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On some Fischer-type determinantal inequalities for accretive-dissipative matrices

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Abstract

In this note, we give some refinements of Fischer-type determinantal inequalities for accretive-dissipative matrices which are due to Lin (Linear Algebra Appl. 438:2808-2812, 2013) and Ikramov (J. Math. Sci. (N.Y.) 121:2458-2464, 2004).

Keywords: accretive-dissipative matrix; Fischer determinantal inequality

1 Introduction

Let $\mathbb{M}_n(\mathbf{C})$ be the space of complex $n \times n$ matrices. For any $A \in \mathbb{M}_n(\mathbf{C})$, the conjugate transpose of A is denoted by A^*. $A \in \mathbb{M}_n(\mathbf{C})$ is said to be accretive-dissipative if it has the Hermitian decomposition

$$A = B + iC, \quad B = B^*, C = C^*, \tag{1.1}$$

where both matrices B and C are positive definite. For simplicity, let A, B, C be partitioned as

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} B_{11} & B_{12} \\ B_{12}^* & B_{22} \end{pmatrix} + i \begin{pmatrix} C_{11} & C_{12} \\ C_{12}^* & C_{22} \end{pmatrix} \tag{1.2}$$

such that the diagonal blocks A_{11} and A_{22} are of order k and l ($k > 0$, $l > 0$ and $k + l = n$), respectively, and let $m = \min\{k, l\}$.

If $B = I_n$ in (1.1), then an accretive-dissipative matrix $A \in \mathbb{M}_n(\mathbf{C})$ is called a Buckley matrix.

If $A \in \mathbb{M}_n(\mathbf{C})$ is partitioned as

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where A_{11} is a nonsingular submatrix, then the matrix $A/A_{11} := A_{22} - A_{21}A_{11}^{-1}A_{12}$ is called the *Schur complement* of the submatrix A_{11} in A. For a nonsingular matrix A, its condition number is denoted by $\kappa(A) := \sqrt{\frac{\lambda_{\max}(A^*A)}{\lambda_{\min}(A^*A)}}$ which is the ratio of largest and smallest singular values of A. For Hermitian matrices $B, C \in \mathbb{M}_n(\mathbf{C})$, we write $B \geq C$ if $B - C$ is positive-semidefinite.

If $A \in \mathbb{M}_n(\mathbf{C})$ is positive definite and partitioned as in (1.2), then the famous Fischer-type determinantal inequality is proved [1, p.478]:

$$\det A \le \det A_{11} \det A_{22}. \tag{1.3}$$

If $A \in \mathbb{M}_n(\mathbf{C})$ is an accretive-dissipative matrix and partitioned as in (1.2), Ikramov [2] first proved the determinantal inequality for A:

$$|\det A| \le 3^m |\det A_{11}||\det A_{22}|. \tag{1.4}$$

Very recently, Lin [3, Theorem 8] got a stronger result than (1.4) as follows.

If $A \in \mathbb{M}_n(\mathbf{C})$ is an accretive-dissipative matrix, then

$$|\det A| \le 2^{\frac{3}{2}m} |\det A_{11}||\det A_{22}|. \tag{1.5}$$

For Buckley matrices, the stronger bound was obtained by Ikramov [2]:

$$|\det A| \le \left(\frac{1+\sqrt{17}}{4}\right)^m |\det A_{11}||\det A_{22}|. \tag{1.6}$$

The purpose of this paper is to give refinements of (1.5) and (1.6). Our main results can be stated as follows.

Theorem 1 *Let $A \in \mathbb{M}_n(\mathbf{C})$ be accretive-dissipative and partitioned as in* (1.2). *Then*

$$|\det A| \le 2^{\frac{1}{2}m} \left(1 + \left(\frac{1-\kappa}{1+\kappa}\right)^2\right)^m |\det A_{11}||\det A_{22}|, \tag{1.7}$$

where κ is the maximum of the condition numbers of B and C.

Because of $2^{\frac{m}{2}}(1 + (\frac{\kappa-1}{\kappa+1})^2)^m \le 2^{\frac{3}{2}m}$, inequality (1.7) is a refinement of inequality (1.5).

Theorem 2 *Let $A \in \mathbb{M}_n(\mathbf{C})$ be a Buckley matrix and partitioned as in* (1.2). *Then*

$$|\det A| \le \left(\frac{d^2 + d\sqrt{16+d^2} + 8}{8}\right)^{\frac{m}{2}} |\det A_{11}||\det A_{22}|, \tag{1.8}$$

where κ is the condition number of C and $d = (\frac{\kappa-1}{\kappa+1})^2 \in [0,1]$.

It is clear that inequality (1.8) improves (1.6). In fact, since the function

$$f(x) = \left(\frac{x^2 + x\sqrt{16+x^2} + 8}{8}\right)^{\frac{m}{2}}$$

is increasing for $x \in [0,1]$, thus we have

$$f(d) = \left(\frac{d^2 + d\sqrt{16+d^2} + 8}{8}\right)^{\frac{m}{2}} \le f(1) = \left(\frac{\sqrt{17}+1}{4}\right)^m,$$

which implies that (1.8) is a refinement of (1.6).

2 Proofs of main results

To achieve the proofs of Theorem 1 and Theorem 2, we need the following lemmas.

Lemma 3 [4, Property 6] *Let* $A \in \mathbb{M}_n(\mathbf{C})$ *be accretive-dissipative and partitioned as in* (1.2). *Then* $A/A_{11} := A_{22} - A_{21}A_{11}^{-1}A_{12}$, *the Schur complement of* A_{11} *in* A *is also accretive-dissipative.*

Lemma 4 [2, Lemma 1] *Let* $A \in \mathbb{M}_n(\mathbf{C})$ *be accretive-dissipative and partitioned as in* (1.2). *Then* $A^{-1} = E - iF$ *with* $E = (B + CB^{-1}C)^{-1}$ *and* $F = (C + BC^{-1}B)^{-1}$.

Lemma 5 [2, Lemma 5] *Let* $A_1, A_2 \in \mathbb{M}_n(\mathbf{C})$ *be accretive-dissipative matrices and let*

$$A_1 = B_1 + iC_1, \qquad A_2 = B_2 + iC_2$$

be the Hermitian decompositions of these matrices. If

$$B_1 \le B_2, \qquad C_1 \le C_2,$$

then

$$|\det A_1| \le |\det A_2|. \tag{2.1}$$

Lemma 6 [3, Lemma 6] *Let* $B, C \in \mathbb{M}_n(\mathbf{C})$ *be positive definite. Then*

$$\left|\det(B + iC)\right| \le \det(B + C) \le 2^{\frac{n}{2}} \left|\det(B + iC)\right|. \tag{2.2}$$

Lemma 7 [5, (6)] *Let* $A \in \mathbb{M}_n(\mathbf{C})$ *be positive definite. Then*

$$A_{12}A_{22}^{-1}A_{21} \le \left(\frac{\lambda_1 - \lambda_n}{\lambda_1 + \lambda_n}\right)^2 A_{11}, \tag{2.3}$$

where λ_1 *and* λ_n *are the largest and the smallest eigenvalues of* A.

Lemma 8 [6, Lemma 3.2] *Let* $B, C \in \mathbb{M}_n(\mathbf{C})$ *be Hermitian and assume that* $B > 0$. *Then*

$$B + CB^{-1}C \ge 2C. \tag{2.4}$$

Remark 1 A stronger inequality than (2.4) was given in Lin [7, Lemma 2.2]: Let $A > 0$ and any Hermitian B. Then $A \sharp (BA^{-1}B) \ge B$.

In what follows, we give the proofs of Theorem 1 and Theorem 2.

Proof of Theorem 1 By Lemma 4, we obtain

$$\begin{aligned} A/A_{11} &= A_{22} - A_{21}A_{11}^{-1}A_{12} \\ &= B_{22} + iC_{22} - \left(B_{12}^* + iC_{12}^*\right)(B_{11} + iC_{11})^{-1}(B_{12} + iC_{12}) \\ &= B_{22} + iC_{22} - \left(B_{12}^* + iC_{12}^*\right)(E_k - iF_k)(B_{12} + iC_{12}). \end{aligned}$$

Furthermore, we have

$$E_k = \left(B_{11} + C_{11}B_{11}^{-1}C_{11}\right)^{-1}, \qquad F_k = \left(C_{11} + B_{11}C_{11}^{-1}B_{11}\right)^{-1},$$

where E_k and F_k are positive definite.

By Lemma 8 and the operator reverse monotonicity of the inverse, we get

$$E_k \le \frac{1}{2}C_{11}^{-1}, \qquad F_k \le \frac{1}{2}B_{11}^{-1}. \tag{2.5}$$

Set $A/A_{11} = R + iS$ with $R = R^*$, $S = S^*$. By Lemma 3, it is easy to know that R, S are positive definite. A simple calculation shows

$$R = B_{22} - B_{12}^*E_kB_{12} + C_{12}^*E_kC_{12} - B_{12}^*F_kC_{12} - C_{12}^*F_kB_{12},$$
$$S = C_{22} + B_{12}^*F_kB_{12} - C_{12}^*F_kC_{12} - C_{12}^*E_kB_{12} - B_{12}^*E_kC_{12}.$$

By the inequalities

$$\left(B_{12}^* \pm C_{12}^*\right)F_k(B_{12} \pm C_{12}) \ge 0, \qquad \left(B_{12}^* \pm C_{12}^*\right)E_k(B_{12} \pm C_{12}) \ge 0,$$

it can be proved that

$$\pm\left(B_{12}^*F_kC_{12} + C_{12}^*F_kB_{12}\right) \le B_{12}^*F_kB_{12} + C_{12}^*F_kC_{12},$$
$$\pm\left(C_{12}^*E_kB_{12} + B_{12}^*E_kC_{12}\right) \le B_{12}^*E_kB_{12} + C_{12}^*E_kC_{12}.$$

Thus

$$R + S \le B_{22} + 2B_{12}^*F_kB_{12} + C_{22}^* + 2C_{12}^*E_kC_{12}. \tag{2.6}$$

Since B, C are positive definite, we have by Lemma 7

$$B_{12}B_{22}^{-1}B_{12}^* \le \left(\frac{\lambda_1 - \lambda_n}{\lambda_1 + \lambda_n}\right)^2 B_{11}, \qquad C_{12}C_{22}^{-1}C_{12}^* \le \left(\frac{\lambda_1' - \lambda_n'}{\lambda_1' + \lambda_n'}\right)^2 C_{11}. \tag{2.7}$$

By (2.7), it is easy to know that

$$B_{12}^*B_{11}^{-1}B_{12} \le \left(\frac{\lambda_1 - \lambda_n}{\lambda_1 + \lambda_n}\right)^2 B_{22}, \qquad C_{12}^*C_{11}^{-1}C_{12} \le \left(\frac{\lambda_1' - \lambda_n'}{\lambda_1' + \lambda_n'}\right)^2 C_{22}. \tag{2.8}$$

In (2.7) and (2.8), λ_1 and λ_n (λ_1' and λ_n') are the largest and the smallest eigenvalues of B (C), respectively.

Note that $f(x) = (\frac{x-1}{x+1})^m$ ($m \ge 1$) is increasing for $x \in [1, \infty)$. Without loss of generality, assume $m = l$. Then we have

$$\begin{aligned} |\det A/A_{11}| &= |\det R + iS| \\ &\le \det(R + S) \quad \text{(by Lemma 6)} \end{aligned}$$

$$
\begin{aligned}
&\le \det\bigl(B_{22} + 2B_{12}^*F_kB_{12} + C_{22} + 2C_{12}^*E_kC_{12}\bigr) \quad \text{(by (2.6))}\\
&\le \det\bigl(B_{22} + B_{12}^*B_{11}^{-1}B_{12} + C_{22} + C_{12}^*C_{11}^{-1}C_{12}\bigr) \quad \text{(by (2.5))}\\
&\le \det\left(B_{22} + C_{22} + \left(\frac{\lambda_1 - \lambda_n}{\lambda_1 + \lambda_n}\right)^2 B_{22} + \left(\frac{\lambda_1' - \lambda_n'}{\lambda_1' + \lambda_n'}\right)^2 C_{22}\right) \quad \text{(by (2.8))}\\
&= \det\left(B_{22} + C_{22} + \left(\frac{\frac{\lambda_1}{\lambda_n} - 1}{\frac{\lambda_1}{\lambda_n} + 1}\right)^2 B_{22} + \left(\frac{\frac{\lambda_1'}{\lambda_n'} - 1}{\frac{\lambda_1'}{\lambda_n'} + 1}\right)^2 C_{22}\right)\\
&\le \left(1 + \left(\frac{\kappa - 1}{\kappa + 1}\right)^2\right)^m \det(B_{22} + C_{22})\\
&\le 2^{\frac{m}{2}}\left(1 + \left(\frac{\kappa - 1}{\kappa + 1}\right)^2\right)^m \bigl|\det(B_{22} + iC_{22})\bigr| \quad \text{(by Lemma 6)}\\
&= 2^{\frac{m}{2}}\left(1 + \left(\frac{\kappa - 1}{\kappa + 1}\right)^2\right)^m |\det A_{22}|,
\end{aligned}
$$

where $\kappa = \max(\frac{\lambda_1}{\lambda_n}, \frac{\lambda_1'}{\lambda_n'}) \ge 1$, *i.e.*, the maximum of the condition numbers of B and C.

By noting

$$|\det A| = |\det A_{11}|\bigl|\det(A/A_{11})\bigr|,$$

the proof is completed. □

Remark 2 In fact, a reverse direction to the inequality of Theorem 1 has been given in Lin [8, Theorem 1.2].

Proof of Theorem 2 The proof is similar to Theorem 1. By Lemma 4, we obtain

$$
\begin{aligned}
A/A_{11} &= A_{22} - A_{21}A_{11}^{-1}A_{12}\\
&= I_l + iC_{22} - C_{12}^*(I_k + iC_{11})^{-1}C_{12}\\
&= I_l + iC_{22} + C_{12}^*(E_k - iF_k)C_{12}
\end{aligned}
$$

with

$$E_k = \bigl(I_k + C_{11}^2\bigr)^{-1}, \qquad F_k = \bigl(C_{11} + C_{11}^{-1}\bigr)^{-1}.$$

By Lemma 8 and the operator reverse monotonicity of the inverse, we get

$$E_k \le \frac{1}{2}C_{11}^{-1}, \qquad F_k \le \frac{1}{2}I_k. \tag{2.9}$$

Set $A/A_{11} = R + iS$ with $R = R^*$, $S = S^*$. By Lemma 3, it is easy to know that R and S are positive definite. A simple calculation shows

$$
\begin{aligned}
R &= I_l + C_{12}^*E_kC_{12},\\
S &= C_{22} - C_{12}^*F_kC_{12},
\end{aligned}
$$

where F_k is positive definite. Therefore

$$S \leq C_{22}.$$

By (2.9), we have

$$R \leq I_l + \frac{1}{2} C_{12}^* C_{11}^{-1} C_{12}.$$

As C is positive definite, we get by (2.8)

$$C_{12}^* C_{11}^{-1} C_{12} \leq \left(\frac{\lambda_1' - \lambda_n'}{\lambda_1' + \lambda_n'} \right)^2 C_{22}, \tag{2.10}$$

where λ_1', λ_n' are the largest and the smallest eigenvalues of C. So we have

$$R \leq I_l + \frac{1}{2} \left(\frac{\lambda_1' - \lambda_n'}{\lambda_1' + \lambda_n'} \right)^2 C_{22}.$$

Without loss of generality, assume $m = l$. Thus we get

$$\begin{aligned} |\det A/A_{11}| &= |\det R + iS| \\ &\leq \det\left(I_l + \frac{1}{2} \left(\frac{\lambda_1' - \lambda_n'}{\lambda_1' + \lambda_n'} \right)^2 C_{22} + iC_{22} \right) \quad \text{(by Lemma 5)} \\ &= \det\left(I_l + \frac{1}{2} \left(\frac{\frac{\lambda_1'}{\lambda_n'} - 1}{\frac{\lambda_1'}{\lambda_n'} + 1} \right)^2 C_{22} + iC_{22} \right) \\ &= \det\left(I_l + \frac{1}{2} \left(\frac{\kappa - 1}{\kappa + 1} \right)^2 C_{22} + iC_{22} \right), \end{aligned}$$

where $\kappa = \frac{\lambda_1'}{\lambda_n'}$.

Let $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_l$ be the eigenvalues of C_{22} and we denote $d = (\frac{\kappa-1}{\kappa+1})^2$. Then it is easy to know that

$$|\det A/A_{11}| \leq \left| \det\left(I_l + \frac{1}{2} \left(\frac{\kappa - 1}{\kappa + 1} \right)^2 C_{22} + iC_{22} \right) \right| \tag{2.11}$$

$$= \prod_{j=1}^{l} \left| \left(1 + \frac{1}{2} d\gamma_j \right) + i\gamma_j \right| \tag{2.12}$$

$$= \prod_{j=1}^{l} \left[\left(1 + \frac{1}{2} d\gamma_j \right)^2 + \gamma_j^2 \right]^{\frac{1}{2}}. \tag{2.13}$$

On the other hand,

$$|\det A_{22}| = \left| \det(I + iC_{22}) \right| = \prod_{j=1}^{l} \left(1 + \gamma_j^2 \right)^{\frac{1}{2}}. \tag{2.14}$$

By (2.13) and (2.14), we have

$$\begin{aligned} |\det A| &= |\det A_{11}|\left|\det(A/A_{11})\right| \\ &= \frac{|\det(A/A_{11})|}{|\det A_{22}|}|\det A_{11}||\det A_{22}| \\ &\le \frac{\prod_{j=1}^{l}[(1+\frac{1}{2}d\gamma_j)^2+\gamma_j^2]^{\frac{1}{2}}}{\prod_{j=1}^{l}(1+\gamma_j^2)^{\frac{1}{2}}}|\det A_{11}||\det A_{22}|. \end{aligned}$$

By noting

$$\max_{x\ge 0}\frac{(1+\frac{d}{2}x)^2+x^2}{1+x^2}=\frac{d^2+d\sqrt{16+d^2}+8}{8},\qquad 0\le d\le 1,$$

the proof is completed. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the manuscript and read and approved the final manuscript.

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