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On interval-valued optimization problems with generalized invex functions

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Abstract

This paper is devoted to study interval-valued optimization problems. Sufficient optimality conditions are established for LU optimal solution concept under generalized $(p, r) - \rho - (\eta, \theta)$-invexity. Weak, strong and strict converse duality theorems for Wolfe and Mond-Weir type duals are derived in order to relate the LU optimal solutions of primal and dual problems.

MSC: 90C46; 90C26; 90C30

Keywords: nonlinear programming; interval-valued functions; $(p, r) - \rho - (\eta, \theta)$-invexity; LU-optimal; sufficiency; duality

1 Introduction

Many real world decision-making problems need to accomplish many objectives: minimize cost, maximize reliability, minimize deviations from desire levels, minimize risk, *etc.* In these cases optimization problems have a large number of applications. The main goal of single objective optimization is to find the best solution which corresponds to the minimum or maximum value of a single objective function.

In many research fields and real world problems, the methodology for solving optimization problems has been used. There are three kinds of methodology that are used for solving optimization problems, namely deterministic optimization problem, stochastic optimization problem and interval-valued optimization problem. The optimization problems with interval coefficients are termed an interval-valued optimization problem. In this problem, the coefficient is taken as closed intervals. The solution concept imposed upon the objective function is the main difference between the above said three kinds of problems.

Interval programming methods have been used to tackle specific issues in multiple objective linear programming: some deal with uncertainty in the objective functions, others handle uncertainty both in the objective functions and in the RHS of the constraints and others deal with uncertainty in all the coefficients of the model. Charnes *et al.* [1] proposed an idea for solving the linear programming problems in which the constraints were assumed as closed intervals. Later, Steuer [2] developed an algorithm to solve the linear programming problem with interval objective functions.

Urli and Nadeau [3] presented a process to solve the multi-objective linear programming problems with interval coefficients. Chanas and Kuchta [4] generalized the solution concepts of the linear programming problem with interval coefficients in the objective function based on preference relations between intervals. By imposing a partial ordering

on the set of all closed intervals, Wu [5] introduced a solution concept in optimization problems with interval-valued objective functions.

There are several approaches to model uncertainty in optimization problems such as stochastic optimization and fuzzy optimization. Here we consider an optimization problem with interval-valued objective function. Stancu-Minasian and Tigan [6, 7] investigated this kind of optimization problem.

Wu [8] formulated Karush-Kuhn-Tucker optimality conditions for an interval-valued objective function. Later, Wu [9, 10] formulated a Wolfe-type dual problem related to the interval-valued optimization problem and established duality theorems by using the concept of nondominated solution employed in vector optimization problems. Zhou and Wang [11] established a sufficient optimality condition and discussed mixed-type duality for a class of nonlinear interval-valued optimization problems. Recently, Bhurjee and Panda [12] developed a methodology to study the existence of the solutions of an interval optimization problem. Very recently, Zhang *et al.* [13] discussed the optimality conditions and duality results for interval-valued optimization problems under generalized preinvexity.

Convexity plays an important role in proving the existence of a solution of a general optimization problem. Hence there is a need to study the convex property of interval optimization problems. One of the most useful generalizations of convexity was introduced by Hanson [14]. For details, readers are advised to see [15]. After the concept of $\rho - (\eta, \theta)$-invex function and (p, r)-invex function had been introduced by Zalmai [16] and Antczak [17], respectively, Mandal and Nahak[18] proposed a new concept of $(p, r) - \rho - (\eta, \theta)$-invex function and established some symmetric duality results. Recently, Jayswal *et al.* [19] derived sufficient optimality conditions and duality theorems for interval-valued optimization problems involving generalized convex functions.

In this paper, we consider an interval-valued optimization problem in which the objective function is an interval-valued function and the constraint functions are real-valued, and derive sufficient optimality conditions and duality theorems under generalized $(p, r) - \rho - (\eta, \theta)$-invexity. The organization of the paper is as follows. In Section 2, we recall some definitions and some basic properties related to interval-valued optimization problems. In Section 3, some sufficient optimality conditions for a class of interval-valued programming problems are discussed. Wolfe and Mond-Weir type duality theorems are obtained in Sections 4 and 5, respectively. Conclusion and future work are proposed in Section 6.

2 Notation and preliminaries

Let I be a class of all closed and bounded intervals in R. Throughout this paper, when we say that A is a closed interval, we mean that A is also bounded in R. If A is a closed interval, we use the notation $A = [a^L, a^U]$, where a^L and a^U mean the lower and upper bounds of A, respectively. If $a^L = a^U = a$, then $A = [a, a] = a$ is a real number. Let $A = [a^L, a^U]$, $B = [b^L, b^U] \in I$, we define

  (i) $A + B = \{a + b : a \in A \text{ and } b \in B\} = [a^L + b^L, a^U + b^U]$,

  (ii) $-A = \{-a : a \in A\} = [-a^U, -a^L]$.

Therefore we see that $A - B = A + (-B) = [a^L - b^U, a^U - b^L]$. We also have

  (i) $k + A = \{k + a : a \in A\} = [k + a^L, k + a^U]$,

(ii) $kA = \{ka : a \in A\} = \begin{cases} [ka^L, ka^U] & \text{if } k \geq 0, \\ [ka^U, ka^L] & \text{if } k < 0, \end{cases}$ where k is a real number.

Let R^n denote an n-dimensional Euclidean space. The function $F : R^n \to I$ is called an interval-valued function, *i.e.*, $F(x) = F(x_1, x_2, \ldots, x_n)$ is a closed interval in R for each $x \in R^n$. The interval-valued function F can also be written as $F(x) = [F^L(x), F^U(x)]$, where $F^L(x)$, $F^U(x)$ are real-valued functions defined on R^n and satisfy the condition $F^L(x) \leq F^U(x)$ for each $x \in R^n$. We note that $[F(x)]^L = F^L(x)$ and $[F(x)]^U = F^U(x)$.

In interval mathematics, an order relation is often used to rank interval numbers and it implies that an interval number is better than another but not that one is larger than another. For $A = [a^L, a^U]$ and $B = [b^L, b^U]$, we write $A \leq_{LU} B$ if and only if $a^L \leq b^L$ and $a^U \leq b^U$. It is easy to see that $\leq_{LU}$ is a partial ordering on I. Also, we can write $A <_{LU} B$ if and only if $A \leq_{LU} B$ and $A \neq B$.

Equivalently, $A <_{LU} B$ if and only if

$$a^L < b^L, \qquad a^U \leq b^U \quad \text{or}$$

$$a^L \leq b^L, \qquad a^U < b^U \quad \text{or}$$

$$a^L < b^L, \qquad a^U < b^U.$$

Definition 2.1 [18] Let $f : R^n \to R$ be a differentiable function and let p, r be arbitrary real numbers. If there exist $\eta : R^n \times R^n \to R^n$, $\theta : R^n \times R^n \to R^n$ and $\rho \in R$ such that the relations

$$\frac{1}{r}\left(e^{r(f(x)-f(y))} - 1\right)(>) \geq \frac{1}{p}\nabla f(y)\left(e^{p\eta(x,y)} - \mathbf{1}\right) + \rho\|\theta(x,y)\|^2 \quad \text{for } p \neq 0, r \neq 0,$$

$$\frac{1}{r}\left(e^{r(f(x)-f(y))} - 1\right)(>) \geq \nabla f(y)\eta(x,y) + \rho\|\theta(x,y)\|^2 \quad \text{for } p = 0, r \neq 0,$$

$$f(x) - f(y)(>) \geq \frac{1}{p}\nabla f(y)\left(e^{p\eta(x,y)} - \mathbf{1}\right) + \rho\|\theta(x,y)\|^2 \quad \text{for } p \neq 0, r = 0,$$

$$f(x) - f(y)(>) \geq \nabla f(y)\eta(x,y) + \rho\|\theta(x,y)\|^2 \quad \text{for } p = 0, r = 0$$

hold, then f is said to be (strictly) $(p, r) - \rho - (\eta, \theta)$-invex at the point y on R^n with respect to η, θ.

Definition 2.2 Let $f : R^n \to R$ be a differentiable function and let p, r be arbitrary real numbers. If there exist $\eta : R^n \times R^n \to R^n$, $\theta : R^n \times R^n \to R^n$ and $\rho \in R$ such that the relations

$$\frac{1}{p}\nabla f(y)\left(e^{p\eta(x,y)} - \mathbf{1}\right) + \rho\|\theta(x,y)\|^2 \geq 0 \Rightarrow \frac{1}{r}\left(e^{r(f(x)-f(y))} - 1\right)(>) \geq 0 \quad \text{for } p \neq 0, r \neq 0,$$

$$\nabla f(y)\eta(x,y) + \rho\|\theta(x,y)\|^2 \geq 0 \Rightarrow \frac{1}{r}\left(e^{r(f(x)-f(y))} - 1\right)(>) \geq 0 \quad \text{for } p = 0, r \neq 0,$$

$$\frac{1}{p}\nabla f(y)\left(e^{p\eta(x,y)} - \mathbf{1}\right) + \rho\|\theta(x,y)\|^2 \geq 0 \Rightarrow f(x) - f(y)(>) \geq 0 \quad \text{for } p \neq 0, r = 0,$$

$$\nabla f(y)\eta(x,y) + \rho\|\theta(x,y)\|^2 \geq 0 \Rightarrow f(x) - f(y)(>) \geq 0 \quad \text{for } p = 0, r = 0$$

hold, then f is said to be (strictly) $(p, r) - \rho - (\eta, \theta)$-pseudo-invex at the point y on R^n with respect to η, θ.

Definition 2.3 Let $f : R^n \to R$ be a differentiable function and let p, r be arbitrary real numbers. If there exist $\eta : R^n \times R^n \to R^n$, $\theta : R^n \times R^n \to R^n$ and $\rho \in R$ such that the relations

$$\frac{1}{r}\big(e^{r(f(x)-f(y))} - 1\big) \le 0 \Rightarrow \frac{1}{p}\nabla f(y)\big(e^{p\eta(x,y)} - \mathbf{1}\big) \le -\rho\big\|\theta(x,y)\big\|^2 \quad \text{for } p \ne 0, r \ne 0,$$

$$\frac{1}{r}\big(e^{r(f(x)-f(y))} - 1\big) \le 0 \Rightarrow \nabla f(y)\eta(x,y) \le -\rho\big\|\theta(x,y)\big\|^2 \quad \text{for } p = 0, r \ne 0,$$

$$f(x) - f(y) \le 0 \Rightarrow \frac{1}{p}\nabla f(y)\big(e^{p\eta(x,y)} - \mathbf{1}\big) \le -\rho\big\|\theta(x,y)\big\|^2 \quad \text{for } p \ne 0, r = 0,$$

$$f(x) - f(y) \le 0 \Rightarrow \nabla f(y)\eta(x,y) \le -\rho\big\|\theta(x,y)\big\|^2 \quad \text{for } p = 0, r = 0$$

hold, then f is said to be $(p, r) - \rho - (\eta, \theta)$-quasi-invex at the point y on R^n with respect to η, θ.

Remark 2.1 It should be noted that the exponentials appearing on the right-hand sides of inequalities above are understood to be taken componentwise and $\mathbf{1} = (1, 1, \ldots, 1) \in R^n$.

Remark 2.2 All theorems in this paper will be proved only in the case when $p \ne 0$, $r \ne 0$ (other cases can be dealt with likewise since the only changes arise in a form of inequality). Moreover, without loss of generality, we shall assume that $r > 0$, $p > 0$ (in the case when $r < 0$, $p < 0$, the direction of some of the inequalities in the proof of the theorems should be changed to the opposite one).

In this paper, we consider the following primal optimization problem with interval-valued objective function:

$$\text{(IVP)} \quad \min F(x) = \big[F^L(x), F^U(x)\big]$$

$$\text{subject to } g_j(x) \le 0, \quad j = 1, 2, \ldots, m,$$

where $F : R^n \to I$ is an interval-valued function and $g_j : R^n \to R$ is a real-valued function. Let $\Omega = \{x \in R^n : g_j(x) \le 0, j = 1, 2, \ldots, m\}$ be the set of all feasible solutions of (IVP). We also denote by $\mathrm{Obj}(F, \Omega) = \{F(x) : x \in \Omega\}$ the set of all objective values of primal problem (IVP).

Definition 2.4 [20] Let x^* be a feasible solution of the primal problem (IVP). We say that x^* is a LU optimal solution of problem (IVP) if there exists no $x_0 \in \Omega$ such that $F(x_0) <_{LU} F(x^*)$.

Theorem 2.1 (Karush-Kuhn-Tucker type conditions [9]) *Assume that x^* is a LU optimal solution of primal problem* (IVP) *and F, g_j, $j = 1, 2, \ldots, m$, are differentiable at x^*. Suppose that the constraint function g_j, $j = 1, 2, \ldots, m$, satisfies the suitable Kuhn-Tucker constraint qualification [9] at x^*. Then there exist multipliers $0 < \xi^L, \xi^U \in R$ and $0 \le \mu_j \in R$, $j = 1, 2, \ldots, m$, such that*

$$\xi^L \nabla F^L\big(x^*\big) + \xi^U \nabla F^U\big(x^*\big) + \sum_{j=1}^{m} \mu_j \nabla g_j\big(x^*\big) = 0, \tag{1}$$

$$\mu_j g_j\big(x^*\big) = 0, \quad j = 1, 2, \ldots, m. \tag{2}$$

3 Sufficient optimality conditions

In this section, we shall establish the following sufficient optimality conditions for (IVP).

Theorem 3.1 (Sufficiency) *Let $x^* \in \Omega$ be a feasible solution of* (IVP). *Suppose that the objective function F and the constraint function g_j, $j = 1, 2, \ldots, m$, are differentiable at x^*. Assume that F^L and F^U are $(p, r) - \rho_1 - (\eta, \theta)$-invex and $(p, r) - \rho_2 - (\eta, \theta)$-invex, respectively, with respect to η, θ and $\sum_{j=1}^{m} \mu_j g_j$ is $(p, r) - \rho_3 - (\eta, \theta)$-invex with respect to η, θ at x^* with $(\xi^L \rho_1 + \xi^U \rho_2 + \rho_3) \geq 0$. If there exist (Lagrange) multipliers $0 < \xi^L, \xi^U \in R$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_m)$, $0 \leq \mu_j \in R$, $j = 1, 2, \ldots, m$, such that (x^*, ξ^L, ξ^U, μ) satisfies* (1) *and* (2), *then x^* is a LU optimal solution to problem* (IVP).

Proof Let x^* be not a LU optimal solution of (IVP). Then there exists a feasible solution $x_0 \in \Omega$ such that

$$F(x_0) <_{LU} F(x^*).$$

That is,

$$\begin{cases} F^L(x_0) < F^L(x^*), \\ F^U(x_0) < F^U(x^*), \end{cases} \quad \text{or} \quad \begin{cases} F^L(x_0) \leq F^L(x^*), \\ F^U(x_0) < F^U(x^*), \end{cases} \quad \text{or} \quad \begin{cases} F^L(x_0) < F^L(x^*), \\ F^U(x_0) \leq F^U(x^*). \end{cases}$$

Since $r > 0$, using the property of an exponential function, we obtain

$$\begin{cases} \frac{1}{r}[e^{r\{F^L(x_0) - F^L(x^*)\}} - 1] < 0, \\ \frac{1}{r}[e^{r\{F^U(x_0) - F^U(x^*)\}} - 1] < 0, \end{cases} \quad \text{or} \quad \begin{cases} \frac{1}{r}[e^{r\{F^L(x_0) - F^L(x^*)\}} - 1] \leq 0, \\ \frac{1}{r}[e^{r\{F^U(x_0) - F^U(x^*)\}} - 1] < 0, \end{cases} \quad \text{or}$$

$$\begin{cases} \frac{1}{r}[e^{r\{F^L(x_0) - F^L(x^*)\}} - 1] < 0, \\ \frac{1}{r}[e^{r\{F^U(x_0) - F^U(x^*)\}} - 1] \leq 0. \end{cases}$$

Using the above inequalities and the $(p, r) - \rho_1 - (\eta, \theta)$-invexity of F^L and the $(p, r) - \rho_2 - (\eta, \theta)$-invexity of F^U at x^*, we get

$$\begin{cases} \frac{1}{p}\{\nabla F^L(x^*)\}(e^{p\eta(x_0, x^*)} - \mathbf{1}) + \rho_1 \|\theta(x_0, x^*)\|^2 < 0, \\ \frac{1}{p}\{\nabla F^U(x^*)\}(e^{p\eta(x_0, x^*)} - \mathbf{1}) + \rho_2 \|\theta(x_0, x^*)\|^2 < 0, \end{cases} \quad \text{or}$$

$$\begin{cases} \frac{1}{p}\{\nabla F^L(x^*)\}(e^{p\eta(x_0, x^*)} - \mathbf{1}) + \rho_1 \|\theta(x_0, x^*)\|^2 \leq 0, \\ \frac{1}{p}\{\nabla F^U(x^*)\}(e^{p\eta(x_0, x^*)} - \mathbf{1}) + \rho_2 \|\theta(x_0, x^*)\|^2 < 0, \end{cases} \quad \text{or}$$

$$\begin{cases} \frac{1}{p}\{\nabla F^L(x^*)\}(e^{p\eta(x_0, x^*)} - \mathbf{1}) + \rho_1 \|\theta(x_0, x^*)\|^2 < 0, \\ \frac{1}{p}\{\nabla F^U(x^*)\}(e^{p\eta(x_0, x^*)} - \mathbf{1}) + \rho_2 \|\theta(x_0, x^*)\|^2 \leq 0. \end{cases}$$

Since $\xi^L > 0$ and $\xi^U > 0$, from the above inequalities, we get

$$\frac{1}{p}\{\xi^L \nabla F^L(x^*) + \xi^U \nabla F^U(x^*)\}(e^{p\eta(x_0, x^*)} - \mathbf{1})$$

$$+ \xi^L \rho_1 \|\theta(x_0, x^*)\|^2 + \xi^U \rho_2 \|\theta(x_0, x^*)\|^2 < 0. \tag{3}$$

On the other hand, from the feasibility of x_0 to (IVP), we have

$$g_j(x_0) \leq 0, \quad j = 1, 2, \ldots, m.$$

Since $\mu_j \geq 0$, $j = 1, 2, \ldots, m$, the above inequality together with (2) yields

$$\sum_{j=1}^{m} \mu_j g_j(x_0) \leq \sum_{j=1}^{m} \mu_j g_j(x^*).$$

As $r > 0$, using the property of an exponential function, we get

$$\frac{1}{r}\left[e^{r\{\sum_{j=1}^{m} \mu_j g_j(x_0) - \sum_{j=1}^{m} \mu_j g_j(x^*)\}} - 1\right] \leq 0,$$

which together with the assumption that $\sum_{j=1}^{m} \mu_j g_j$ is $(p, r) - \rho_3 - (\eta, \theta)$-invex at x^* gives

$$\frac{1}{p}\sum_{j=1}^{m} \mu_j \nabla g_j(x^*)\left(e^{p\eta(x_0, x^*)} - 1\right) + \rho_3\left\|\theta(x_0, x^*)\right\|^2 \leq 0. \tag{4}$$

On adding (3) and (4), we get

$$\frac{1}{p}\left(e^{p\eta(x_0, x^*)} - 1\right)\left[\xi^L \nabla F^L(x^*) + \xi^U \nabla F^U(x^*) + \sum_{j=1}^{m} \mu_j \nabla g_j(x^*)\right]$$

$$+ \left(\xi^L \rho_1 + \xi^U \rho_2 + \rho_3\right)\left\|\theta(x_0, x^*)\right\|^2 < 0.$$

Therefore, from the hypothesis that $(\xi^L \rho_1 + \xi^U \rho_2 + \rho_3) \geq 0$ and the above inequality, we get

$$\frac{1}{p}\left(e^{p\eta(x_0, x^*)} - 1\right)\left[\xi^L \nabla F^L(x^*) + \xi^U \nabla F^U(x^*) + \sum_{j=1}^{m} \mu_j \nabla g_j(x^*)\right] < 0,$$

which contradicts (1). Therefore x^* is a LU optimal solution of (IVP). This completes the proof. □

Theorem 3.2 (Sufficiency) *Let $x^* \in \Omega$ be a feasible solution of* (IVP). *Suppose that the objective function F and the constraint function g_j, $j = 1, 2, \ldots, m$, are differentiable at x^*. Assume that $(\xi^L F^L + \xi^U F^U)$ is $(p, r) - \rho_1 - (\eta, \theta)$-pseudo-invex with respect to η, θ and $\sum_{j=1}^{m} \mu_j g_j$ is $(p, r) - \rho_2 - (\eta, \theta)$-quasi-invex with respect to η, θ at x^* with $(\rho_1 + \rho_2) \geq 0$. If there exist (Lagrange) multipliers $0 < \xi^L, \xi^U \in R$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_m)$, $0 \leq \mu_j \in R$, $j = 1, 2, \ldots, m$, such that (x^*, ξ^L, ξ^U, μ) satisfies (1) and (2), then x^* is a LU optimal solution to problem* (IVP).

Proof Let x^* be not a LU optimal solution of (IVP). Then there exists a feasible solution $x_0 \in \Omega$ such that

$$F(x_0) <_{LU} F(x^*).$$

That is,

$$
\begin{cases} F^L(x_0) < F^L(x^*), \\ F^U(x_0) < F^U(x^*), \end{cases} \quad \text{or} \quad \begin{cases} F^L(x_0) \le F^L(x^*), \\ F^U(x_0) < F^U(x^*), \end{cases} \quad \text{or} \quad \begin{cases} F^L(x_0) < F^L(x^*), \\ F^U(x_0) \le F^U(x^*). \end{cases}
$$

Since $\xi^L > 0$ and $\xi^U > 0$, from the above inequalities, we have

$$
\xi^L F^L(x_0) + \xi^U F^U(x_0) < \xi^L F^L(x^*) + \xi^U F^U(x^*).
$$

As $r > 0$, using the property of an exponential function, we get

$$
\frac{1}{r}\left[e^{r\{(\xi^L F^L(x_0) + \xi^U F^U(x_0)) - (\xi^L F^L(x^*) + \xi^U F^U(x^*))\}} - 1\right] < 0,
$$

which together with the assumption that $\xi^L F^L + \xi^U F^U$ is $(p, r) - \rho_1 - (\eta, \theta)$-pseudo-invex at x^* gives

$$
\frac{1}{p}\left\{\xi^L \nabla F^L(x^*) + \xi^U \nabla F^U(x^*)\right\}\left(e^{p\eta(x_0, x^*)} - 1\right) + \rho_1\left\|\theta(x_0, x^*)\right\|^2 < 0. \tag{5}
$$

On the other hand, from the feasibility of x_0 to (IVP), we have

$$
g_j(x_0) \le 0, \quad j = 1, 2, \ldots, m.
$$

Since $\mu_j \ge 0$, $j = 1, 2, \ldots, m$, the above inequality together with (2) yields

$$
\sum_{j=1}^{m} \mu_j g_j(x_0) \le \sum_{j=1}^{m} \mu_j g_j(x^*).
$$

As $r > 0$, using the property of an exponential function, we get

$$
\frac{1}{r}\left[e^{r\{\sum_{j=1}^{m} \mu_j g_j(x_0) - \sum_{j=1}^{m} \mu_j g_j(x^*)\}} - 1\right] \le 0,
$$

which together with the assumption that $\sum_{j=1}^{m} \mu_j g_j$ is $(p, r) - \rho_2 - (\eta, \theta)$-quasi-invex at x^* gives

$$
\frac{1}{p}\sum_{j=1}^{m} \mu_j \nabla g_j(x^*)\left(e^{p\eta(x_0, x^*)} - 1\right) + \rho_2\left\|\theta(x_0, x^*)\right\|^2 \le 0. \tag{6}
$$

On adding (5) and (6), we get

$$
\frac{1}{p}\left(e^{p\eta(x_0, x^*)} - 1\right)\left[\xi^L \nabla F^L(x^*) + \xi^U \nabla F^U(x^*) + \sum_{j=1}^{m} \mu_j \nabla g_j(x^*)\right]
$$

$$
+ (\rho_1 + \rho_2)\left\|\theta(x_0, x^*)\right\|^2 < 0.
$$

Since $(\rho_1 + \rho_2) \ge 0$, we have

$$
\frac{1}{p}\left(e^{p\eta(x_0, x^*)} - 1\right)\left[\xi^L \nabla F^L(x^*) + \xi^U \nabla F^U(x^*) + \sum_{j=1}^{m} \mu_j \nabla g_j(x^*)\right] < 0,
$$

which contradicts (1). Therefore x^* is a LU optimal solution of (IVP). This completes the proof. □

4 Wolfe-type duality

In this section, we consider the following Wolfe-type dual problem:

$$\text{(WD)} \quad \max F(y) + \sum_{j=1}^{m} \mu_j g_j(y)$$

subject to

$$\xi^L \nabla F^L(y) + \xi^U \nabla F^U(y) + \sum_{j=1}^{m} \mu_j \nabla g_j(y) = 0, \tag{7}$$

$$\xi^L > 0, \xi^U > 0, \mu_j \geq 0, j = 1, 2, \ldots, m, \tag{8}$$

where $F(y) + \sum_{j=1}^{m} \mu_j g_j(y) = [F^L(y) + \sum_{j=1}^{m} \mu_j g_j(y), F^U(y) + \sum_{j=1}^{m} \mu_j g_j(y)]$ is an interval-valued function.

Definition 4.1 Let $(y^*, \xi^{*L}, \xi^{*U}, \mu^*)$ be a feasible solution of dual problem (WD). We say that $(y^*, \xi^{*L}, \xi^{*U}, \mu^*)$ is a LU optimal solution of dual problem (WD) if there exists no $(y, \xi^{*L}, \xi^{*U}, \mu^*)$ such that $F(y^*) + \sum_{j=1}^{m} \mu_j^* g_j(y^*) <_{LU} F(y) + \sum_{j=1}^{m} \mu_j^* g_j(y)$.

Theorem 4.1 (Weak duality) *Let X_0 be an open subset of R^n. Let F and g_j, $j = 1, 2, \ldots, m$, be differentiable on X_0. Suppose that x and (y, ξ^L, ξ^U, μ) are the feasible solutions to* (IVP) *and* (WD), *respectively. Further assume that $\xi^L > 0$, $\xi^U > 0$ and $\mu_j \geq 0$, $j = 1, 2, \ldots, m$, such that $\xi^L F^L + \xi^U F^U + \sum_{j=1}^{m} \mu_j g_j$ is $(p, r) - \rho - (\eta, \theta)$-invex with respect to η, θ at y with $\xi^L + \xi^U = 1$ and $\rho \geq 0$. Then*

$$F(x) \geq_{LU} F(y) + \sum_{j=1}^{m} \mu_j g_j(y).$$

Proof Suppose contrary to the result that

$$F(x) <_{LU} F(y) + \sum_{j=1}^{m} \mu_j g_j(y).$$

That is,

$$\begin{cases} F^L(x) < F^L(y) + \sum_{j=1}^{m} \mu_j g_j(y), \\ F^U(x) < F^U(y) + \sum_{j=1}^{m} \mu_j g_j(y), \end{cases} \quad \text{or} \quad \begin{cases} F^L(x) \leq F^L(y) + \sum_{j=1}^{m} \mu_j g_j(y), \\ F^U(x) < F^U(y) + \sum_{j=1}^{m} \mu_j g_j(y), \end{cases} \quad \text{or}$$

$$\begin{cases} F^L(x) < F^L(y) + \sum_{j=1}^{m} \mu_j g_j(y), \\ F^U(x) \leq F^U(y) + \sum_{j=1}^{m} \mu_j g_j(y). \end{cases}$$

Since $\xi^L > 0$, $\xi^U > 0$ and $\xi^L + \xi^U = 1$, the above inequalities together with the feasibility of x to (IVP) become

$$\xi^L F^L(x) + \xi^U F^U(x) + \sum_{j=1}^{m} \mu_j g_j(x) < \xi^L F^L(y) + \xi^U F^U(y) + \sum_{j=1}^{m} \mu_j g_j(y). \tag{9}$$

From the assumption that $\xi^L F^L + \xi^U F^U + \sum_{j=1}^{m} \mu_j g_j$ is $(p,r) - \rho - (\eta,\theta)$-invex at y, we have

$$\frac{1}{r}\left[e^{r\{(\xi^L F^L(x)+\xi^U F^U(x)+\sum_{j=1}^{m}\mu_j g_j(x))-(\xi^L F^L(y)+\xi^U F^U(y)+\sum_{j=1}^{m}\mu_j g_j(y))\}} - 1\right]$$

$$\geq \frac{1}{p}\left[\xi^L \nabla F^L(y) + \xi^U \nabla F^U(y) + \sum_{j=1}^{m}\mu_j \nabla g_j(y)\right]\left(e^{p\eta(x,y)} - 1\right) + \rho\|\theta(x,y)\|^2.$$

The above inequality together with (7) and $\rho \geq 0$ yields

$$\frac{1}{r}\left[e^{r\{(\xi^L F^L(x)+\xi^U F^U(x)+\sum_{j=1}^{m}\mu_j g_j(x))-(\xi^L F^L(y)+\xi^U F^U(y)+\sum_{j=1}^{m}\mu_j g_j(y))\}} - 1\right] \geq 0.$$

Since $r > 0$, using the property of an exponential function, we get

$$\xi^L F^L(x) + \xi^U F^U(x) + \sum_{j=1}^{m}\mu_j g_j(x) \geq \xi^L F^L(y) + \xi^U F^U(y) + \sum_{j=1}^{m}\mu_j g_j(y),$$

which contradicts the inequality (9). This completes the proof. □

Theorem 4.2 (Strong duality) *Let x^* be a LU optimal solution to* (IVP) *at which Kuhn-Tucker constraints qualification are satisfied. Then there exist $\xi^{*L} > 0$, $\xi^{*U} > 0$ and $\mu^* \geq 0$ such that $(x^*, \xi^{*L}, \xi^{*U}, \mu^*)$ is feasible for* (WD) *and the two objectives have the same value. Further, if the hypothesis of weak duality Theorem 4.1 holds for all feasible solutions $(y^*, \xi^{*L}, \xi^{*U}, \mu^*)$, then $(x^*, \xi^{*L}, \xi^{*U}, \mu^*)$ is a LU optimal solution to* (WD).

Proof Since x^* is a LU optimal solution to (IVP) and the Kuhn-Tucker constraints qualification are satisfied at x^*, then by Theorem 2.1, there exist multipliers $\xi^{*L} > 0$, $\xi^{*U} > 0$ and $\mu_j^* \geq 0$, $j = 1, 2, \ldots, m$, such that

$$\xi^{*L}\nabla F^L\left(x^*\right) + \xi^{*U}\nabla F^U\left(x^*\right) + \sum_{j=1}^{m}\mu_j^*\nabla g_j\left(x^*\right) = 0,$$

$$\mu_j^* g_j\left(x^*\right) = 0,$$

which yields that $(x^*, \xi^{*L}, \xi^{*U}, \mu^*)$ is a feasible solution for (WD) and corresponding objective values are the same. Further, if $(x^*, \xi^{*L}, \xi^{*U}, \mu^*)$ is not a LU optimal solution of (WD), then there exists a feasible solution $(y^*, \xi^{*L}, \xi^{*U}, \mu^*)$ for (WD) such that

$$F\left(x^*\right) <_{LU} F\left(y^*\right) + \sum_{j=1}^{m}\mu_j^* g_j\left(y^*\right),$$

which contradicts weak duality Theorem 4.1. Hence $(x^*, \xi^{*L}, \xi^{*U}, \mu^*)$ is a LU optimal solution to (WD). □

Theorem 4.3 (Strict converse duality) *Let X_0 be an open subset of R^n. Let F and g_j, $j = 1, 2, \ldots, m$, be differentiable on X_0. Suppose that x^* and $(y^*, \xi^{*L}, \xi^U, \mu^*)$ are the feasible solutions to* (IVP) *and* (WD), *respectively. Assume that $\xi^{*L} F^L + \xi^{*U} F^U + \sum_{j=1}^{m}\mu_j^* g_j$ is*

strictly $(p,r) - \rho - (\eta,\theta)$-*invex with respect to* η, θ *at* y^* *with* $\rho \geq 0$ *and*

$$\xi^{*L}F^L(x^*) + \xi^{*U}F^U(x^*) + \sum_{j=1}^{m}\mu_j^*g_j(x^*) \leq \xi^{*L}F^L(y^*) + \xi^{*U}F^U(y^*) + \sum_{j=1}^{m}\mu_j^*g_j(y^*). \quad (10)$$

Then $x^* = y^*$.

Proof Now we assume that $x^* \neq y^*$ and exhibit a contradiction. From the assumption that $\xi^{*L}F^L + \xi^{*U}F^U + \sum_{j=1}^{m}\mu_j^*g_j$ is strictly $(p,r) - \rho - (\eta,\theta)$-invex at y^*, we have

$$\frac{1}{r}\left[e^{r\{(\xi^{*L}F^L(x^*)+\xi^{*U}F^U(x^*)+\sum_{j=1}^{m}\mu_j^*g_j(x^*))-(\xi^{*L}F^L(y^*)+\xi^{*U}F^U(y^*)+\sum_{j=1}^{m}\mu_j^*g_j(y^*))\}} - 1\right]$$

$$> \frac{1}{p}\left[\xi^{*L}\nabla F^L(y^*) + \xi^{*U}\nabla F^U(y^*) + \sum_{j=1}^{m}\mu_j^*\nabla g_j(y^*)\right](e^{p\eta(x^*,y^*)} - 1) + \rho\|\theta(x^*,y^*)\|^2.$$

From the feasibility of $(y^*, \xi^{*L}, \xi^{*U}, \mu^*)$ to (WD), the above inequality together with the hypothesis $\rho \geq 0$ yields

$$\frac{1}{r}\left[e^{r\{(\xi^{*L}F^L(x^*)+\xi^{*U}F^U(x^*)+\sum_{j=1}^{m}\mu_j^*g_j(x^*))-(\xi^{*L}F^L(y^*)+\xi^{*U}F^U(y^*)+\sum_{j=1}^{m}\mu_j^*g_j(y^*))\}} - 1\right] > 0.$$

Since $r > 0$, using the property of an exponential function, we get

$$\xi^{*L}F^L(x^*) + \xi^{*U}F^U(x^*) + \sum_{j=1}^{m}\mu_j^*g_j(x^*) > \xi^{*L}F^L(y^*) + \xi^{*U}F^U(y^*) + \sum_{j=1}^{m}\mu_j^*g_j(y^*),$$

which contradicts (10). This completes the proof. □

5 Mond-Weir type duality

In this section, we consider the following Mond-Weir type dual problem for (IVP):

(MWD) $\max F(y) = \left[F^L(y), F^U(y)\right]$

subject to

$$\xi^L\nabla F^L(y) + \xi^U\nabla F^U(y) + \sum_{j=1}^{m}\mu_j\nabla g_j(y) = 0, \quad (11)$$

$$\mu_j g_j(y) \geq 0, j = 1, 2, \ldots, m, \quad (12)$$

$$\xi^L > 0, \xi^U > 0, \mu_j \geq 0, j = 1, 2, \ldots, m. \quad (13)$$

Definition 5.1 Let $(y^*, \xi^{*L}, \xi^{*U}, \mu^*)$ be a feasible solution of dual problem (MWD). We say that $(y^*, \xi^{*L}, \xi^{*U}, \mu^*)$ is a LU optimal solution of dual problem (MWD) if there exists no $(y, \xi^{*L}, \xi^{*U}, \mu^*)$ such that $F(y^*) <_{LU} F(y)$.

Theorem 5.1 (Weak duality) *Let* X_0 *be an open subset of* R^n. *Let* F *and* g_j, $j = 1, 2, \ldots, m$, *be differentiable on* X_0. *Suppose that* x *and* (y, ξ^L, ξ^U, μ) *are the feasible solutions to* (IVP) *and* (MWD), *respectively. Further assume that there exist* $\xi^L > 0$ *and* $\xi^U > 0$ *and* $\mu_j \geq 0, j =$

$1, 2, \ldots, m$, *such that* $(\xi^L F^L + \xi^U F^U)$ *is* $(p, r) - \rho_1 - (\eta, \theta)$-*pseudo-invex with respect to* η, θ *and* $\sum_{j=1}^{m} \mu_j g_j$ *is* $(p, r) - \rho_2 - (\eta, \theta)$-*quasi-invex with respect to* η, θ *at* y *with* $(\rho_1 + \rho_2) \geq 0$. *Then*

$$F(x) \geq_{LU} F(y).$$

Proof Suppose contrary to the result that

$$F(x) <_{LU} F(y).$$

That is,

$$\begin{cases} F^L(x) < F^L(y), \\ F^U(x) < F^U(y), \end{cases} \quad \text{or} \quad \begin{cases} F^L(x) \leq F^L(y), \\ F^U(x) < F^U(y), \end{cases} \quad \text{or} \quad \begin{cases} F^L(x) < F^L(y), \\ F^U(x) \leq F^U(y). \end{cases}$$

Since $\xi^L > 0$ and $\xi^U > 0$, from the above inequalities, we have

$$\xi^L F^L(x) + \xi^U F^U(x) < \xi^L F^L(y) + \xi^U F^U(y). \tag{14}$$

On the other hand, since $\mu_j \geq 0$, $j = 1, 2, \ldots, m$, from the feasibility of x and (y, ξ^L, ξ^U, μ) to (IVP) and (MWD), respectively, we obtain

$$\sum_{j=1}^{m} \mu_j g_j(x) \leq \sum_{j=1}^{m} \mu_j g_j(y).$$

Since $r > 0$, using the property of an exponential function, we get

$$\frac{1}{r} \Big[e^{r\{\sum_{j=1}^{m} \mu_j g_j(x) - \sum_{j=1}^{m} \mu_j g_j(y)\}} - 1 \Big] \leq 0,$$

which together with the assumption that $\sum_{j=1}^{m} \mu_j g_j$ is $(p, r) - \rho_2 - (\eta, \theta)$-quasi-invex at y, gives

$$\frac{1}{p} \sum_{j=1}^{m} \mu_j \nabla g_j(y) \big(e^{p\eta(x,y)} - \mathbf{1} \big) + \rho_2 \big\| \theta(x, y) \big\|^2 \leq 0.$$

Therefore, from (11) and the hypothesis that $(\rho_1 + \rho_2) \geq 0$, the above inequality yields

$$\frac{1}{p} \big(e^{p\eta(x,y)} - \mathbf{1} \big) \big[\xi^L \nabla F^L(y) + \xi^U \nabla F^U(y) \big] + \rho_1 \big\| \theta(x, y) \big\|^2 \geq 0,$$

which together with the assumption that $\xi^L F^L + \xi^U F^U$ is $(p, r) - \rho_1 - (\eta, \theta)$-pseudo-invex at y gives

$$\frac{1}{r} \Big[e^{r\{(\xi^L F^L(x) + \xi^U F^U(x)) - (\xi^L F^L(y) + \xi^U F^U(y))\}} - 1 \Big] \geq 0.$$

Since $r > 0$, using the property of an exponential function, we get

$$\xi^L F^L(x) + \xi^U F^U(x) \geq \xi^L F^L(y) + \xi^U F^U(y),$$

which contradicts (14). This completes the proof. □

Theorem 5.2 (Strong duality) *Let x^* be a LU optimal solution to* (IVP) *at which Kuhn-Tucker constraints qualification are satisfied. Then there exist $\xi^{*L} > 0$, $\xi^{*U} > 0$ and $\mu^* \geq 0$ such that $(x^*, \xi^{*L}, \xi^{*U}, \mu^*)$ is feasible for* (MWD) *and the two objectives have the same value. Further, if the hypothesis of weak duality Theorem 5.1 holds for all feasible solutions $(y^*, \xi^{*L}, \xi^{*U}, \mu^*)$, then $(x^*, \xi^{*L}, \xi^{*U}, \mu^*)$ is a LU optimal solution to* (MWD).

Proof Since x^* is a LU optimal solution to (IVP) and the Kuhn-Tucker constraints qualification are satisfied at x^*, then by Theorem 2.1, there exist multipliers $\xi^{*L} > 0$, $\xi^{*U} > 0$, $\mu_j^* \geq 0$, $j = 1, 2, \ldots, m$, such that

$$\xi^{*L} \nabla F^L(x^*) + \xi^{*U} \nabla F^U(x^*) + \sum_{j=1}^{m} \mu_j^* \nabla g_j(x^*) = 0,$$

$$\mu_j^* g_j(x^*) = 0,$$

which yields that $(x^*, \xi^{*L}, \xi^{*U}, \mu^*)$ is a feasible solution for (MWD) and corresponding objective values are same. Further, if $(x^*, \xi^{*L}, \xi^{*U}, \mu^*)$ is not a LU optimal solution of (MWD), then there exists a feasible solution $(y^*, \xi^{*L}, \xi^{*U}, \mu^*)$ for (MWD) such that

$$F(x^*) <_{LU} F(y^*),$$

which contradicts weak duality Theorem 5.1. Hence $(x^*, \xi^{*L}, \xi^{*U}, \mu^*)$ is a LU optimal solution to (MWD). □

Theorem 5.3 (Strict converse duality) *Let X_0 be an open subset of R^n. Let F and g_j, $j = 1, 2, \ldots, m$, be differentiable on X_0. Suppose that x^* and $(y^*, \xi^{*L}, \xi^{*U}, \mu^*)$ are the feasible solutions to* (IVP) *and* (MWD), *respectively. Assume that $\xi^{*L} F^L + \xi^{*U} F^U$ is strictly $(p, r) - \rho_1 - (\eta, \theta)$-pseudo-invex and $\sum_{j=1}^{m} \mu_j^* g_j$ is $(p, r) - \rho_2 - (\eta, \theta)$-quasi-invex with respect to η, θ at y^* with $(\rho_1 + \rho_2) \geq 0$ and*

$$\xi^{*L} F^L(x^*) + \xi^{*U} F^U(x^*) \leq \xi^{*L} F^L(y^*) + \xi^{*U} F^U(y^*). \tag{15}$$

Then $x^* = y^*$.

Proof Now we assume that $x^* \neq y^*$ and exhibit a contradiction. From the assumption that $(y^*, \xi^{*L}, \xi^{*U}, \mu^*)$ is a feasible solution for (MWD), we get

$$\xi^{*L} \nabla F^L(y^*) + \xi^{*U} \nabla F^U(y^*) + \sum_{j=1}^{m} \mu_j^* \nabla g_j(y^*) = 0. \tag{16}$$

Since $\mu_j^* \geq 0$, $j = 1, 2, \ldots, m$, from the feasibility of x^* and $(y^*, \xi^{*L}, \xi^{*U}, \mu^*)$ to (IVP) and (MWD), respectively, we obtain

$$\sum_{j=1}^{m} \mu_j^* g_j(x^*) \leq \sum_{j=1}^{m} \mu_j^* g_j(y^*).$$

As $r > 0$, using the property of an exponential function, we get

$$\frac{1}{r}\big[e^{r\{\sum_{j=1}^{m}\mu_j^*g_j(x^*)-\sum_{j=1}^{m}\mu_j^*g_j(y^*)\}} - 1\big] \leq 0,$$

which together with the assumption that $\sum_{j=1}^{m}\mu_j^*g_j$ is $(p,r) - \rho_2 - (\eta,\theta)$-quasi-invex at y^* gives

$$\frac{1}{p}\sum_{j=1}^{m}\mu_j^*\nabla g_j(y^*)\big(e^{p\eta(x^*,y^*)} - \mathbf{1}\big) + \rho_2\big\|\theta(x^*,y^*)\big\|^2 \leq 0.$$

Therefore, from (16) and the hypothesis that $(\rho_1 + \rho_2) \geq 0$, the above inequality yields

$$\frac{1}{p}\big(e^{p\eta(x^*,y^*)} - \mathbf{1}\big)\big[\xi^{*L}\nabla F^L(y^*) + \xi^{*U}\nabla F^U(y^*)\big] + \rho_1\big\|\theta(x^*,y^*)\big\|^2 \geq 0,$$

which together with the assumption that $\xi^{*L}F^L + \xi^{*U}F^U$ is strictly $(p,r) - \rho_1 - (\eta,\theta)$-pseudo-invex at y^* gives

$$\frac{1}{r}\big[e^{r\{(\xi^{*L}F^L(x^*)+\xi^{*U}F^U(x^*))-(\xi^{*L}F^L(y^*)+\xi^{*U}F^U(y^*))\}} - 1\big] > 0.$$

Since $r > 0$, using the property of an exponential function, we get

$$\xi^{*L}F^L(x^*) + \xi^{*U}F^U(x^*) > \xi^{*L}F^L(y^*) + \xi^{*U}F^U(y^*),$$

which contradicts (15). This completes the proof. $\qquad\square$

6 Conclusion

In this paper, we have derived sufficient optimality conditions for a class of interval-valued optimization problems under generalized invex functions. Weak, strong and strict converse duality theorems are discussed for two types of the dual models. It will be interesting to obtain the optimality and duality theorem for a class of interval-valued programming under generalized invexity assumptions in which the involved functions are non-smooth. Moreover, it will also be interesting to see whether the second-order duality results for a class of interval-valued programming problem hold or not. This will orient the future research of the authors.

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