hudzik and Maligranda proved that if $s \in (0,1)$, $f \in K_s^2$ implies $f([0,\infty)) \subseteq [0,\infty)$, *i.e.*, they proved that all functions from K_s^2, $s \in (0,1)$, are nonnegative.

Example 1 [1] Let $s \in (0,1)$ and $a, b, c \in \mathbb{R}$. We define the function $f : [0,\infty) \to \mathbb{R}$ as

$$f(t) = \begin{cases} a, & t = 0, \\ bt^s + c, & t > 0. \end{cases}$$

It can be easily checked that

(i) if $b \geq 0$ and $0 \leq c \leq a$, then $f \in K_s^2$,

(ii) if $b > 0$ and $c < 0$, then $f \notin K_s^2$.

Many important inequalities are established for the class of convex functions, but one of the most famous is the so-called Hermite-Hadamard's inequality (or Hadamard's inequality). This double inequality is stated as follows: Let f be a convex function on $[a,b] \subset \mathbb{R}$, where $a \neq b$. Then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}.$$

For several recent results concerning Hadamard's inequality, we refer the interested reader to [2–5].

In [6] Dragomir and Fitzpatrick proved a variant of Hadamard's inequality which holds for s-convex functions in the second sense.

Theorem 1 *Suppose that $f : [0,\infty) \to [0,\infty)$ is an s-convex function in the second sense, where $s \in (0,1)$, and let $a,b \in [0,\infty)$, $a < b$. If $f \in L([a,b])$, then the following inequalities hold*:

$$2^{s-1}f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{s+1}. \tag{1.1}$$

The constant $k = 1/(s+1)$ is best possible in the second inequality in [7].

The above inequalities are sharp. For recent results and generalizations concerning s-convex functions, see [8–13].

Along this paper, we consider a real interval $I \subset \mathbb{R}$, and we denote that I° is the interior of I.

The main aim of this paper is to establish new inequalities of Hermite-Hadamard type for the class of functions whose second derivatives at certain powers are s-convex functions in the second sense.

2 Main results

To prove our main results, we consider the following lemma.

Lemma 1 *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° where $a,b \in I$ with $a < b$. If $f'' \in L[a,b]$, then the following equality holds*:

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right) \\
&\quad= \frac{(b-a)^2}{16}\left[\int_0^1 t^2 f''\left(t\frac{a+b}{2} + (1-t)a\right)dt \right. \\
&\qquad \left. + \int_0^1 (t-1)^2 f''\left(tb + (1-t)\frac{a+b}{2}\right)dt\right].
\end{aligned} \tag{2.1}$$

Proof By integration by parts, we have the following identity:

$$\begin{aligned}
I_1 &= \int_0^1 t^2 f''\left(t\frac{a+b}{2} + (1-t)a\right) dt \\
&= t^2 \frac{2}{b-a} f'\left(t\frac{a+b}{2} + (1-t)a\right)\Big|_0^1 - \frac{4}{b-a}\int_0^1 tf'\left(t\frac{a+b}{2} + (1-t)a\right) dt \\
&= \frac{2}{b-a} f'\left(\frac{a+b}{2}\right) - \frac{4}{b-a}\left[t\frac{2}{b-a} f\left(t\frac{a+b}{2} + (1-t)a\right)\Big|_0^1 \right. \\
&\quad \left. - \frac{2}{b-a}\int_0^1 f\left(t\frac{a+b}{2} + (1-t)a\right) dt\right] \\
&= \frac{2}{b-a} f'\left(\frac{a+b}{2}\right) - \frac{8}{(b-a)^2} f\left(\frac{a+b}{2}\right) \\
&\quad + \frac{8}{(b-a)^2}\int_0^1 f\left(t\frac{a+b}{2} + (1-t)a\right) dt. \qquad (2.2)
\end{aligned}$$

Using the change of the variable $x = t\frac{a+b}{2} + (1-t)a$ for $t \in [0,1]$ and multiplying the both sides (2.2) by $\frac{(b-a)^2}{16}$, we obtain

$$\begin{aligned}
&\frac{(b-a)^2}{16}\int_0^1 t^2 f''\left(t\frac{a+b}{2} + (1-t)a\right) dt \\
&\quad = \frac{b-a}{8} f'\left(\frac{a+b}{2}\right) - \frac{1}{2} f\left(\frac{a+b}{2}\right) + \frac{1}{b-a}\int_a^{\frac{a+b}{2}} f(x)\,dx. \qquad (2.3)
\end{aligned}$$

Similarly, we observe that

$$\begin{aligned}
&\frac{(b-a)^2}{16}\int_0^1 (t-1)^2 f''\left(tb + (1-t)\frac{a+b}{2}\right) dt \\
&\quad = -\frac{b-a}{8} f'\left(\frac{a+b}{2}\right) - \frac{1}{2} f\left(\frac{a+b}{2}\right) + \frac{1}{b-a}\int_{\frac{a+b}{2}}^{b} f(x)\,dx. \qquad (2.4)
\end{aligned}$$

Thus, adding (2.3) and (2.4), we get the required identity (2.1). □

Theorem 2 *Let $f : I \subset [0,\infty) \to \mathbb{R}$ be a differentiable mapping on I° such that $f'' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f|$ is s-convex on $[a,b]$, for some fixed $s \in (0,1]$, then the following inequality holds*:

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
&\quad \le \frac{(b-a)^2}{8(s+1)(s+2)(s+3)}\left\{\left|f''(a)\right| + (s+1)(s+2)\left|f''\left(\frac{a+b}{2}\right)\right| + \left|f''(b)\right|\right\} \qquad (2.5) \\
&\quad \le \frac{[1+(s+2)2^{1-s}](b-a)^2}{8(s+1)(s+2)(s+3)}\left\{\left|f''(a)\right| + \left|f''(b)\right|\right\}. \qquad (2.6)
\end{aligned}$$

Proof From Lemma 1, we have

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
&\quad\leq \frac{(b-a)^2}{16}\left[\int_0^1 t^2\left|f''\left(t\frac{a+b}{2}+(1-t)a\right)\right|dt\right. \\
&\qquad\left.+\int_0^1 (t-1)^2\left|f''\left(tb+(1-t)\frac{a+b}{2}\right)\right|dt\right] \\
&\quad\leq \frac{(b-a)^2}{16}\int_0^1 t^2\left[t^s\left|f''\left(\frac{a+b}{2}\right)\right|+(1-t)^s\left|f''(a)\right|\right]dt \\
&\qquad+\frac{(b-a)^2}{16}\int_0^1 (t-1)^2\left[t^s\left|f''(b)\right|+(1-t)^s\left|f''\left(\frac{a+b}{2}\right)\right|\right]dt \\
&\quad=\frac{(b-a)^2}{16}\left[\frac{1}{s+3}\left|f''\left(\frac{a+b}{2}\right)\right|+\frac{2}{(s+1)(s+2)(s+3)}\left|f''(a)\right|\right] \\
&\qquad+\frac{(b-a)^2}{16}\left[\frac{2}{(s+1)(s+2)(s+3)}\left|f''(b)\right|+\frac{1}{s+3}\left|f''\left(\frac{a+b}{2}\right)\right|\right] \\
&\quad=\frac{(b-a)^2}{8(s+1)(s+2)(s+3)}\left\{\left|f''(a)\right|+(s+1)(s+2)\left|f''\left(\frac{a+b}{2}\right)\right|+\left|f''(b)\right|\right\}, \qquad (2.7)
\end{aligned}$$

where we have used the fact that

$$\begin{aligned}
&\int_0^1 t^2(1-t)^s\,dt=\int_0^1 (t-1)^2 t^s\,dt=\frac{2}{(s+1)(s+2)(s+3)}, \\
&\int_0^1 t^{s+2}\,dt=\int_0^1 (1-t)^{s+2}\,dt=\frac{1}{s+3}.
\end{aligned}$$

This proves inequality (2.5). To prove (2.6), and since $|f''|$ is s-convex on $[a,b]$, for any $t\in[0,1]$, then by (1.1) we have

$$2^{s-1}f\left(\frac{a+b}{2}\right)\leq\frac{f(a)+f(b)}{s+1}. \qquad (2.8)$$

Combining (2.7) and (2.8), we have

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
&\quad\leq\frac{(b-a)^2}{8(s+1)(s+2)(s+3)}\left\{\left|f''(a)\right|+(s+1)(s+2)\left|f''\left(\frac{a+b}{2}\right)\right|+\left|f''(b)\right|\right\} \\
&\quad\leq\frac{(b-a)^2}{8(s+1)(s+2)(s+3)}\left\{\left|f''(a)\right|+(s+1)(s+2)2^{1-s}\frac{f(a)+f(b)}{s+1}+\left|f''(b)\right|\right\} \\
&\quad=\frac{[1+(s+2)2^{1-s}](b-a)^2}{8(s+1)(s+2)(s+3)}\left\{\left|f''(a)\right|+\left|f''(b)\right|\right\},
\end{aligned}$$

which proves inequality (2.6), and thus the proof is completed. □

Corollary 1 *In Theorem* 2, *if we choose* $s = 1$, *we have*

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
&\quad\le \frac{(b-a)^2}{192}\left\{\left|f''(a)\right|+6\left|f''\left(\frac{a+b}{2}\right)\right|+\left|f''(b)\right|\right\} \\
&\quad\le \frac{(b-a)^2}{48}\left\{\left|f''(a)\right|+\left|f''(b)\right|\right\}.
\end{aligned}\tag{2.9}$$

The next theorem gives a new upper bound of the left Hadamard inequality for s-convex mappings.

Theorem 3 *Let* $f : I \subset [0,\infty) \to \mathbb{R}$ *be a differentiable mapping on* I° *such that* $f'' \in L[a,b]$, *where* $a, b \in I$ *with* $a < b$. *If* $|f''|^q$ *is* s-*convex on* $[a,b]$, *for some fixed* $s \in (0,1]$ *and* $q > 1$ *with* $\frac{1}{p}+\frac{1}{q}=1$, *then the following inequality holds*:

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
&\quad\le \frac{(b-a)^2}{16}\left(\frac{1}{2p+1}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}} \\
&\qquad\times\left[\left(\left|f''(a)\right|^q+\left|f''\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}+\left(\left|f''\left(\frac{a+b}{2}\right)\right|^q+\left|f''(b)\right|^q\right)^{\frac{1}{q}}\right].
\end{aligned}\tag{2.10}$$

Proof Suppose that $p > 1$. From Lemma 1 and using the Hölder inequality, we have

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
&\quad\le \frac{(b-a)^2}{16}\left[\int_0^1 t^2\left|f''\left(t\frac{a+b}{2}+(1-t)a\right)\right|dt\right. \\
&\qquad\left.+\int_0^1 (t-1)^2\left|f''\left(tb+(1-t)\frac{a+b}{2}\right)\right|dt\right] \\
&\quad\le \frac{(b-a)^2}{16}\left(\int_0^1 t^{2p}\,dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f''\left(t\frac{a+b}{2}+(1-t)a\right)\right|^q dt\right)^{\frac{1}{q}} \\
&\qquad+\frac{(b-a)^2}{16}\left(\int_0^1 (t-1)^{2p}\,dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f''\left(tb+(1-t)\frac{a+b}{2}\right)\right|^q dt\right)^{\frac{1}{q}}.
\end{aligned}$$

Because $|f''|^q$ is s-convex, we have

$$\int_0^1\left|f''\left(t\frac{a+b}{2}+(1-t)a\right)\right|^q dt \le \frac{1}{s+1}\left\{\left|f''(a)\right|^q+\left|f''\left(\frac{a+b}{2}\right)\right|^q\right\}$$

and

$$\int_0^1\left|f''\left(tb+(1-t)\frac{a+b}{2}\right)\right|^q dt \le \frac{1}{s+1}\left\{\left|f''\left(\frac{a+b}{2}\right)\right|^q+\left|f''(b)\right|^q\right\}.$$

By a simple computation,

$$\int_0^1 t^{2p}\,dt = \frac{1}{2p+1}$$

and

$$\int_0^1 (t-1)^{2p}\,dt = \int_0^1 (1-t)^{2p}\,dt = \frac{1}{2p+1}.$$

Therefore, we have

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
&\quad\le \frac{(b-a)^2}{16}\left(\frac{1}{2p+1}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}} \\
&\quad\quad\times\left[\left(\left|f''(a)\right|^q + \left|f''\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}} + \left(\left|f''\left(\frac{a+b}{2}\right)\right|^q + \left|f''(b)\right|^q\right)^{\frac{1}{q}}\right].
\end{aligned}$$

This completes the proof. □

Corollary 2 *Let* $f : I \subset [0,\infty) \to \mathbb{R}$ *be a differentiable mapping on* I° *such that* $f'' \in L[a,b]$, *where* $a, b \in I$ *with* $a < b$. *If* $|f''|^q$ *is s-convex on* $[a,b]$, *for some fixed* $s \in (0,1]$ *and* $q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$, *then the following inequality holds*:

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
&\quad\le \frac{(b-a)^2}{16}\left(\frac{1}{2p+1}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{2}{q}} \\
&\quad\quad\times\left\{2^{\frac{1-s}{q}} + \left(2^{1-s}+s+1\right)^{\frac{1}{q}}\right\}\left[\left|f''(a)\right| + \left|f''(b)\right|\right].
\end{aligned}$$

Proof We consider inequality (2.10), and since $|f''|^q$ is s-convex on $[a,b]$, then by (1.1) we have

$$2^{s-1}f\left(\frac{a+b}{2}\right) \le \frac{f(a)+f(b)}{s+1}.$$

Therefore

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
&\quad\le \frac{(b-a)^2}{16}\left(\frac{1}{2p+1}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{2}{q}} \\
&\quad\quad\times\left[\left(\left\{2^{1-s}+s+1\right\}\left|f''(a)\right|^q + 2^{1-s}\left|f''(b)\right|^q\right)^{\frac{1}{q}}\right. \\
&\quad\quad\left.+\left(2^{1-s}\left|f''(a)\right|^q + \left\{2^{1-s}+s+1\right\}\left|f''(b)\right|^q\right)^{\frac{1}{q}}\right].
\end{aligned}$$

We let $a_1 = (2^{1-s} + s + 1)|f''(a)|^q$, $b_1 = 2^{1-s}|f''(b)|^q$, $a_2 = 2^{1-s}|f''(a)|^q$ and $b_2 = (2^{1-s} + s + 1)|f''(b)|^q$. Here, $0 < 1/q < 1$ for $q > 1$. Using the fact

$$\sum_{i=1}^{n}(a_i + b_i)^r \leq \sum_{i=1}^{n} a_i^r + \sum_{i=1}^{n} b_i^r$$

for $0 < r < 1$, $a_1, a_2, \dots, a_n \geq 0$ and $b_1, b_2, \dots, b_n \geq 0$, we obtain the inequalities

$$\begin{aligned}
&\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,dx \right| \\
&\quad\leq \frac{(b-a)^2}{16}\left(\frac{1}{2p+1}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{2}{q}} \\
&\quad\quad\times \left[\left(\left\{2^{1-s} + s + 1\right\}\left|f''(a)\right|^q + 2^{1-s}\left|f''(b)\right|^q\right)^{\frac{1}{q}}\right. \\
&\quad\quad\left. + \left(2^{1-s}\left|f''(a)\right|^q + \left\{2^{1-s} + s + 1\right\}\left|f''(b)\right|^q\right)^{\frac{1}{q}}\right] \\
&\quad\leq \frac{(b-a)^2}{16}\left(\frac{1}{2p+1}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{2}{q}} \\
&\quad\quad\times \left\{2^{\frac{1-s}{q}} + \left(2^{1-s} + s + 1\right)^{\frac{1}{q}}\right\}\left[\left|f''(a)\right| + \left|f''(b)\right|\right].
\end{aligned}$$

□

Theorem 4 *Let $f : I \subset [0,\infty) \to \mathbb{R}$ be a differentiable mapping on I° such that $f'' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f''|^q$, $q \geq 1$ is s-convex on $[a,b]$, for some fixed $s \in (0,1]$, then the following inequality holds*:

$$\begin{aligned}
&\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,dx \right| \\
&\quad\leq \frac{(b-a)^2}{16}\left(\frac{1}{3}\right)^{\frac{1}{p}}\left\{\left(\frac{2}{(s+1)(s+2)(s+3)}\left|f''(a)\right|^q + \frac{1}{s+3}\left|f''\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\right. \\
&\quad\quad\left. + \left(\frac{1}{s+3}\left|f''\left(\frac{a+b}{2}\right)\right|^q + \frac{2}{(s+1)(s+2)(s+3)}\left|f''(b)\right|^q\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

Proof Suppose that $p \geq 1$. From Lemma 1 and using the power mean inequality, we have

$$\begin{aligned}
&\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,dx \right| \\
&\quad\leq \frac{(b-a)^2}{16}\left[\int_0^1 t^2\left|f''\left(t\frac{a+b}{2} + (1-t)a\right)\right|dt\right. \\
&\quad\quad\left. + \int_0^1 (t-1)^2\left|f''\left(tb + (1-t)\frac{a+b}{2}\right)\right|dt\right] \\
&\quad\leq \frac{(b-a)^2}{16}\left(\int_0^1 t^2\,dt\right)^{\frac{1}{p}}\left(\int_0^1 t^2\left|f''\left(t\frac{a+b}{2} + (1-t)a\right)\right|^q dt\right)^{\frac{1}{q}} \\
&\quad\quad + \frac{(b-a)^2}{16}\left(\int_0^1 (t-1)^2\,dt\right)^{\frac{1}{p}}\left(\int_0^1 (t-1)^2\left|f''\left(tb + (1-t)\frac{a+b}{2}\right)\right|^q dt\right)^{\frac{1}{q}}.
\end{aligned}$$

Because $|f''|^q$ is s-convex, we have

$$\int_0^1 t^2\left|f''\left(t\frac{a+b}{2}+(1-t)a\right)\right|^q dt \le \frac{2}{(s+1)(s+2)(s+3)}\left|f''(a)\right|^q + \frac{1}{s+3}\left|f''\left(\frac{a+b}{2}\right)\right|^q$$

and

$$\begin{aligned}&\int_0^1 (t-1)^2\left|f''\left(tb+(1-t)\frac{a+b}{2}\right)\right|^q dt\\ &\quad\le \frac{1}{s+3}\left|f''\left(\frac{a+b}{2}\right)\right|^q + \frac{2}{(s+1)(s+2)(s+3)}\left|f''(b)\right|^q.\end{aligned}$$

Therefore, we have

$$\begin{aligned}&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\\ &\quad\le \frac{(b-a)^2}{16}\left(\frac{1}{3}\right)^{\frac{1}{p}}\left\{\left(\frac{2}{(s+1)(s+2)(s+3)}\left|f''(a)\right|^q + \frac{1}{s+3}\left|f''\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\right.\\ &\qquad\left.+\left(\frac{1}{s+3}\left|f''\left(\frac{a+b}{2}\right)\right|^q + \frac{2}{(s+1)(s+2)(s+3)}\left|f''(b)\right|^q\right)^{\frac{1}{q}}\right\}.\end{aligned}$$ □

Corollary 3 *In Theorem* 4, *if we choose* $s=1$, *we have*

$$\begin{aligned}&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\\ &\quad\le \frac{(b-a)^2}{48}\left(\frac{3}{4}\right)^{\frac{1}{q}}\left\{\left(\frac{|f''(a)|^q}{3}+\left|f''\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\right.\\ &\qquad\left.+\left(\left|f''\left(\frac{a+b}{2}\right)\right|^q+\frac{|f''(b)|^q}{3}\right)^{\frac{1}{q}}\right\}.\end{aligned}\tag{2.11}$$

Now, we give the following Hadamard-type inequality for s-concave mappings.

Theorem 5 *Let* $f: I\subset[0,\infty)\to\mathbb{R}$ *be a differentiable mapping on* I° *such that* $f''\in L[a,b]$, *where* $a,b\in I$ *with* $a<b$. *If* $|f''|^q$ *is s-concave on* $[a,b]$, *for some fixed* $s\in(0,1]$ *and* $q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$, *then the following inequality holds*:

$$\begin{aligned}&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\\ &\quad\le \frac{(b-a)^2}{16}\frac{2^{\frac{s-1}{q}}}{(2p+1)^{1/p}}\left\{\left|f''\left(\frac{3a+b}{4}\right)\right|+\left|f''\left(\frac{a+3b}{4}\right)\right|\right\}.\end{aligned}$$

Proof From Lemma 1 and using the Hölder inequality for $q>1$ and $\frac{1}{p}+\frac{1}{q}=1$, we obtain

$$\begin{aligned}&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\\ &\quad\le \frac{(b-a)^2}{16}\left[\int_0^1 t^2\left|f''\left(t\frac{a+b}{2}+(1-t)a\right)\right|dt+\int_0^1(t-1)^2\left|f''\left(tb+(1-t)\frac{a+b}{2}\right)\right|dt\right]\end{aligned}$$

$$\leq \frac{(b-a)^2}{16}\left(\int_0^1 t^{2p}\,dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f''\left(t\frac{a+b}{2}+(1-t)a\right)\right|^q dt\right)^{\frac{1}{q}}$$
$$+\frac{(b-a)^2}{16}\left(\int_0^1 (t-1)^{2p}\,dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f''\left(tb+(1-t)\frac{a+b}{2}\right)\right|^q dt\right)^{\frac{1}{q}}. \tag{2.12}$$

Since $|f''|^q$ is s-concave, using inequality (1.1), we have

$$\int_0^1\left|f''\left(t\frac{a+b}{2}+(1-t)a\right)\right|^q dt \leq 2^{s-1}\left|f''\left(\frac{3a+b}{4}\right)\right|^q \tag{2.13}$$

and

$$\int_0^1\left|f''\left(tb+(1-t)\frac{a+b}{2}\right)\right|^q dt \leq 2^{s-1}\left|f''\left(\frac{a+3b}{4}\right)\right|^q. \tag{2.14}$$

From (2.12)-(2.14), we get

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq \frac{(b-a)^2}{16}\frac{2^{\frac{s-1}{q}}}{(2p+1)^{1/p}}\left\{\left|f''\left(\frac{3a+b}{4}\right)\right|+\left|f''\left(\frac{a+3b}{4}\right)\right|\right\},$$

which completes the proof. □

Corollary 4 *In Theorem* 5, *if we choose* $s=1$ *and* $\frac{1}{3}<(\frac{1}{2p+1})^{\frac{1}{p}}<1$, $p>1$, *we have*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq \frac{(b-a)^2}{16}\left\{\left|f''\left(\frac{3a+b}{4}\right)\right|+\left|f''\left(\frac{a+3b}{4}\right)\right|\right\}. \tag{2.15}$$

3 Applications to special means

We now consider the means for arbitrary real numbers α, β ($\alpha \neq \beta$). We take:

(1) *Arithmetic mean*:

$$A(\alpha,\beta)=\frac{\alpha+\beta}{2}, \quad \alpha,\beta\in\mathbb{R}^+;$$

(2) *Logarithmic mean*:

$$L(\alpha,\beta)=\frac{\alpha-\beta}{\ln|\alpha|-\ln|\beta|}, \quad |\alpha|\neq|\beta|, \alpha,\beta\neq 0, \alpha,\beta\in\mathbb{R}^+;$$

Generalized log-mean:

$$L_n(\alpha,\beta)=\left[\frac{\beta^{n+1}-\alpha^{n+1}}{(n+1)(\beta-\alpha)}\right]^{\frac{1}{n}}, \quad n\in\mathbb{Z}\backslash\{-1,0\}, \alpha,\beta\in\mathbb{R}^+.$$

Now, using the results of Section 2, we give some applications to special means of real numbers.

Proposition 1 *Let* $0 < a < b$ *and* $s \in (0,1)$. *Then we have*

$$\left|A^s(a,b) - L_s^s(a,b)\right| \leq \frac{|s(s-1)|(b-a)^2}{192}\left\{a^{s-2} + 6\left(\frac{a+b}{2}\right)^{s-2} + b^{s-2}\right\}.$$

Proof The assertion follows from (2.9) applied to the s-convex function in the second sense $f : [0,1] \to [0,1]$, $f(x) = x^s$. □

Proposition 2 *Let* $0 < a < b$ *and* $s \in (0,1)$. *Then we have*

$$\left|A^s(a,b) - L_s^s(a,b)\right| \leq \frac{|s(s-1)|(b-a)^2}{48}\left(\frac{3}{4}\right)^q\left\{\left[\frac{a^{q(s-2)}}{3} + \left(\frac{a+b}{2}\right)^{q(s-2)}\right]^{\frac{1}{q}} + \left[\left(\frac{a+b}{2}\right)^{q(s-2)} + \frac{b^{q(s-2)}}{3}\right]^{\frac{1}{q}}\right\}.$$

Proof The assertion follows from (2.11) applied to the s-convex function in the second sense $f : [0,1] \to [0,1]$, $f(x) = x^s$. □

Proposition 3 *Let* $0 < a < b$ *and* $p > 1$. *Then we have*

$$\left|A^s(a,b) - L_s^s(a,b)\right| \leq (b-a)^2\left\{\frac{1}{(3a+b)^2} + \frac{1}{(a+3b)^2}\right\}.$$

Proof The inequality follows from (2.15) applied to the concave function in the second sense $f : [a;b] \to \mathbb{R}$, $f(x) = \ln x$. The details are omitted. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
ÇY, AOA and ES carried out the design of the study and performed the analysis. MEÖ participated in its design and coordination. All authors read and approved the final manuscript.

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