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On the strong and △-convergence of SP-iteration on $CAT(0)$ space

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Abstract

In this paper, we study the strong and △-convergence theorems of SP-iteration for nonexpansive mappings on a $CAT(0)$ space. Our results extend and improve many results in the literature.

MSC: 47H09; 47H10

Keywords: $CAT(0)$ space; nonexpansive mapping; strong convergence; △-convergence; SP-iteration; fixed point

1 Introduction

A $CAT(0)$ space plays a fundamental role in various areas of mathematics (see Bridson and Haefliger [1], Burago *et al.* [2], Gromov [3]). Moreover, there are applications in biology and computer science as well [4, 5]. A metric space X is a $CAT(0)$ *space* if it is geodesically connected and if every geodesic triangle in X is at least as 'thin' as its comparison triangle in the Euclidean plane. It is well known that any complete, simply connected Riemannian manifold having non-positive sectional curvature is a $CAT(0)$ space. The complex Hilbert ball with a hyperbolic metric is a $CAT(0)$ space (see [6]). Other examples include pre-Hilbert spaces, R-trees (see [1]) and Euclidean buildings (see [7]).

Fixed point theory in a $CAT(0)$ space has been first studied by Kirk (see [8, 9]). He showed that every nonexpansive mapping defined on a bounded closed convex subset of a complete $CAT(0)$ space always has a fixed point. Since then the fixed point theory in a $CAT(0)$ space has been rapidly developed and a lot of papers have appeared (see, *e.g.*, [8–16]).

The Noor iteration (see [17]) is defined by $x_1 \in K$ and

$$\begin{cases} z_n = (1-\gamma_n)x_n + \gamma_n Tx_n, \\ y_n = (1-\beta_n)x_n + \beta_n Tz_n, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n Ty_n \end{cases} \tag{1.1}$$

for all $n \geq 1$, where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0,1]$. If we take $\beta_n = \gamma_n = 0$ for all n, (1.1) reduces to the Mann iteration (see [18]), and we take $\gamma_n = 0$ for all n, (1.1) reduces to the Ishikawa iteration (see [19]).

The new two-step iteration (see [20]) is defined by $x_1 \in K$ and

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n Tx_n, \\ x_{n+1} = (1-\alpha_n)y_n + \alpha_n Ty_n \end{cases} \tag{1.2}$$

for all $n \geq 1$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$.

Recently, Phuengrattana and Suantai (see [21]) defined the SP-iteration as follows:

$$\begin{cases} z_n = (1-\gamma_n)x_n + \gamma_n Tx_n, \\ y_n = (1-\beta_n)z_n + \beta_n Tz_n, \\ x_{n+1} = (1-\alpha_n)y_n + \alpha_n Ty_n \end{cases} \tag{1.3}$$

for all $n \geq 1$, where $x_1 \in K$, $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0,1]$. They showed that the Mann, Ishikawa, Noor and SP-iterations are equivalent and the SP-iteration converges better than the others for the class of continuous and nondecreasing functions. Clearly, the new two-step and Mann iterations are special cases of the SP-iteration.

Now, we apply SP-iteration (1.3) in a $CAT(0)$ space for nonexpansive mappings as follows:

$$\begin{cases} z_n = (1-\gamma_n)x_n \oplus \gamma_n Tx_n, \\ y_n = (1-\beta_n)z_n \oplus \beta_n Tz_n, \\ x_{n+1} = (1-\alpha_n)y_n \oplus \alpha_n Ty_n \end{cases} \tag{1.4}$$

for all $n \geq 1$, where K is a nonempty convex subset of a $CAT(0)$ space, $x_1 \in K$, $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0,1]$.

In this paper, we study the SP-iteration for a nonexpansive mapping in a $CAT(0)$ space. This paper contains three sections. In Section 2, we first collect some known preliminaries and lemmas that will be used in the proofs of our main theorems. In Section 3, we give the main results which are related to the strong and $\triangle$-convergence theorems of the SP-iteration in a $CAT(0)$ space. It is worth mentioning that our results in a $CAT(0)$ space can be applied to any $CAT(k)$ space with $k \leq 0$ since any $CAT(k)$ space is a $CAT(k')$ space for every $k' \geq k$ (see [1], p.165).

2 Preliminaries and lemmas

Let us recall some definitions and known results in the existing literature on this concept.

Let K be a nonempty subset of a $CAT(0)$ space X and let $T : K \to K$ be a mapping. A point $x \in K$ is called a fixed point of T if $Tx = x$. We will denote the set of fixed points of T by $F(T)$. The mapping T is said to be *nonexpansive* if

$$d(Tx, Ty) \leq d(x,y) \quad \text{for all } x, y \in K.$$

Let (X,d) be a metric space. A *geodesic path* joining $x \in X$ to $y \in X$ (or more briefly, a *geodesic* from x to y) is a map c from a closed interval $[0,l] \subset R$ to X such that $c(0) = x$, $c(l) = y$ and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0,l]$. In particular, c is an isometry and $d(x,y) = l$. The image of c is called a *geodesic* (or *metric*) *segment* joining x and y. When it is unique, this geodesic is denoted by $[x,y]$. The space (X,d) is said to be a *geodesic space* if every two points of X are joined by a geodesic, and X is said to be a *uniquely geodesic space* if there is exactly one geodesic joining x to y for each $x, y \in X$.

A *geodesic triangle* $\triangle(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points in X (the vertices of $\triangle$) and a geodesic segment between each pair of vertices (the edges of $\triangle$). A *comparison triangle* for the geodesic triangle $\triangle(x_1, x_2, x_3)$ in (X, d) is a triangle $\overline{\triangle}(x_1, x_2, x_3) = \triangle(\overline{x}_1, \overline{x}_2, \overline{x}_3)$ in the Euclidean plane $\mathbb{R}^2$ such that $d_{\mathbb{R}^2}(\overline{x}_i, \overline{x}_j) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$.

A geodesic metric space is said to be a $CAT(0)$ space [1] if all geodesic triangles of appropriate size satisfy the following comparison axiom.

$CAT(0)$: Let $\triangle$ be a geodesic triangle in X and let $\overline{\triangle}$ be a comparison triangle for $\triangle$. Then $\triangle$ is said to satisfy the $CAT(0)$ *inequality* if for all $x, y \in \triangle$ and all comparison points $\overline{x}, \overline{y} \in \overline{\triangle}$,

$$d(x,y) \le d_{\mathbb{R}^2}(\overline{x}, \overline{y}).$$

Finally, we observe that if x, y_1, y_2 are points of a $CAT(0)$ space and if y_0 is the midpoint of the segment $[y_1, y_2]$, then the $CAT(0)$ inequality implies

$$d(x,y_0)^2 \le \frac{1}{2} d(x,y_1)^2 + \frac{1}{2} d(x,y_2)^2 - \frac{1}{4} d(y_1,y_2)^2. \tag{2.1}$$

The equality holds for the Euclidean metric. In fact (see [1], p.163), a geodesic metric space is a $CAT(0)$ space if and only if it satisfies inequality (2.1) (which is known as the *CN* inequality of Bruhat and Tits [22]).

The following lemmas can be found in [12].

Lemma 1 ([12], Lemma 2.4) *Let X be a $CAT(0)$ space. Then*

$$d\big((1-t)x \oplus ty, z\big) \le (1-t)\, d(x,z) + t\, d(y,z)$$

for all $t \in [0,1]$ and $x, y, z \in X$.

Lemma 2 ([12], Lemma 2.5) *Let X be a $CAT(0)$ space. Then*

$$d\big((1-t)x \oplus ty, z\big)^2 \le (1-t)\, d(x,z)^2 + t\, d(y,z)^2 - t(1-t)\, d(x,y)^2$$

for all $t \in [0,1]$ and $x, y, z \in X$.

Now, we recall some definitions.

Let X be a complete $CAT(0)$ space and let $\{x_n\}$ be a bounded sequence in X. For $x \in X$, set

$$r\big(x, \{x_n\}\big) = \limsup_{n\to\infty} d(x, x_n).$$

The *asymptotic radius* $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r\big(\{x_n\}\big) = \inf\big\{r\big(x, \{x_n\}\big) : x \in X\big\}.$$

The *asymptotic center* $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}.$$

It is known that in a complete $CAT(0)$ space, $A(\{x_n\})$ consists of exactly one point ([10], Proposition 7). Also, every $CAT(0)$ space has the *Opial* property, *i.e.*, if $\{x_n\}$ is a sequence in K and $\triangle\text{-}\lim_{n\to\infty} x_n = x$, then for each $y(\neq x) \in K$,

$$\limsup_{n\to\infty} d(x_n, x) < \limsup_{n\to\infty} d(x_n, y).$$

Definition 1 ([16], Definition 3.1) A sequence $\{x_n\}$ in a $CAT(0)$ space X is said to be $\triangle$-convergent to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case, we write $\triangle\text{-}\lim_{n\to\infty} x_n = x$ and x is called the $\triangle$-limit of $\{x_n\}$.

The notion of $\triangle$-convergence in a general metric space was introduced by Lim [23]. Recently, Kirk and Panyanak [16] used the concept of $\triangle$-convergence introduced by Lim [23] to prove on the $CAT(0)$ space analogous of some Banach space results which involve weak convergence. Further, Dhompongsa and Panyanak [12] obtained $\triangle$-convergence theorems for the Picard, Mann and Ishikawa iterations in a $CAT(0)$ space.

Lemma 3 ([12], Lemma 2.7)

(i) *Every bounded sequence in a complete* $CAT(0)$ *space always has a* $\triangle$*-convergent subsequence.*

(ii) *Let* K *be a nonempty closed convex subset of a complete* $CAT(0)$ *space and let* $\{x_n\}$ *be a bounded sequence in* K*. Then the asymptotic center of* $\{x_n\}$ *is in* K*.*

(iii) *Let* K *be a nonempty closed convex subset of a complete* $CAT(0)$ *space* X *and let* $f : K \to X$ *be a nonexpansive mapping. Then the conditions,* $\{x_n\}$ $\triangle$*-converges to* x *and* $d(x_n, f(x_n)) \to 0$*, imply* $x \in K$ *and* $f(x) = x$*.*

3 Main results

We start with proving the lemma for later use in this section.

Lemma 4 *Let* K *be a nonempty closed convex subset of a complete* $CAT(0)$ *space* X *and let* $T : K \to K$ *be a nonexpansive mapping with* $F(T) \neq \emptyset$*. Let* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *be sequences in* $[0,1]$, $\{\gamma_n\}$ *be a sequence in* $[\epsilon, 1-\epsilon]$ *for some* $\epsilon \in (0,1)$ *and* $\{x_n\}$ *be defined by the iteration process* (1.4)*. Then*

(i) $\lim_{n\to\infty} d(x_n, x^\star)$ *exists for all* $x^\star \in F(T)$*.*

(ii) $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

Proof (i) Let $x^\star \in F(T)$. By (1.4) and Lemma 1, we have

$$\begin{aligned} d(z_n, x^\star) &= d((1-\gamma_n)x_n \oplus \gamma_n Tx_n, x^\star) \\ &\leq (1-\gamma_n)\, d(x_n, x^\star) + \gamma_n\, d(Tx_n, x^\star) \\ &\leq (1-\gamma_n)\, d(x_n, x^\star) + \gamma_n\, d(x_n, x^\star) \\ &= d(x_n, x^\star). \end{aligned} \tag{3.1}$$

Also, we get

$$\begin{aligned} d(y_n, x^\star) &= d((1-\beta_n)z_n \oplus \beta_n T z_n, x^\star) \\ &\le (1-\beta_n)\, d(z_n, x^\star) + \beta_n\, d(T z_n, x^\star) \\ &\le (1-\beta_n)\, d(z_n, x^\star) + \beta_n\, d(z_n, x^\star) \\ &= d(z_n, x^\star). \end{aligned} \tag{3.2}$$

Then we obtain

$$d(y_n, x^\star) \le d(x_n, x^\star). \tag{3.3}$$

Using (1.4) and Lemma 1, we have

$$\begin{aligned} d(x_{n+1}, x^\star) &= d((1-\alpha_n)y_n \oplus \alpha_n T y_n, x^\star) \\ &\le (1-\alpha_n)\, d(y_n, x^\star) + \alpha_n\, d(T y_n, x^\star) \\ &\le (1-\alpha_n)\, d(y_n, x^\star) + \alpha_n\, d(y_n, x^\star) \\ &= d(y_n, x^\star). \end{aligned} \tag{3.4}$$

Combining (3.3) and (3.4), we get

$$d(x_{n+1}, x^\star) \le d(x_n, x^\star).$$

This implies that the sequence $\{d(x_n, x^\star)\}$ is nonincreasing and bounded below, and so $\lim_{n\to\infty} d(x_n, x^\star)$ exists for all $x^\star \in F(T)$. This completes the proof of part (i).

(ii) Let

$$\lim_{n\to\infty} d(x_n, x^\star) = c. \tag{3.5}$$

Firstly, we will prove that $\lim_{n\to\infty} d(y_n, x^\star) = c$. By (3.4) and (3.5),

$$\liminf_{n\to\infty} d(y_n, x^\star) \ge c.$$

Also, from (3.3) and (3.5),

$$\limsup_{n\to\infty} d(y_n, x^\star) \le c.$$

Then we obtain

$$\lim_{n\to\infty} d(y_n, x^\star) = c. \tag{3.6}$$

Secondly, we will prove that $\lim_{n\to\infty} d(z_n, x^\star) = c$. From (3.1) and (3.2), we have

$$d(y_n, x^\star) \le d(z_n, x^\star) \le d(x_n, x^\star).$$

This gives

$$\lim_{n\to\infty} d(z_n, x^\star) = c. \tag{3.7}$$

Next, by Lemma 2,

$$\begin{aligned} d(z_n, x^\star)^2 &= d((1-\gamma_n)x_n \oplus \gamma_n Tx_n, x^\star)^2 \\ &\le (1-\gamma_n)\, d(x_n, x^\star)^2 + \gamma_n\, d(Tx_n, x^\star)^2 - \gamma_n(1-\gamma_n)d(x_n, Tx_n)^2 \\ &\le (1-\gamma_n)\, d(x_n, x^\star)^2 + \gamma_n\, d(x_n, x^\star)^2 - \gamma_n(1-\gamma_n)d(x_n, Tx_n)^2 \\ &= d(x_n, x^\star)^2 - \gamma_n(1-\gamma_n)d(x_n, Tx_n)^2. \end{aligned}$$

Thus,

$$\gamma_n(1-\gamma_n)d(x_n, Tx_n)^2 \le d(x_n, x^\star)^2 - d(z_n, x^\star)^2,$$

so that

$$\begin{aligned} d(x_n, Tx_n)^2 &\le \frac{1}{\gamma_n(1-\gamma_n)}\big[d(x_n, x^\star)^2 - d(z_n, x^\star)^2\big] \\ &\le \frac{1}{\epsilon^2}\big[d(x_n, x^\star)^2 - d(z_n, x^\star)^2\big]. \end{aligned}$$

Now using (3.5) and (3.7), $\limsup_{n\to\infty} d(x_n, Tx_n) \le 0$ and hence,

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0.$$

This completes the proof of part (ii). □

Now, we give the $\triangle$-convergence theorem of the SP-iteration on a $CAT(0)$ space.

Theorem 1 *Let X, K, T, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{x_n\}$ satisfy the hypotheses of Lemma 4. Then the sequence $\{x_n\}$ $\triangle$-converges to a fixed point of T.*

Proof By Lemma 4, we have $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Also, $\lim_{n\to\infty} d(x_n, x^\star)$ exists for all $x^\star \in F(T)$. Thus $\{x_n\}$ is bounded. Let $W_\triangle(x_n) = \cup A(\{u_n\})$, where the union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$. We claim that $W_\triangle(x_n) \subseteq F(T)$. Let $u \in W_\triangle(x_n)$. Then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. By Lemma 3(i) and (ii), there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\triangle\text{-}\lim_{n\to\infty} v_n = v \in K$. By Lemma 3(iii), $v \in F(T)$. By Lemma 4(i), $\lim_{n\to\infty} d(x_n, v)$ exists. Now, we claim that $u = v$. On the contrary, assume that $u \ne v$. Then, by the uniqueness of asymptotic centers, we have

$$\begin{aligned} \limsup_{n\to\infty} d(v_n, v) &< \limsup_{n\to\infty} d(v_n, u) \\ &\le \limsup_{n\to\infty} d(u_n, u) \\ &< \limsup_{n\to\infty} d(u_n, v) \end{aligned}$$

$$\begin{aligned}
&= \limsup_{n\to\infty} d(x_n, v) \\
&= \limsup_{n\to\infty} d(v_n, v).
\end{aligned} \tag{3.8}$$

This is a contradiction. Thus $u = v \in F(T)$ and $W_\triangle(x_n) \subseteq F(T)$. To show that the sequence $\{x_n\}$ $\triangle$-converges to a fixed point of T, we will show that $W_\triangle(x_n)$ consists of exactly one point. Let $\{u_n\}$ be a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and let $A(\{x_n\}) = \{x\}$. We have already seen that $u = v$ and $v \in F(T)$. Finally, we claim that $x = v$. If not, then the existence of $\lim_{n\to\infty} d(x_n, v)$ and the uniqueness of asymptotic centers imply that there exists a contradiction as (3.8) and hence $x = v \in F(T)$. Therefore, $W_\triangle(x_n) = \{x\}$. As a result, the sequence $\{x_n\}$ $\triangle$-converges to a fixed point of T. □

We give the strong convergence theorem on a $CAT(0)$ space as follows.

Theorem 2 *Let X, K, T, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{x_n\}$ satisfy the hypotheses of Lemma* 4. *Then the sequence $\{x_n\}$ converges strongly to a fixed point of T if and only if*

$$\liminf_{n\to\infty} d\big(x_n, F(T)\big) = 0,$$

where $d(x, F(T)) = \inf\{d(x,p) : p \in F(T)\}$.

Proof Necessity is obvious. Conversely, suppose that $\liminf_{n\to\infty} d(x_n, F(T)) = 0$. As proved in Lemma 4(i),

$$d\big(x_{n+1}, x^\star\big) \le d\big(x_n, x^\star\big)$$

for all $x^\star \in F(T)$. This implies that

$$d\big(x_{n+1}, F(T)\big) \le d\big(x_n, F(T)\big).$$

Since the sequence $\{d(x_n, F(T))\}$ is nonincreasing and bounded below, $\lim_{n\to\infty} d(x_n, F(T))$ exists. Thus, by the hypothesis, $\lim_{n\to\infty} d(x_n, F(T)) = 0$.

Next, we will show that $\{x_n\}$ is a Cauchy sequence in K. Let $\varepsilon > 0$ be arbitrarily chosen. Since $\lim_{n\to\infty} d(x_n, F(T)) = 0$, there exists a constant n_0 such that for all $n \ge n_0$, we have

$$d\big(x_n, F(T)\big) < \frac{\varepsilon}{4}.$$

In particular, $\inf\{d(x_{n_0}, p) : p \in F(T)\} < \frac{\varepsilon}{4}$. Thus there exists $p^\star \in F(T)$ such that

$$d\big(x_{n_0}, p^\star\big) < \frac{\varepsilon}{2}.$$

Now, for all $m, n \ge n_0$, we have

$$\begin{aligned}
d(x_{n+m}, x_n) &\le d\big(x_{n+m}, p^\star\big) + d\big(x_n, p^\star\big) \\
&\le 2d\big(x_{n_0}, p^\star\big) \\
&< 2\left(\frac{\varepsilon}{2}\right) = \varepsilon.
\end{aligned}$$

Hence $\{x_n\}$ is a Cauchy sequence in a closed subset K of a complete $CAT(0)$ space X, it must be convergent to a point in K. Let $\lim_{n\to\infty} x_n = x^\star \in K$. Now, $\lim_{n\to\infty} d(x_n, F(T)) = 0$ gives that $d(x^\star, F(T)) = 0$ and the closedness of $F(T)$ forces $x^\star$ to be in $F(T)$. Therefore, the sequence $\{x_n\}$ converges strongly to a fixed point $x^\star$ of T. □

Senter and Dotson [24] introduced *Condition* (I) as follows.

Definition 2 ([24], p.375) A mapping $T : K \to K$ is said to satisfy *Condition* (I) if there exists a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r > 0$ such that

$$d(x, Tx) \geq f\big(d\big(x, F(T)\big)\big) \quad \text{for all } x \in K.$$

With respect to the above definition, we have the following theorem.

Theorem 3 *Let X, K, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{x_n\}$ satisfy the hypotheses of Lemma 4 and let $T : K \to K$ be a nonexpansive mapping satisfying Condition (I). Then the sequence $\{x_n\}$ converges strongly to a fixed point of T.*

Proof By Lemma 4(i), $\lim_{n\to\infty} d(x_n, x^\star)$ exists for all $x^\star \in F(T)$. Let this limit be c, where $c \geq 0$. If $c = 0$, there is nothing to prove. Suppose that $c > 0$. Now,

$$d\big(x_{n+1}, x^\star\big) \leq d\big(x_n, x^\star\big)$$

gives

$$\inf_{x^\star\in F(T)} d\big(x_{n+1}, x^\star\big) \leq \inf_{x^\star\in F(T)} d\big(x_n, x^\star\big),$$

which means that $d(x_{n+1}, F(T)) \leq d(x_n, F(T))$ and so $\lim_{n\to\infty} d(x_n, F(T))$ exists. Also, by Lemma 4(ii), we have $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. It follows from *Condition* (I) that

$$\lim_{n\to\infty} f\big(d\big(x_n, F(T)\big)\big) \leq \lim_{n\to\infty} d(x_n, Tx_n) = 0.$$

That is,

$$\lim_{n\to\infty} f\big(d\big(x_n, F(T)\big)\big) = 0.$$

Since $f : [0,\infty) \to [0,\infty)$ is a nondecreasing function satisfying $f(0) = 0$, $f(t) > 0$ for all $t \in (0,\infty)$, therefore we obtain

$$\lim_{n\to\infty} d\big(x_n, F(T)\big) = 0.$$

The conclusion now follows from Theorem 2. □

It is worth noting that, in the case of a nonexpansive mapping, *Condition* (I) is weaker than the compactness of K.

Since the SP-iteration reduces to the new two-step iteration when $\alpha_n = 0$ for all $n \in \mathbb{N}$ and to the Mann iteration when $\alpha_n = \beta_n = 0$ for all $n \in \mathbb{N}$, we have the following corollaries.

Corollary 1 *Let X, K, T, $\{\gamma_n\}$ satisfy the hypotheses of Lemma* 4 *and let $\{x_n\}$ be defined by the iteration process* (1.2). *Then the sequence $\{x_n\}$ $\triangle$-converges to a fixed point of T. Further, if $\{x_n\}$ is defined by the iteration process* (1.1), *the sequence $\{x_n\}$ $\triangle$-converges to a fixed point of T.*

Corollary 2 *Let X, K, $\{\gamma_n\}$ satisfy the hypotheses of Lemma* 4, *let $T : K \to K$ be a nonexpansive mapping satisfying Condition* (I) *and let $\{x_n\}$ be defined by the iteration process* (1.2). *Then the sequence $\{x_n\}$ converges strongly to a fixed point of T. Also, if $\{x_n\}$ is defined by the iteration process* (1.1), *the sequence $\{x_n\}$ converges strongly to a fixed point of T.*

Conclusions

The SP-iteration reduces to the new two-step and Mann iterations. Then these results presented in this paper extend and generalize some works for $CAT(0)$ space in the literature.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors contributed equally and significantly in writing this article. Both authors read and approved the final manuscript.

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