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Weighted estimates for vector-valued multilinear operators with non-smooth kernels

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Abstract

Let T be the multilinear Calderón-Zygmund operator with non-smooth kernel and let T^* be its corresponding maximal operator. In this paper, vector-valued weighted norm inequalities for T and T^* are established. As applications, weighted strong type estimates for vector-valued commutators associated with T and T^* are deduced respectively.

1 Introduction and main results

Let T be a multilinear operator initially defined on the m-fold product of Schwartz spaces and taking values in the space of tempered distributions,

$$T:\mathcal{S}(\mathbb{R}^n)\times\cdots\times\mathcal{S}(\mathbb{R}^n)\to\mathcal{S}'(\mathbb{R}^n).$$

Following [1], we say that T is a multilinear Calderón-Zygmund operator if, for some $1\le q_j<\infty$, it extends to a bounded multilinear operator from $L^{q_1}\times\cdots\times L^{q_m}$ to L^q, where $\frac{1}{q}=\frac{1}{q_1}+\cdots+\frac{1}{q_m}$, and if there exists a function K, defined off the diagonal $x=y_1=\cdots=y_m$ in $(\mathbb{R}^n)^{m+1}$, satisfying

$$T(\vec{f})(x)=T(f_1,\ldots,f_m)(x)=\int_{(\mathbb{R}^n)^m}K(x,y_1,\ldots,y_m)f_1(y_1)\cdots f_m(y_m)\,dy_1\cdots dy_m, \tag{1.1}$$

for all $x\notin\bigcap_{j=1}^m \operatorname{supp} f_j$;

$$\bigl|K(y_0,y_1,\ldots,y_m)\bigr|\le\frac{A}{(\sum_{k,l=0}^m|y_k-y_l|)^{mn}}; \tag{1.2}$$

and

$$\bigl|K(y_0,y_1,\ldots,y_j,\ldots,y_m)-K\bigl(y_0,y_1,\ldots,y_j',\ldots,y_m\bigr)\bigr|\le\frac{A|y_j-y_j'|^\epsilon}{(\sum_{k,l=0}^m|y_k-y_l|)^{mn+\epsilon}} \tag{1.3}$$

for some $\epsilon>0$ and all $0\le j\le m$, whenever $2|y_j-y_j'|\le\max_{0\le k\le m}|y_j-y_k|$. Such kernels are called m-linear Calderón-Zygmund kernels, and the collection of such functions is denoted by m-$CZK(A,\epsilon)$ in [1]. As in [2], we define the maximal multilinear operator by

$$T^*(\vec{f})(x) = \sup_{\delta>0}\bigl|T_\delta(f_1,\ldots,f_m)(x)\bigr|,$$

where T_δ is the smooth truncation of T given by

$$T_\delta(\vec{f})(x) = \int_{|x-y_1|^2+\cdots+|x-y_m|^2>\delta^2} K(x,y_1,\ldots,y_m)f_1(y_1)\cdots f_m(y_m)\,dy_1\cdots dy_m.$$

The vector-valued multilinear Calderón-Zygmund operator T_q and vector-valued maximal multilinear operator T^*_q associated with T are defined and studied in [3, 4].

$$\begin{aligned} T_q(\vec{f})(x) &= \bigl|T(f_1,\ldots,f_m)(x)\bigr|_q = \bigl\|T(f_{1\cdot},\ldots,f_{m\cdot})(x)\bigr\|_{l^q} \\ &= \Biggl(\sum_{k=1}^{\infty}\bigl|T(f_{1k},\ldots,f_{mk})(x)\bigr|^q\Biggr)^{1/q}, \\ T^*_q(\vec{f})(x) &= \bigl|T^*(f_1,\ldots,f_m)(x)\bigr|_q = \bigl\|T^*(f_{1\cdot},\ldots,f_{m\cdot})(x)\bigr\|_{l^q} \\ &= \Biggl(\sum_{k=1}^{\infty}\bigl|T^*(f_{1k},\ldots,f_{mk})(x)\bigr|^q\Biggr)^{1/q}, \end{aligned}$$

where $f_i = \{f_{ik}\}_{k=1}^\infty$ for $i = 1,\ldots,m$.

Theorem A [3] *Assume that T is a multilinear Calderón-Zygmund operator. Let $1 < p_1,\ldots,p_m < \infty$, $1 < q_1,\ldots,q_m < \infty$ and $1/m < p,q < \infty$ such that $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, $\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m}$. If $(\omega_1^{p_1},\ldots,\omega_m^{p_m}) \in (A_{p_1},\ldots,A_{p_m})$, then there exists a constant $C > 0$ such that*

$$\bigl\|T_q(\vec{f})\bigr\|_{L^p(\omega_1^p\cdots\omega_m^p)} \le C\prod_{j=1}^m\bigl\||f_j|_{q_j}\bigr\|_{L^{p_j}(\omega_j^{p_j})}.$$

Theorem B [4] *Assume that T is a multilinear Calderón-Zygmund operator. Let $1 \le p_1,\ldots,p_m < \infty$, $1 < q_1,\ldots,q_m < \infty$ and $0 < p,q < \infty$ such that $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, $\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m}$.*

(i) *If $1 < p_1,\ldots,p_m < \infty$ and $\omega \in A_{p_1} \cap \cdots \cap A_{p_m}$, then there exists a constant $C > 0$ such that*

$$\bigl\|T_q(\vec{f})\bigr\|_{L^p(\omega)} \le C\prod_{j=1}^m\bigl\||f_j|_{q_j}\bigr\|_{L^{p_j}(\omega)}.$$

(ii) *If at least one $p_j = 1$ and $\omega \in A_1$, then there exists a constant $C > 0$ such that*

$$\bigl\|T_q(\vec{f})\bigr\|_{L^{p,\infty}(\omega)} \le C\prod_{j=1}^m\bigl\||f_j|_{q_j}\bigr\|_{L^{p_j}(\omega)}.$$

It is worth noting that similar results hold for T^* in Theorems A and B.

We will replace (1.3) by a weaker regularity condition on the kernel K. Assume that operators A_t are associated with kernels $a_t(x,y)$ in the sense that

$$A_t f(x) = \int_{\mathbb{R}^n} a_t(x,y) f(y)\, dy$$

for every function $f \in L^p(\mathbb{R}^n)$, $1 \le p \le \infty$, and $a_t(x,y)$ satisfy the following size condition:

$$\left|a_t(x,y)\right| \le h_t(x,y) = t^{-n/s} h\left(\frac{|x-y|^s}{t}\right), \tag{1.4}$$

where s is a positive fixed constant and h is a positive, bounded, decreasing function satisfying that for some $\eta > 0$,

$$\lim_{r\to\infty} r^{n+\eta} h\left(r^s\right) = 0. \tag{1.5}$$

The jth transpose $T^{*,j}$ of T is defined via

$$\left\langle T^{*,j}(f_1,\ldots,f_m), g\right\rangle = \left\langle T(f_1,\ldots,f_{j-1},g,f_{j+1},\ldots,f_m), f_j\right\rangle$$

for all $f_1,\ldots,f_m$, g in $\mathcal{S}(\mathbb{R}^n)$. It is easy to check that the kernel $K^{*,j}$ of $T^{*,j}$ is related to the kernel K of T via the identity

$$K^{*,j}(x,y_1,\ldots,y_{j-1},y_j,y_{j+1},\ldots,y_m) = K(y_j,y_1,\ldots,y_{j-1},x,y_{j+1},\ldots,y_m).$$

To maintain uniform notation, we may occasionally denote $T = T^{*,0}$ and $K = K^{*,0}$.

Assumption (H0) We always assume that there exist some $1 \le q_1,\ldots,q_m < \infty$ and some $0 < q < \infty$ with $\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m}$ such that both T^* and T map $L^{q_1} \times \cdots \times L^{q_m}$ to $L^{q,\infty}$.

Assumption (H1) Assume that there exist operators $\{A_t^{(i)}\}_{t>0}$ with kernels $a_t^{(i)}(x,y)$ that satisfy the conditions (1.4) and (1.5) with constants s and η for each $i = 1,\ldots,m$ and that for every $j = 0,1,2,\ldots,m$, there exists kernel $K_t^{*,j,(i)}(x,y_1,\ldots,y_m)$ such that

$$\begin{aligned}&\left\langle T^{*,j}\left(f_1,\ldots,A_t^{(i)} f_i,\ldots,f_m\right), g\right\rangle \\ &\quad = \int_{\mathbb{R}^n}\int_{(\mathbb{R}^n)^m} K_t^{*,j,(i)}(x,y_1,\ldots,y_m) f_1(y_1)\cdots f_m(y_m) g(x)\, dy_1\cdots dy_m\, dx,\end{aligned}$$

for all $f_1,\ldots,f_m$ in $\mathcal{S}(\mathbb{R}^n)$ with $\bigcap_{k=1}^m \operatorname{supp} f_k \cap \operatorname{supp} g = \phi$. Also assume that there exist a non-negative function $\phi \in C(\mathbb{R})$ with $\operatorname{supp}\phi \in [-1,1]$ and a constant $\epsilon > 0$ so that for every $j \in \{0,1,\ldots,m\}$ and every $i \in \{1,2,\ldots,m\}$, all $t > 0$ and all $x, y_1,\ldots,y_n \in \mathbb{R}^n$, we have

$$\begin{aligned}&\left|K^{*,j}(x,y_1,\ldots,y_m) - K_t^{*,j,(i)}(x,y_1,\ldots,y_m)\right| \\ &\quad \le \frac{A}{(|x-y_1| + \cdots + |x-y_m|)^{mn}} \sum_{k=1,k\neq i}^m \phi\left(\frac{|y_i - y_k|}{t^{1/s}}\right) \\ &\qquad + \frac{A t^{\epsilon/s}}{(|x-y_1| + \cdots + |x-y_m|)^{mn+\epsilon}},\end{aligned}$$

whenever $2t^{1/s} \le |x - y_i|$.

Kernels K that satisfy the size estimate (1.2) and Assumption (H1) with parameters m, A, s, η, ε are called generalized Calderón-Zygmund kernels, and their collection is denoted by m-$GCZK(A,s,\eta,\varepsilon)$. We say that T is of class m-$GCZO(A,s,\eta,\varepsilon)$ if T has an associated kernel K in m-$GCZK(A,s,\eta,\varepsilon)$.

Assumption (H2) Assume that there exist operators $\{A_t\}_{t>0}$ with kernels $a_t(x,y)$ that satisfy conditions (1.4) and (1.5) with constants s and η, and there exist kernels $K_t^{(0)}(x,y_1,\dots,y_m)$ such that the representation is valid

$$K_t^{(0)}(x,y_1,\dots,y_m) = \int_{\mathbb{R}^n} K(z,y_1,\dots,y_m)a_t(x,z)\,dz$$

and that there exist a non-negative function $\phi \in \mathcal{C}(\mathbb{R})$ and $\operatorname{supp}\phi \subset [-1,1]$ and a positive constant ε such that

$$\begin{aligned}&\bigl|K(x,y_1,\dots,y_m) - K_t^{(0)}(x,y_1,\dots,y_m)\bigr| \\ &\quad\le \frac{A}{(\sum_{k=1}^m |x-y_k|)^{mn}} \sum_{\substack{k=1\\k\ne j}}^m \phi\left(\frac{|x-y_k|}{t^{1/s}}\right) + \frac{At^{\varepsilon/s}}{(\sum_{k=1}^m |x-y_k|)^{mn+\varepsilon}}\end{aligned} \tag{1.6}$$

for some $A > 0$, whenever $2t^{1/s} \le \max_{1\le j\le m}|x-y_j|$. Moreover, assume that for all $x, y_1,\dots,y_m \in \mathbb{R}^n$,

$$\bigl|K_t^{(0)}(x,y_1,\dots,y_m)\bigr| \le \frac{A}{(\sum_{k=1}^m |x-y_k|)^{mn}},$$

whenever $2t^{1/s} \le \min_{1\le j\le m}|x-y_j|$, and for all $x, x', y_1,\dots,y_m \in \mathbb{R}^n$,

$$\bigl|K(x,y_1,\dots,y_m) - K_t^{(0)}(x',y_1,\dots,y_m)\bigr| \le \frac{At^{\varepsilon/s}}{(\sum_{k=1}^m |x-y_k|)^{mn+\varepsilon}},$$

whenever $2t^{1/s} \le \min_{1\le j\le m}|x-y_j|$ and $2|x-x'| \le t^{1/s}$.

The commutators associated with T and T^* are defined respectively by

$$\begin{aligned}T_{\Pi\vec{b}}(\vec{f})(x) &= \bigl[b_1,\bigl[b_2,\dots\bigl[b_{l-1},[b_l,T]_l\bigr]_{l-1}\cdots\bigr]_2\bigr]_1(\vec{f})(x) \\ &= \int_{(\mathbb{R}^n)^m} \prod_{j=1}^l \bigl(b_j(x)-b_j(y_j)\bigr)K(x,y_1,\dots,y_m)\prod_{i=1}^m f_i(y_i)\,d\vec{y},\end{aligned}$$

and

$$\begin{aligned}T^*_{\Pi\vec{b}}(\vec{f})(x) &= \sup_{\delta>0}\bigl|\bigl[b_1,\bigl[b_2,\dots\bigl[b_{l-1},[b_l,T_\delta]_l\bigr]_{l-1}\cdots\bigr]_2\bigr]_1(\vec{f})(x)\bigr| \\ &= \sup_{\delta>0}\left|\int_{|x-y_1|^2+\cdots+|x-y_m|^2>\delta^2} \prod_{j=1}^l \bigl(b_j(x)-b_j(y_j)\bigr)K(x,y_1,\dots,y_m)\prod_{i=1}^m f_i(y_i)\,d\vec{y}\right|.\end{aligned}$$

Here and subsequently, we often write $\vec{y} = (y_1,\dots,y_m)$ and $d\vec{y} = dy_1\cdots dy_m$.

For simplicity of notation, we often write $\vec{f} = (f_1,\dots,f_m)$ with $f_j = \{f_{jk}\}_{k=1}^\infty$. For the sequence $\{\vec{f}_k\}_{k=1}^\infty = \{f_{1k},\dots,f_{mk}\}_{k=1}^\infty$ of vector functions, the commutators associated with vector-valued T_q and T_q^* can be defined by

$$T_{\Pi\vec{b},q}(\vec{f})(x) = \left|T_{\Pi\vec{b}}(\vec{f})(x)\right|_q = \left\|T_{\Pi\vec{b}}(f_{1\cdot},\ldots,f_{m\cdot})(x)\right\|_{l^q} = \left(\sum_{k=1}^{\infty}\left|T_{\Pi\vec{b}}(\vec{f}_k)(x)\right|^q\right)^{1/q},$$

$$T^*_{\Pi\vec{b},q}(\vec{f})(x) = \left|T^*_{\Pi\vec{b}}(\vec{f})(x)\right|_q = \left\|T^*_{\Pi\vec{b}}(f_{1\cdot},\ldots,f_{m\cdot})(x)\right\|_{l^q} = \left(\sum_{k=1}^{\infty}\left|T^*_{\Pi\vec{b}}(\vec{f}_k)(x)\right|^q\right)^{1/q}.$$

From now on, we always assume that T is a multilinear operator in m-$GCZO(A,s,\eta,\varepsilon)$ and its kernel satisfies Assumption (H2). Recently, if $l = m$, Peng *et al.* [5] obtained the following weighted strong type estimates for $T_{\Pi\vec{b}}$ and $T^*_{\Pi\vec{b}}$ with multiple weights (see Definition 2.1).

Theorem C [5] *Let $\vec{b} \in BMO^m$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$ with $1 < p_j < \infty$, $j = 1,\ldots,m$. Then we have*

(i) *There exists a constant C such that*

$$\left\|T^*_{\Pi\vec{b}}(\vec{f})\right\|_{L^p(\nu_{\vec{\omega}})} \le C\prod_{i=1}^{m}\|b_i\|_{BMO}\prod_{i=1}^{m}\|f_i\|_{L^{p_i}(M\omega_i)}.$$

(ii) *If each $\omega_i \in A_{p_j}$, then there exists a constant C such that*

$$\left\|T^*_{\Pi\vec{b}}(\vec{f})\right\|_{L^p(\nu_{\vec{\omega}})} \le C\prod_{i=1}^{m}\|b_i\|_{BMO}\prod_{i=1}^{m}\|f_i\|_{L^{p_i}(\omega_i)},$$

where $\nu_{\vec{\omega}} = \prod_{i=1}^{m}\omega_i^{p/p_j}$. Similar results still hold for $T_{\Pi\vec{b}}$, which extend the results in [6] *significantly.*

In this work, we first pursue results parallel to Theorems A and B, then extend Theorem C to a vector-valued version. The main results can be stated as follows.

Theorem 1.1 *Let $1 < p_1,\ldots,p_m < \infty$, $1 < q_1,\ldots,q_m < \infty$ and $1/m < p,q < \infty$ such that $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, $\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m}$. If $(\omega_1^{p_1},\ldots,\omega_m^{p_m}) \in (A_{p_1},\ldots,A_{p_m})$, then there exists a constant $C > 0$ such that*

$$\left\|T_q(\vec{f})\right\|_{L^p(\omega_1^p\cdots\omega_m^p)} \le C\prod_{j=1}^{m}\left\||f_j|_{q_j}\right\|_{L^{p_j}(\omega_j^{p_j})}.$$

Moreover, similar estimates hold for T^*.

Theorem 1.2 *Let $1 \le p_1,\ldots,p_m < \infty$, $1 < q_1,\ldots,q_m < \infty$ and $0 < p,q < \infty$ such that $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, $\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m}$.*

(i) *If $1 < p_1,\ldots,p_m < \infty$ and $\omega \in A_{p_1}\cap\cdots\cap A_{p_m}$, then there exists a constant $C > 0$ such that*

$$\left\|T_q(\vec{f})\right\|_{L^p(\omega)} \le C\prod_{j=1}^{m}\left\||f_j|_{q_j}\right\|_{L^{p_j}(\omega)}.$$

(ii) *If at least one* $p_j = 1$ *and* $\omega \in A_1$, *then there exists a constant* $C > 0$ *such that*

$$\big\| T_q(\vec{f})\big\|_{L^{p,\infty}(\omega)} \leq C \prod_{j=1}^{m} \big\| |f_j|_{q_j}\big\|_{L^{p_j}(\omega)}.$$

Moreover, similar estimates hold for T^*.

Theorem 1.3 *Let* $1/m < p < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$ *with* $1 < p_1, \dots, p_m < \infty$, $1/m < q < \infty$, *and* $\frac{1}{q_1} + \cdots + \frac{1}{q_m} = \frac{1}{q}$ *with* $1 < q_1, \dots, q_m < \infty$. *Suppose that* $\vec{\omega} \in A_{\vec{p}}$, $\nu_{\vec{\omega}} = \prod_{i=1}^{m} \omega_i^{\frac{p}{p_i}}$ *and* $\vec{b} \in (BMO)^l$. *Then we have*

(i) *There then exists a constant* $C > 0$ *such that*

$$\big\| T_{\Pi\vec{b},q}(\vec{f})\big\|_{L^p(\nu_{\vec{\omega}})} \leq \prod_{j=1}^{l} \|b_j\|_{BMO} \prod_{j=1}^{m} \big\| |f_j|_{q_j}\big\|_{L^{p_j}(Mw_j)}.$$

(ii) *If* $\omega_j \in A_{p_j}$, *then there exists a constant* $C > 0$ *such that*

$$\big\| T_{\Pi\vec{b},q}(\vec{f})\big\|_{L^p(\nu_{\vec{\omega}})} \leq \prod_{j=1}^{l} \|b_j\|_{BMO} \prod_{j=1}^{m} \big\| |f_j|_{q_j}\big\|_{L^{p_j}(w_j)}.$$

Moreover, similar estimates hold for T^*.

Remark 1.4 If $l = 1$ and $l = m$, Theorem 1.3 can be seen as the vector-valued extension of Theorem 4.5 in [6] and Theorem C, respectively.

2 Proofs of Theorem 1.1 and Theorem 1.2

Let us begin with the definition of Hardy-Littlewood maximal operator, that is

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q \big|f(y)\big|\, dy.$$

The sharp maximal function is defined by

$$M^{\sharp}f(x) = \sup_{Q \ni x} \inf_{c} \frac{1}{|Q|} \int_Q \big|f(y) - c\big|\, dy \approx \sup_{Q \ni x} \frac{1}{|Q|} \int_Q \big|f(y) - f_Q\big|\, dy.$$

For $\delta > 0$, we also need the maximal function $M_\delta f = M(|f|^\delta)^{\frac{1}{\delta}}$ and $M_\delta^{\sharp} f = M^{\sharp}(|f|^\delta)^{\frac{1}{\delta}}$. The new maximal function $\mathcal{M}$ can be defined by

$$\mathcal{M}(\vec{f})(x) = \sup_{Q \ni x} \prod_{j=1}^{m} \frac{1}{|Q|} \int_Q \big|f_j(y_j)\big|\, dy_j.$$

For $1 \leq l \leq m$, as in [7], a modified maximal function $\mathcal{M}_l$ is given by

$$\mathcal{M}_l(\vec{f})(x) = \sup_{Q \ni x} \sum_{k=0}^{\infty} 2^{-knl} \left(\prod_{j=1}^{l} \frac{1}{|Q|} \int_Q \big|f_j(y_j)\big|\, dy_j \right) \left(\prod_{j=l+1}^{m} \frac{1}{|2^k Q|} \int_{2^k Q} \big|f_j(y_j)\big|\, dy_j \right).$$

For exponents $p_1,\dots,p_m$, we will often write p for the number given by $1/p = 1/p_1 + \cdots + 1/p_m$, and $\vec{p}$ for the vector $\vec{p} = (p_1,\dots,p_m)$. Let us recall the definition of $A_{\vec{p}}$ weights.

Definition 2.1 Let $1 \le p_1,\dots,p_m < \infty$. Given $\vec{\omega} = (\omega_1,\dots,\omega_m)$, set $\nu_{\vec{\omega}} = \prod_{i=1}^m \omega_i^{p/p_i}$. We say that $\vec{\omega}$ satisfies the $A_{\vec{p}}$ condition if

$$\sup_Q \left(\frac{1}{|Q|}\int_Q \prod_{i=1}^m \omega_i^{\frac{p}{p_i}}\right)^{\frac{1}{p}} \prod_{i=1}^m \left(\frac{1}{|Q|}\int_Q \omega_i^{1-p_i'}\right)^{\frac{1}{p_i'}} < \infty,$$

when $p_i = 1$, $(\frac{1}{|Q|}\int_Q \omega_i^{1-p_i'})^{\frac{1}{p_i'}}$ is understood as $(\inf_Q \omega_i)^{-1}$.

We will use the following lemmas in the proof of Theorem 1.1 and Theorem 1.2.

Lemma 2.2 [3] *Let $\mathcal{T}$ be an m-linear operator, and let $1 < q_1,\dots,q_m < \infty$ and $\frac{1}{m} < q < \infty$ be fixed indices such that $\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m}$. For $(\omega_1^{q_1},\dots,\omega_m^{q_m}) \in (A_{q_1},\dots,A_{q_m})$, the following estimate holds*:

$$\|\mathcal{T}(\vec{f})\|_{L^q(\omega_1^q\cdots\omega_m^q)} \le C\prod_{j=1}^m \|f_j\|_{L^{q_j}(\omega_j^{q_j})}.$$

Then, for all indices, $1 < p_1,\dots,p_m < \infty$ and $\frac{1}{m} < p < \infty$ satisfy $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, $1 < s_1,\dots,s_m < \infty$ and $\frac{1}{m} < s < \infty$ such that $\frac{1}{s} = \frac{1}{s_1} + \cdots + \frac{1}{s_m}$, and all $(\omega_1^{p_1},\dots,\omega_m^{p_m}) \in (A_{p_1},\dots,A_{p_m})$. Then the following inequality holds:

$$\left\|\left(\sum_k |\mathcal{T}(f_{1k},\dots,f_{mk})|^s\right)^{\frac{1}{s}}\right\|_{L^p(\omega_1^p\cdots\omega_m^p)} \le C\prod_{j=1}^m \left\|\left(\sum_k |f_{jk}|^{s_j}\right)^{\frac{1}{s_j}}\right\|_{L^{p_j}(\omega_j^{p_j})}.$$

Lemma 2.3 [7] *Suppose that for some $1 \le q_1, q_2,\dots,q_{m-1} \le \infty$, $q_m \in (1,\infty)$ and $q \in (0,\infty)$ satisfying $\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m}$, T maps $L^{q_1} \times \cdots \times L^{q_m}$ to L^q. Let $1 \le p_1,\dots,p_m < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, $\vec{\omega} = (\omega_1,\dots,\omega_m) \in A_{\vec{p}}$ and $\vec{p} = (p_1,\dots,p_m)$. Then*

(i) *T^* can be extended to a bounded operator from $L^{p_1}(\omega_1) \times \cdots \times L^{p_m}(\omega_m)$ to $L^p(\nu_{\vec{\omega}})$ if all the exponents p_j are strictly greater than* 1.

(ii) *T^* can be extended to a bounded operator from $L^{p_1}(\omega_1) \times \cdots \times L^{p_m}(\omega_m)$ to $L^{p,\infty}(\nu_{\vec{\omega}})$ if some exponents p_j equal* 1.

Similar results hold for T.

Note that if each $\omega_j \in A_{p_j}$, then $\prod_{j=1}^m A_{p_j} \subset A_{\vec{p}}$ and this inclusion is strict (see [8] for details). This fact together with Lemma 2.3 yields the following weighted estimates.

Lemma 2.4 *Consider an m-tuple $(\omega_1^{p_1},\dots,\omega_m^{p_m}) \in (A_{p_1},\dots,A_{p_m})$, where $1 < p_1,\dots,p_m < \infty$ and $\frac{1}{m} < p < \infty$ satisfy $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$. Then there exists a constant C such that*

$$\|T(\vec{f})\|_{L^p(\omega_1^p\cdots\omega_m^p)} \le C\prod_{j=1}^m \|f_j\|_{L^{p_j}(\omega_j^{p_j})},$$

and

$$\left\| T^*(\vec{f}) \right\|_{L^p(\omega_1^p \cdots \omega_m^p)} \le C \prod_{j=1}^m \|f_j\|_{L^{p_j}(\omega_j^{p_j})}.$$

Proof of Theorem 1.1 *and Theorem* 1.2 As a consequence of Lemma 2.2 and Lemma 2.4, we obtain Theorem 1.1 (see the proof of Corollary 3 in [3]). From [7] we know that for all $1 \le l \le m$, $\vec{f} = (f_1, \ldots, f_m)$ and $x \in \mathbb{R}^n$, the following two inequalities hold:

$$\mathcal{M}(\vec{f})(x) \le \mathcal{M}_l(\vec{f})(x) \le 2 \prod_{j=1}^m M(f_j)(x),$$

$$\|T\vec{f}\|_{L^p(\nu_{\vec{\omega}})} \le C \left\| \sum_{l=1}^m \mathcal{M}_l(\vec{f}) \right\|_{L^p(\nu_{\vec{\omega}})}.$$

So, we get $\|T\vec{f}\|_{L^p(\nu_{\vec{\omega}})} \le C \| \prod_{j=1}^m M(f_j)\|_{L^p(\nu_{\vec{\omega}})}$. A similar inequality still holds for T^*. Theorem 1.2 follows by repeating the same steps as in Corollary 3.3 in [4]. In fact, we apply Theorem 2.1 in [4] to the families

$$\mathcal{F}\left(T(f_1, \ldots, f_m), \prod_{j=1}^m Mf_j \right), \qquad \mathcal{F}\left(T^*(f_1, \ldots, f_m), \prod_{j=1}^m Mf_j \right).$$

Hölder's inequality and the normal inequalities for the maximal operator yield the desired results. □

3 Proof of Theorem 1.3

We begin with some lemmas which will be used in the proof of Theorem 1.3.

Lemma 3.1 [9] *Let* $0 < p, \delta < \infty$ *and let* ω *be a weight in* A_∞. *Then there exists* $C > 0$ (*depending upon the* A_∞ *condition of* ω) *such that*

$$\int_{\mathbb{R}^n} \left(M_\delta f(x) \right)^p \omega(x)\,dx \le C \int_{\mathbb{R}^n} \left(M_\delta^\sharp f(x) \right)^p \omega(x)\,dx \tag{3.1}$$

for every function such that the left-hand side is finite.

Lemma 3.2 *Let* $0 < \delta < 1/m$, $1/m < q < \infty$ *and* $1/q = 1/q_1 + \cdots + 1/q_m$ *with* $1 < q_1, \ldots, q_m < \infty$. *Then there exists a constant* $C > 0$ *such that*

$$M_\delta^\sharp \left(T_q(\vec{f}) \right)(x) \le C \prod_{j=1}^m M\left(|f_j|_{q_j} \right)(x)$$

for any smooth vector function $\{\vec{f}_k\}_{k=1}^\infty$ *for any* $x \in \mathbb{R}^n$.

Proof Fix a point $x \in \mathbb{R}^n$ and a cube Q centered at x. Set $\vec{f}_j = \vec{f}_j^0 + \vec{f}_j^\infty$, where $\vec{f}_j^0 = \vec{f}_j \chi_{Q^*}$. Let $\vec{f}^\alpha = f_1^{\alpha_1} \cdots f_m^{\alpha_m}$ and $Q^* = (8\sqrt{n} + 4)Q$. It is easy to see

$$\left| T_q(\vec{f})(z) - \mathcal{C} \right| \le \left| T_q(\vec{f}^0)(z) \right| + \sum_{\alpha_1, \ldots, \alpha_m} \left| T(\vec{f}^\alpha)(z) - T(\vec{f}^\alpha)(x) \right|_q,$$

where $\mathcal{C} = \sum_{\alpha_1,\ldots,\alpha_m} |T(f_1^{\alpha_1}\cdots f_m^{\alpha_m})(x)|_q$ and in the last sum each $\alpha_j = 0$ or ∞ and in each term there is at least one $\alpha_j = \infty$. Since $0 < \delta < 1/m < 1$, it follows that

$$\begin{aligned}
&\left(\frac{1}{|Q|}\int_Q \big||T_q(\vec{f})(z)|^\delta - |\mathcal{C}|^\delta\big|\,dz\right)^{1/\delta} \\
&\quad\le C\left(\frac{1}{|Q|}\int_Q \big|T(\vec{f})(z) - c\big|_q^\delta\,dz\right)^{1/\delta} \\
&\quad\le C\left(\frac{1}{|Q|}\int_Q \big|T(\vec{f}^0)(z)\big|_q^\delta\,dz\right)^{1/\delta} + C\sum_{\alpha_1,\ldots,\alpha_m}\left(\frac{1}{|Q|}\int_Q \big|T(\vec{f}^\alpha)(z) - c\big|_q^\delta\,dz\right)^{1/\delta} \\
&\quad\triangleq P_1 + P_2,
\end{aligned}$$

where $\mathcal{C} = |c|_q = (\sum_{k\ge1}|c_k|^q)^{1/q}$.

Applying Kolmogorov's inequality and Theorem 1.2 to P_1, we have

$$\begin{aligned}
\left(\frac{1}{|Q|}\int_Q \big|T_q(\vec{f}^0)(z)\big|^\delta\,dz\right)^{1/\delta} &\le C\big\|T_q(\vec{f}^0)\big\|_{L^{1/m,\infty}(Q,\frac{dz}{|Q|})} \\
&\le C\prod_{j=1}^m \frac{1}{|Q|}\int_Q \big|f_j(z)\big|_{q_j}\,dz \\
&\le C\prod_{j=1}^m M\big(|f_j|_{q_j}\big)(x).
\end{aligned}$$

We proceed to the estimate for P_2. We can take $t = [2\sqrt{n}l(Q)]^s$. If $\alpha_1 = \cdots = \alpha_m = \infty$, we have

$$\begin{aligned}
&\left(\frac{1}{|Q|}\int_Q \big|T(\vec{f})(z) - c\big|_q^\delta\,dz\right)^{1/\delta} \\
&\quad\le \frac{C}{|Q|}\int_Q \big|T(\vec{f}^\infty)(z) - c\big|_q\,dz \\
&\quad\le \frac{C}{|Q|}\int_Q \left(\sum_{k=1}^\infty \big|T(\vec{f}_k^\infty)(z) - T(\vec{f}_k^\infty)(x)\big|^q\right)^{1/q} dz \\
&\quad\le \frac{C}{|Q|}\int_Q \left(\sum_{k=1}^\infty \left|\int_{(\mathbb{R}^n\setminus Q^*)^m} \big|K(z,\vec{y}) - K(x,\vec{y})\big||f_{1k}\cdots f_{mk}|\,d\vec{y}\right|^q\right)^{1/q} dz \\
&\quad\le \frac{C}{|Q|}\int_Q \left(\sum_{k=1}^\infty \left|\int_{(\mathbb{R}^n\setminus Q^*)^m} \big|K(z,\vec{y}) - K_t^{(0)}(z,\vec{y})\big||f_{1k}\cdots f_{mk}|\,d\vec{y}\right|^q\right)^{1/q} dz \\
&\qquad + \frac{C}{|Q|}\int_Q \left(\sum_{k=1}^\infty \left|\int_{(\mathbb{R}^n\setminus Q^*)^m} \big|K_t^{(0)}(z,\vec{y}) - K(x,\vec{y})\big||f_{1k}\cdots f_{mk}|\,d\vec{y}\right|^q\right)^{1/q} dz \\
&\quad= P_{21} + P_{22}.
\end{aligned}$$

Since $z \in Q$ and $y_j \in \mathbb{R}^n \setminus (8\sqrt{n}+4)Q$, we get $|y_j - z| > (4\sqrt{n}+1)l(Q) > 2t^{1/s}$ for all $j = 1,\ldots,m$. Applying Assumption (H2), we obtain

$$\begin{aligned} P_{21} &\le \frac{C}{|Q|}\int_Q \int_{(\mathbb{R}^n\setminus Q^*)^m} \frac{At^{\varepsilon/s}}{(|z-y_1|+\cdots+|z-y_m|)^{mn+\varepsilon}} |f_1|_{q_1}\cdots|f_m|_{q_m}\,d\vec{y}\,dz \\ &\le \sum_{k=1}^{\infty}\frac{1}{2^{k\varepsilon}}\prod_{j=1}^{m}\frac{1}{2^{(k+1)n}|Q^*|}\int_{2^{k+1}Q^*}|f_j|_{q_j}\,dy_j \\ &\le C\prod_{j=1}^{m} M\big(|f_j|_{q_j}\big)(x). \end{aligned}$$

Since $x, z \in Q$, $|z-x| \le \sqrt{n}l(Q) \le \frac{1}{2}t^{1/s}$. Note that $|y_j - z| > (4\sqrt{n}+1)l(Q) > 2t^{1/s}$, for all $j = 1,\ldots,m$, hence $\phi(\frac{|y_j-z|}{t^{1/s}}) = 0$. Similarly, we get $P_{22} \le \prod_{j=1}^{m} M(|f_j|_{q_j})(x)$.

Now we estimate the typical representative of P_2, that is, $\alpha_1 = \cdots = \alpha_l = \infty$ and $\alpha_{l+1} = \cdots = \alpha_m = 0$.

$$\begin{aligned} &\big|T\big(f_1^\infty,\ldots,f_l^\infty,f_{l+1}^0,\ldots,f_m^0\big)(z) - T\big(f_1^\infty,\ldots,f_l^\infty,f_{l+1}^0,\ldots,f_m^0\big)(x)\big|_q \\ &\quad\le \int_{(\mathbb{R}^n)^m}\big|K(z,\vec{y}) - K_t^{(0)}(z,\vec{y})\big| + \big|K_t^{(0)}(z,\vec{y}) - K(x,\vec{y})\big||f_1|_{q_1}\cdots|f_m|_{q_m}\,d\vec{y} \\ &\quad= \int_{(\mathbb{R}^n)^m}\big|K(z,\vec{y}) - K_t^{(0)}(z,\vec{y})\big||f_1|_{q_1}\cdots|f_m|_{q_m}\,d\vec{y} \\ &\qquad+ \int_{(\mathbb{R}^n)^m}\big|K_t^{(0)}(z,\vec{y}) - K(x,\vec{y})\big||f_1|_{q_1}\cdots|f_m|_{q_m}\,d\vec{y}. \end{aligned}$$

Thus, we get

$$\begin{aligned} &\left(\frac{1}{|Q|}\int_Q \big|T\big(f_1^\infty,\ldots,f_l^\infty,f_{l+1}^0,\ldots,f_m^0\big)(z) - c\big|_q^\delta\,dz\right)^{1/\delta} \\ &\quad\le \frac{C}{|Q|}\int_Q \big|T\big(f_1^\infty,\ldots,f_l^\infty,f_{l+1}^0,\ldots,f_m^0\big)(z) - T\big(f_1^\infty,\ldots,f_l^\infty,f_{l+1}^0,\ldots,f_m^0\big)(x)\big|_q\,dz \\ &\quad\le \frac{C}{|Q|}\int_Q\int_{(\mathbb{R}^n)^m}\big|K(z,\vec{y}) - K_t^{(0)}(z,\vec{y})\big||f_1|_{q_1}\cdots|f_m|_{q_m}\,d\vec{y}\,dz \\ &\qquad+ \frac{C}{|Q|}\int_Q\int_{(\mathbb{R}^n)^m}\big|K_t^{(0)}(z,\vec{y}) - K(x,\vec{y})\big||f_1|_{q_1}\cdots|f_m|_{q_m}\,d\vec{y}\,dz \\ &\quad= P_{23} + P_{24}. \end{aligned}$$

For P_{23}, by Assumption (H2), we have

$$\begin{aligned} P_{23} &\le \frac{C}{|Q|}\int_Q \left(\int_{(\mathbb{R}^n\setminus Q^*)^l}\frac{t^{\varepsilon/s}\prod_{j=1}^{l}|f_j(y_j)|_{q_j}\,dy_j}{(\sum_{j\in\{1,2,\ldots,l\}}|z-y_j|)^{mn+\varepsilon}}\right. \\ &\qquad \left.+ \int_{(\mathbb{R}^n\setminus Q^*)^l}\frac{\prod_{j=1}^{l}|f_j(y_j)|_{q_j}\,dy_j}{(\sum_{j\in\{1,2,\ldots,l\}}|z-y_j|)^{mn}}\right)\prod_{j=l+1}^{m}\int_{Q^*}\big|f_j(y_j)\big|_{q_j}\,dy_j\,dz \\ &\le \left(\sum_{k=1}^{\infty}\frac{|Q^*|^{\varepsilon/n}}{(2^k|Q^*|^{1/n})^{mn+\varepsilon}}\int_{(\mathbb{R}^n\setminus Q^*)^l}\prod_{j=1}^{l}\big|f_j(y_j)\big|_{q_j}\,dy_j\right. \end{aligned}$$

$$+\sum_{k=1}^{\infty}\frac{1}{(2^k|Q^*|^{1/n})^{mn}}\int_{(2^{k+1}Q^*\setminus 2^kQ^*)^l}\prod_{j=1}^{l}\bigl|f_j(y_j)\bigr|_{q_j}\,dy_j\Biggr)\prod_{j=l+1}^{m}\int_{Q^*}\bigl|f_j(y_j)\bigr|_{q_j}\,dy_j$$
$$\le C\prod_{j=1}^{m}M\bigl(|f_j|_{q_j}\bigr)(x).$$

By a similar argument, we deduce that $P_{24}\le C\prod_{j=1}^{m}M(|f_j|_{q_j})(x)$. In other case, we can also deduce the same estimate with minor modifications on the above arguments. We have thus proved Lemma 3.2. □

Lemma 3.3 *Let $0<\delta<\varepsilon<1/m$, $1/m<q<\infty$ and $1/q=1/q_1+\cdots+1/q_m$ with $1<q_1,\dots,q_m<\infty$. Suppose that $\vec{b}\in(BMO)^l$. There then exists a constant $C>0$ depending only on δ and ε such that*

$$M_\delta^\sharp(T_{\Pi\vec{b},q}\vec{f})(x)\le C\prod_{j=1}^{l}\|b_j\|_{BMO}\Biggl(\prod_{j=1}^{m}M_{L(\log L)}\bigl(|f_j|_{q_j}\bigr)(x)+M_\varepsilon(T_q\vec{f})(x)\Biggr)$$
$$+C\sum_{j=1}^{l-1}\sum_{\sigma\in C_j^l}\prod_{i\in\sigma}\|b_i\|_{BMO}M_\varepsilon(T_{\Pi b_{\sigma'},q}\vec{f})(x)$$

for any smooth vector function $\{\vec{f}_k\}_{k=1}^{\infty}$ for any $x\in\mathbb{R}^n$, where $\sigma'=\{1,\dots,l\}\setminus\sigma$.

Proof For simplicity of notation, we write $F(\vec{y})$ instead of the product of m functions $f_1(y_1)\cdots f_m(y_m)$ and let $\lambda_j=\frac{1}{|2Q|}\int_{2Q}b_j(z)\,dz$, for $j=1,\dots,l$. Let $x\in\mathbb{R}^n$ and Q be a cube centered at x. Then we have

$$T_{\Pi\vec{b}}(\vec{f})(x)=\int_{(\mathbb{R}^n)^m}\bigl(b_1(x)-b_1(y_1)\bigr)\cdots\bigl(b_l(x)-b_l(y_l)\bigr)K(x,\vec{y})F(\vec{y})\,d\vec{y}$$
$$=\int_{(\mathbb{R}^n)^m}\bigl(\bigl(b_1(x)-\lambda_1\bigr)-\bigl(b_1(y)-\lambda_1\bigr)\bigr)\cdots\bigl(\bigl(b_l(x)-\lambda_l\bigr)-\bigl(b_l(y)-\lambda_l\bigr)\bigr)$$
$$\times K(x,\vec{y})F(\vec{y})\,d\vec{y}$$
$$=\sum_{i=0}^{l}\sum_{\sigma\in C_i^l}(-1)^{l-j}\prod_{j\in\sigma}\bigl(b_j(x)-\lambda_j\bigr)\int_{(\mathbb{R}^n)^m}\prod_{j\in\sigma'}\bigl(b_j(y_j)-\lambda_j\bigr)K(x,\vec{y})F(\vec{y})\,d\vec{y}$$
$$=\bigl(b_1(x)-\lambda_1\bigr)\cdots\bigl(b_l(x)-\lambda_l\bigr)T(\vec{f})(x)+T\bigl(\bigl(b_1(\cdot_1)-\lambda_1\bigr)\cdots\bigl(b_l(\cdot_l)-\lambda_l\bigr)\vec{f}\bigr)(x)$$
$$+\sum_{i=1}^{l-1}\sum_{\sigma\in C_i^l}(-1)^{l-j}\prod_{j\in\sigma}\bigl(b_j(x)-\lambda_j\bigr)\int_{(\mathbb{R}^n)^m}\prod_{j\in\sigma'}\bigl(b_j(y_j)-\lambda_j\bigr)K(x,\vec{y})F(\vec{y})\,d\vec{y}.$$

Noting that $\prod_{j\in\sigma'}(b_j(y_j)-\lambda_j)=\prod_{j\in\sigma'}[(b_j(y_j)-b_j(x))-(b_j(x)-\lambda_j)]$. Then we get

$$T_{\Pi\vec{b},q}\vec{f}(x)\le\bigl|\bigl(b_1(x)-\lambda_1\bigr)\cdots\bigl(b_l(x)-\lambda_l\bigr)\bigr|T_q(\vec{f})(x)$$
$$+\bigl|T\bigl(\bigl(b_1(\cdot_1)-\lambda_1\bigr)\cdots\bigl(b_l(\cdot_l)-\lambda_l\bigr)\vec{f}\bigr)(x)\bigr|_q$$
$$+C\sum_{i=1}^{l-1}\sum_{\sigma\in C_i^l}\prod_{j\in\sigma}\bigl|b_j(x)-\lambda_j\bigr|T_{\Pi b_{\sigma'},q}\vec{f}(x).$$

Since $0 < \delta < 1/m < 1$, it follows that

$$\begin{aligned}
&\left(\frac{1}{|Q|}\int_Q \left|\left|T_{\Pi\vec{b},q}(\vec{f})(z)\right|^{\delta} - |\mathcal{C}|^{\delta}\right| dz\right)^{1/\delta} \\
&\quad\le C\left(\frac{1}{|Q|}\int_Q \left|T_{\Pi\vec{b}}(\vec{f})(z) - c\right|_q^{\delta} dz\right)^{1/\delta} \\
&\quad\le C\left(\frac{1}{|Q|}\int_Q \left|\big(b_1(z)-\lambda_1\big)\cdots\big(b_l(z)-\lambda_l\big)\left|T(\vec{f})(z)\right|_q^{\delta}\right| dz\right)^{1/\delta} \\
&\qquad + C\sum_{i=1}^{l-1}\sum_{\sigma\in C_i^l}\left(\frac{1}{|Q|}\int_Q \prod_{j\in\sigma}\left(\left|b_j(z)-\lambda_j\right| T_{\Pi b_{\sigma'},q}\vec{f}(z)\right)^{\delta} dz\right)^{1/\delta} \\
&\qquad + C\left(\frac{1}{|Q|}\int_Q \left|T\big(\big(b_1(\cdot_1)-\lambda_1\big)\cdots\big(b_l(\cdot_l)-\lambda_l\big)\vec{f}\big)(z) - c\right|_q^{\delta} dz\right)^{1/\delta} \\
&\quad\triangleq I + II + III,
\end{aligned}$$

where $\mathcal{C} = |c|_q = (\sum_{k\ge1}|c_k|^q)^{1/q}$. We can choose $1 < p_1, \ldots, p_l < \infty$ with $\frac{1}{p_1} + \cdots + \frac{1}{p_l} + \frac{1}{\varepsilon} = \frac{1}{\delta}$. Since $0 < \delta < \varepsilon < 1/m$, Hölder's inequality gives

$$I \le C\prod_{j=1}^{l}\|b_j\|_{BMO} M_\varepsilon(T_q\vec{f})(x),$$

$$II \le C\sum_{i=1}^{l-1}\sum_{\sigma\in C_i^l}\prod_{j\in\sigma}\|b_j\|_{BMO} M_\varepsilon(T_{\Pi b_{\sigma'},q}\vec{f})(x).$$

Let us estimate term III. Set $\vec{f}_j = \vec{f}_j^{0} + \vec{f}_j^{\infty}$, where $\vec{f}_j^{0} = \vec{f}_j\chi_{Q^*}$. Let $\vec{f}^{\alpha} = f_1^{\alpha_1}\cdots f_m^{\alpha_m}$ and $Q^* = (8\sqrt{n}+4)Q$. Taking $\mathcal{C} = \sum_{\alpha_1,\ldots,\alpha_m}|T((b_1(\cdot_1)-\lambda_1)\cdots(b_l(\cdot_l)-\lambda_l)f_1^{\alpha_1}\cdots f_m^{\alpha_m})(x)|_q$, we have

$$\begin{aligned}
&\left|T_{\Pi\vec{b},q}(\vec{f})(z) - \mathcal{C}\right| \\
&\quad\le T_q\big(\big(b_1(\cdot_1)-\lambda_1\big)\cdots\big(b_l(\cdot_l)-\lambda_l\big)\vec{f}^{0}\big)(z) \\
&\qquad + C\sum_{\alpha_1,\ldots,\alpha_m}\Big|\Big(T\big(\big(b_1(\cdot_1)-\lambda_1\big)\cdots\big(b_l(\cdot_l)-\lambda_l\big)\vec{f}^{\alpha}\big)\Big)(z) \\
&\qquad - \Big(T\big(\big(b_1(\cdot_1)-\lambda_1\big)\cdots\big(b_l(\cdot_l)-\lambda_l\big)\vec{f}^{\alpha}\big)\Big)(x)\Big|_q,
\end{aligned}$$

where in the last sum each $\alpha_j = 0$ or ∞ and in each term there is at least one $\alpha_j = \infty$.

If $\alpha_1 = \cdots = \alpha_m = 0$, applying Kolmogorov's inequality and Theorem 1.2, we get

$$\begin{aligned}
&\left(\frac{1}{|Q|}\int_Q \left|T_q\big(\big(b_1(\cdot_1)-\lambda_1\big)\cdots\big(b_l(\cdot_l)-\lambda_l\big)\vec{f}^{0}\big)(z)\right|^{\delta} dz\right)^{1/\delta} \\
&\quad\le C\left\|T_q\big(\big(b_1(\cdot_1)-\lambda_1\big)\cdots\big(b_l(\cdot_l)-\lambda_l\big)\vec{f}^{0}\big)\right\|_{L^{1/m,\infty}(Q,\frac{dz}{|Q|})} \\
&\quad\le C\prod_{j=1}^{l}\frac{1}{|Q|}\int_Q \left|b_j(y_j)-\lambda_j\right|\left|f_j(z)\right|_{q_j} dz \prod_{j=l+1}^{m}\frac{1}{|Q|}\int_Q \left|f_j(z)\right|_{q_j} dz
\end{aligned}$$

$$\begin{aligned}
&\le C\prod_{j=1}^{l}\|b_j\|_{BMO}\big\||f_j|_{q_j}\big\|_{L(\log L),Q}\prod_{j=l+1}^{m}\frac{1}{|Q|}\int_Q\big|f_j(z)\big|_{q_j}\,dz\\
&\le C\prod_{j=1}^{l}\|b_j\|_{BMO}\prod_{j=1}^{m}M_{L(\log L)}\big(|f_j|_{q_j}\big)(x).
\end{aligned}$$

If $\alpha_1 = \cdots = \alpha_m = \infty$, we have

$$\begin{aligned}
&\left(\frac{1}{|Q|}\int_Q\big|T_{\Pi\vec{b}}(\vec{f}^{\infty})(z)-c\big|_q^{\delta}\,dz\right)^{1/\delta}\\
&\quad\le\frac{C}{|Q|}\int_Q\big|T_{\Pi\vec{b}}(\vec{f}^{\infty})(z)-c\big|_q\,dz\\
&\quad\le\frac{C}{|Q|}\int_Q\left(\sum_{k=1}^{\infty}\big|T_{\Pi\vec{b}}(\vec{f}_k^{\infty})(z)-T_{\Pi\vec{b}}(\vec{f}_k^{\infty})(x)\big|^q\right)^{1/q}dz\\
&\quad\le\frac{C}{|Q|}\int_Q\left(\sum_{k=1}^{\infty}\bigg|\int_{(\mathbb{R}^n\setminus Q^*)^m}\big|K(z,\vec{y})-K(x,\vec{y})\big|\right.\\
&\qquad\left.\times\big|\big(b_1(y_1)-\lambda_1\big)\cdots\big(b_l(y_l)-\lambda_l\big)f_{1k}\cdots f_{mk}\big|\,d\vec{y}\bigg|^q\right)^{1/q}dz\\
&\quad\le\frac{C}{|Q|}\int_Q\left(\sum_{k=1}^{\infty}\bigg|\int_{(\mathbb{R}^n\setminus Q^*)^m}\big|K(z,\vec{y})-K_t^{(0)}(z,\vec{y})\big|\right.\\
&\qquad\left.\times\big|\big(b_1(y_1)-\lambda_1\big)\cdots\big(b_l(y_l)-\lambda_l\big)f_{1k}\cdots f_{mk}\big|\,d\vec{y}\bigg|^q\right)^{1/q}dz\\
&\qquad+\frac{C}{|Q|}\int_Q\left(\sum_{k=1}^{\infty}\bigg|\int_{(\mathbb{R}^n\setminus Q^*)^m}\big|K_t^{(0)}(z,\vec{y})-K(x,\vec{y})\big|\right.\\
&\qquad\left.\times\big|\big(b_1(y_1)-\lambda_1\big)\cdots\big(b_l(y_l)-\lambda_l\big)f_{1k}\cdots f_{mk}\big|\,d\vec{y}\bigg|^q\right)^{1/q}dz\\
&\quad=III_1+III_2.
\end{aligned}$$

Consider now the term III_1. Taking $t=[2\sqrt{n}l(Q)]^s$, we have by Assumption (H2)

$$\begin{aligned}
III_1&\le\frac{C}{|Q|}\int_Q\int_{(\mathbb{R}^n\setminus Q^*)^m}\sum_{k=1}^{\infty}\frac{|Q^*|^{\varepsilon/n}}{(2^k|Q^*|^{1/n})^{mn+\varepsilon}}\\
&\quad\times\big|\big(b_1(y_1)-\lambda_1\big)\cdots\big(b_l(y_l)-\lambda_l\big)\big||f_1|_{q_1}\cdots|f_m|_{q_m}\,d\vec{y}\,dz\\
&\le C\prod_{j=1}^{l}\sum_{k=1}^{\infty}\frac{1}{2^{k\varepsilon}}\frac{1}{2^{(k+1)n}|Q^*|}\\
&\quad\times\int_{2^{k+1}Q^*}\big|b_j(y_j)-\lambda_j\big||f_j|_{q_j}\,dy_j\prod_{j=l+1}^{m}\frac{1}{2^{(k+1)n}|Q^*|}\int_{2^{k+1}Q^*}|f_j|_{q_j}\,dy_j\\
&\le C\prod_{j=1}^{l}\sum_{k=1}^{\infty}\frac{k}{2^{k\varepsilon}}\|b_j\|_{BMO}\big\||f_j|_{q_j}\big\|_{L(\log L),2^{k+1}Q^*}\prod_{j=l+1}^{m}\frac{1}{2^{(k+1)n}|Q^*|}\int_{2^{k+1}Q^*}|f_j|_{q_j}\,dy_j
\end{aligned}$$

$$\le C\prod_{j=1}^{l}\|b_j\|_{BMO}\big\||f_j|_{q_j}\big\|_{L(\log L),2^{k+1}Q^*}\prod_{j=l+1}^{m}\frac{1}{2^{(k+1)n}|Q^*|}\int_{2^{k+1}Q^*}|f_j|_{q_j}\,dy_j$$
$$\le C\prod_{j=1}^{l}\|b_j\|_{BMO}\prod_{j=1}^{m}M_{L(\log L)}\big(|f_j|_{q_j}\big)(x).$$

Similarly, we get $III_2\le\prod_{j=1}^{l}\|b_j\|_{BMO}\prod_{j=1}^{m}M_{L(\log L)}(|f_j|_{q_j})(x)$. Now we consider only the typical representative of III. Similar to the estimates of P_2 in Lemma 3.2, we have

$$\begin{aligned}
&\left(\frac{1}{|Q|}\int_Q\big|T\big((b_1(\cdot_1)-\lambda_1)\cdots(b_l(\cdot_l)-\lambda_l)f_1^\infty,\dots,f_l^\infty,f_{l+1}^0,\dots,f_m^0\big)(z)-c\big|_q^\delta\,dz\right)^{1/\delta}\\
&\quad\le\frac{C}{|Q|}\int_Q\big|T\big((b_1(\cdot_1)-\lambda_1)\cdots(b_l(\cdot_l)-\lambda_l)f_1^\infty,\dots,f_l^\infty,f_{l+1}^0,\dots,f_m^0\big)(z)\\
&\qquad-T\big((b_1(\cdot_1)-\lambda_1)\cdots(b_l(\cdot_l)-\lambda_l)f_1^\infty,\dots,f_l^\infty,f_{l+1}^0,\dots,f_m^0\big)(x)\big|_q\,dz\\
&\quad\le\prod_{j=1}^{l}\bigg(\int_{(\mathbb{R}^n\setminus Q^*)^l}\frac{t^{\varepsilon/s}|(b_1(y_1)-\lambda_1)\cdots(b_l(y_l)-\lambda_l)|\prod_{j=1}^{l}|f_j(y_j)|_{q_j}\,dy_j}{(\sum_{j\in\{1,2,\dots,l\}}|z-y_j|)^{mn+\varepsilon}}\\
&\qquad+\int_{(\mathbb{R}^n\setminus Q^*)^l}\frac{|(b_1(y_1)-\lambda_1)\cdots(b_l(y_l)-\lambda_l)|\prod_{j=1}^{l}|f_j(y_j)|_{q_j}\,dy_j}{(\sum_{j\in\{1,2,\dots,l\}}|z-y_j|)^{mn}}\bigg)\prod_{j=l+1}^{m}\int_{Q^*}\big|f_j(y_j)\big|_{q_j}\,dy_j\\
&\quad\le C\prod_{j=1}^{l}\|b_j\|_{BMO}\big\||f_j|_{q_j}\big\|_{L(\log L),2^{k+1}Q^*}\prod_{j=l+1}^{m}\frac{1}{|Q^*|}\int_{Q^*}\big|f_j(z)\big|_{q_j}\,dz\\
&\quad\le C\prod_{j=1}^{l}\|b_j\|_{BMO}\prod_{j=1}^{m}M_{L(\log L)}\big(|f_j|_{q_j}\big)(x).
\end{aligned}$$

Then Lemma 3.3 is proved. □

Lemma 3.4 *Let* $0<p<\infty$, $1/m<q<\infty$, *and* $\frac{1}{q_1}+\cdots+\frac{1}{q_m}=\frac{1}{q}$ *with* $1<q_1,\dots,q_m<\infty$ *and let* $w\in A_\infty$. *Suppose that* $\vec{b}\in(BMO)^l$. *Then there exists a constant* $C>0$ *such that*

$$\int_{\mathbb{R}^n}|T_{\Pi\vec{b},q}\vec{f}|^p w(x)\,dx\le C\prod_{j=1}^{l}\|b_j\|_{BMO}^p\int_{\mathbb{R}^n}\left(\prod_{j=1}^{m}M_{L(\log L)}\big(|f_j|_{q_j}\big)(x)\right)^p w(x)\,dx \tag{3.2}$$

for any smooth function $\vec{f}$ *with compact support.*

Proof We assume that the right-hand side of (3.2) is finite, since otherwise there is nothing to be proved. For $l=1$, by using Lemma 3.1 and Lemma 3.3, we obtain

$$\begin{aligned}
\|T_{\Pi\vec{b},q}\vec{f}\|_{L^p(\omega)}&\le\|M_\delta T_{\Pi\vec{b},q}\vec{f}\|_{L^p(\omega)}\le C\big\|M_\delta^\sharp T_{\Pi\vec{b},q}\vec{f}\big\|_{L^p(\omega)}\\
&\le C\|b_1\|_{BMO}\left[\big\|M_\varepsilon(T_q\vec{f})\big\|_{L^p(\omega)}+\left\|\prod_{j=1}^{m}M_{L(\log L)}\big(|f_j|_{q_j}\big)\right\|_{L^p(\omega)}\right]\\
&\le C\|b_1\|_{BMO}\left[\|T_q\vec{f}\|_{L^p(\omega)}+\left\|\prod_{j=1}^{m}M_{L(\log L)}\big(|f_j|_{q_j}\big)\right\|_{L^p(\omega)}\right]
\end{aligned}$$

$$\begin{aligned}&\le C\|b_1\|_{BMO}\left[\left\|\prod_{j=1}^m M(|f_j|_{q_j})\right\|_{L^p(\omega)}+\left\|\prod_{j=1}^m M_{L(\log L)}(|f_j|_{q_j})\right\|_{L^p(\omega)}\right]\\&\le C\|b_1\|_{BMO}\left\|\prod_{j=1}^m M_{L(\log L)}(|f_j|_{q_j})\right\|_{L^p(\omega)}.\end{aligned}\tag{3.3}$$

For the general case $l \ge 2$, similarly to the case for $l = 1$, we have

$$\|T_{\Pi\vec{b},q}\vec{f}\|_{L^p(\omega)} \le C\prod_{j=1}^l \|b_j\|_{BMO}\left\|\prod_{j=1}^m M_{L(\log L)}(|f_j|_{q_j})\right\|_{L^p(\omega)}.$$

To apply the Fefferman-Stein inequality in (3.3), one needs to verify now that $\|T_{\Pi\vec{b},q}\vec{f}\|_{L^p(\omega)} < \infty$ and $\|T_q\vec{f}\|_{L^p(\omega)} < \infty$. We will only show the first one since the proof of another one is very similar but easier.

Suppose that the symbols b_j and the weight ω are bounded functions. Since $\vec{f}$ has compact support, we may assume $\operatorname{supp} f_j \subset B(0,R)$. Then we have

$$\begin{aligned}\|T_{\Pi\vec{b},q}\vec{f}\|_{L^p(\omega)} &= \int_{|x|\le 2R}\left|T_{\Pi\vec{b},q}\vec{f}(x)\right|^p\omega(x)\,dx + \int_{|x|>2R}\left|T_{\Pi\vec{b},q}\vec{f}(x)\right|^p\omega(x)\,dx\\&= I_1 + I_2.\end{aligned}$$

We choose $s > p$ and $s_1,\dots,s_m > 1$ such that $1/s = 1/s_1 + \cdots + 1/s_m$. Theorem 1.1 and Hölder's inequality imply

$$\begin{aligned}I_1 &\le C\int_{|x|\le 2R}|T_{\Pi\vec{b},q}\vec{f}|^p\,dx \le C\left(\int_{\mathbb{R}^n}|T_q\vec{f}|^s\,dx\right)^{p/s}R^{n(s-p)/s}\\&\le C\left(\prod_{j=1}^m \left\||f_j|_{q_j}\right\|_{L^{s_j}}\right)^p R^{n(s-p)/s} < \infty.\end{aligned}$$

Consider now the term I_2. Since $|x| > 2R$, we have by Assumption (H2)

$$\begin{aligned}\left|T_{\Pi\vec{b},q}\vec{f}(x)\right| &\le C\left|\int_{(B(0,2R))^m}\left|K(x,\vec{y})\right|\left|b_1(x)-b_1(y_1)\right|\cdots\left|b_l(x)-b_l(y_l)\right|\right.\\&\qquad\left.\times\left|f_1(y_1)\right|\cdots\left|f_m(y_m)\right|d\vec{y}\right|_q\\&\le C\int_{(B(0,2R))^m}\left|K(x,\vec{y})-K_t^{(0)}(x,\vec{y})\right|\prod_{j=1}^m\left|f_j(y_j)\right|_{q_j}d\vec{y}\\&\quad+\int_{(B(0,2R))^m}\left|K_t^{(0)}(x,\vec{y})\right|\prod_{j=1}^m\left|f_j(y_j)\right|_{q_j}d\vec{y}\\&\le C\prod_{j=1}^m\frac{1}{|x|^n}\int_{B(0,|x|)}\left|f_j(y_j)\right|_{q_j}dy_j \le \prod_{j=1}^m M_{L(\log L)}(|f_j|_{q_j})(x).\end{aligned}$$

Thus, $I_2 \le C\int_{\mathbb{R}^n}(\prod_{j=1}^m M_{L(\log L)}(|f_j|_{q_j})(x))^p w(x)\,dx < \infty$.

For the general case, we can check the limit (see the proof of [10], Theorem 1.1, we omit the details here). Thus, (3.2) is proved. □

Proof of Theorem 1.3 (i) Since $\nu_{\vec{\omega}} \in A_{mp} \subset A_\infty$, Lemma 3.4 gives

$$\int_{R^n} \left|T_{\Pi\vec{b},q}(\vec{f})(x)\right|^p \nu_{\vec{\omega}}\,dx \le C \int_{R^n} \prod_{j=1}^m M^p_{L(\log L)}\left(|f_j|_{q_j}\right)(x)\nu_{\vec{\omega}}\,dx.$$

It follows from [8] that there exists $r > 1$ such that $\nu_{\vec{\omega}} \in A_{(\frac{p_1}{r},\dots,\frac{p_m}{r})}$. On the other hand, since $\Phi(t) = t(1 + \log^+ t) \le t^r$ for all $t > 1$, the generalized Jensen inequality yields that

$$\|f_j\|_{L(\log L),Q} \le C\left(\frac{1}{|Q|}\int_Q \left|f_j(y)\right|^r dy\right)^{1/r}$$

for all j. One sees immediately that

$$\left\|T_{\Pi\vec{b},q}(\vec{f})\right\|_{L^p(\nu_{\vec{\omega}})} \le C\left\|\prod_{j=1}^m M_r\left(|f_j|_{q_j}\right)\right\|_{L^p(\nu_{\vec{\omega}})}.$$

Since $\nu_{\vec{\omega}} \in A_{(\frac{p_1}{r},\dots,\frac{p_m}{r})}$, it follows from Hölder's inequality and the well-known inequality of Fefferman-Stein [9] that

$$\left\|\prod_{j=1}^m M\left(|f_j|_{q_j}\right)\right\|_{L^{p/r}(\nu_{\vec{\omega}})} \le C\prod_{j=1}^m \left\||f_j|_{q_j}\right\|_{L^{p_j/r}(M\omega_j)}.$$

We then have

$$\left\|T_{\Pi\vec{b},q}(\vec{f})\right\|_{L^p(\nu_{\vec{\omega}})} \le C\prod_{j=1}^l \|b_j\|_{BMO}\prod_{j=1}^m \left\||f_j|_{q_j}\right\|_{L^{p_j}(M\omega_j)}.$$

This is the desired conclusion.

(ii) Since $\omega_j \in A_{p_j}$, there exists $r > 0$ such that $\omega_j \in A_{p_j/r}$. Analysis similar to that in the proof of Theorem 1.3 (i) shows that

$$\left\|T_{\Pi\vec{b},q}(\vec{f})\right\|_{L^p(\nu_{\vec{\omega}})} \le C\left\|\prod_{j=1}^m M_r\left(|f_j|_{q_j}\right)\right\|_{L^p(\nu_{\vec{\omega}})}.$$

The Hölder inequality implies

$$\begin{aligned}\left\|\prod_{j=1}^m M_r\left(|f_j|_{q_j}\right)\right\|_{L^p(\nu_{\vec{\omega}})} &= \left(\left\|\prod_{j=1}^m M\left(|f_j|^r_{q_j}\right)\right\|_{L^{p/r}(\nu_{\vec{\omega}})}\right)^{1/r} \\ &\le \left(\prod_{j=1}^m \left\|M\left(|f_j|^r_{q_j}\right)\right\|_{L^{p_j/r}(\omega)}\right)^{1/r} \\ &\le \prod_{j=1}^m \left\||f_j|_{q_j}\right\|_{L^{p_j}(\omega_j)}. \end{aligned} \tag{3.4}$$

To prove Theorem 1.3 holds for T^*, it suffices to prove Lemmas 3.2-3.4 hold for T^*. The proof follows from similar steps in [11] and combines the argument we used in the above lemmas. The key for tackling the new complexities is a very careful deal with the supremum, we refer the reader to [11]. This concludes the proof of the theorem. □

Competing interests
The author declares that he has no competing interests.

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