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Optimality conditions of *E*-convex programming for an *E*-differentiable function

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Abstract

In this paper we introduce a new definition of an *E*-differentiable convex function, which transforms a non-differentiable function to a differentiable function under an operator $E: R^n \to R^n$. By this definition, we can apply Kuhn-Tucker and Fritz-John conditions for obtaining the optimal solution of mathematical programming with a non-differentiable function.

Keywords: *E*-convex set; *E*-convex function; semi *E*-convex function; *E*-differentiable function

1 Introduction

The concepts of E-convex sets and an E-convex function have been introduced by Youness in [1, 2], and they have some important applications in various branches of mathematical sciences. Youness in [1] introduced a class of sets and functions which is called E-convex sets and E-convex functions by relaxing the definition of convex sets and convex functions. This kind of generalized convexity is based on the effect of an operator $E: R^n \to R^n$ on the sets and the domain of the definition of functions. Also, in [2] Youness discussed the optimality criteria of E-convex programming. Xiusu Chen [3] introduced a new concept of semi E-convex functions and discussed its properties. Yu-Ru Syan and Stanelty [4] introduced some properties of an E-convex function, while Emam and Youness in [5] introduced a new class of E-convex sets and E-convex functions, which are called strongly E-convex sets and strongly E-convex functions, by taking the images of two points x and y under an operator $E: R^n \to R^n$ besides the two points themselves. In [6] Megahed *et al.* introduced a combined interactive approach for solving E-convex multiobjective nonlinear programming. Also, in [7, 8] Iqbal and *et al.* introduced geodesic E-convex sets, geodesic E-convex and some properties of geodesic semi-E-convex functions.

In this paper we present the concept of an E-differentiable convex function which transforms a non-differentiable convex function to a differentiable function under an operator $E: R^n \to R^n$, for which we can apply the Fritz-John and Kuhn-Tucker conditions [9, 10] to find a solution of mathematical programming with a non-differentiable function.

In the following, we present the definitions of E-convex sets, E-convex functions, and semi E-convex functions.

Definition 1 [1] A set M is said to be an E-convex set with respect to an operator $E: R^n \to R^n$ if and only if $\lambda E(x) + (1-\lambda)E(y) \in M$ for each $x, y \in M$ and $\lambda \in [0,1]$.

Definition 2 [1] A function $f : R^n \to R$ is said to be an E-convex function with respect to an operator $E : R^n \to R^n$ on an E-convex set $M \subseteq R^n$ if and only if

$$f\bigl(\lambda E(x) + (1-\lambda)E(y)\bigr) \le \lambda(f \circ E)(x) + (1-\lambda)(f \circ E)(y)$$

for each $x, y \in M$ and $\lambda \in [0,1]$.

Definition 3 [3] A real-valued function $f : M \subseteq R^n \to R$ is said to be semi E-convex function with respect to an operator $E : R^n \to R^n$ on M if M is an E-convex set and

$$f\bigl(\lambda E(x) + (1-\lambda)E(y)\bigr) \le \lambda f(x) + (1-\lambda)f(y)$$

for each $x, y \in M$ and $\lambda \in [0,1]$.

Proposition 4 [1] 1- *Let a set $M \subseteq R^n$ be an E-convex set with respect to an operator E, $E : R^n \to R^n$, then $E(M) \subseteq M$.*

2- *If $E(M)$ is a convex set and $E(M) \subseteq M$, then M is an E-convex set.*

3- *If M_1 and M_2 are E-convex sets with respect to E, then $M_1 \cap M_2$ is an E-convex set with respect to E.*

Lemma 5 [1] *Let $M \subseteq R^n$ be an E_1- and E_2-convex set, then M is an $(E_1 \circ E_2)$- and $(E_2 \circ E_1)$-convex set.*

Lemma 6 [1] *Let $E : R^n \to R^n$ be a linear map and let $M_1, M_2 \subset R^n$ be E-convex sets, then $M_1 + M_2$ is an E-convex set.*

Definition 7 [1] Let $S \subset R^n \times R$ and $E : R^n \to R^n$, we say that the set S is E-convex if for each $(x, \alpha), (y, \beta) \in S$ and each $\lambda \in [0,1]$, we have

$$\bigl(\lambda Ex + (1-\lambda)Ey, \lambda\alpha + (1-\lambda)\beta\bigr) \in S.$$

2 Generalized *E*-convex function

Definition 8 [1] Let $M \subseteq R^n$ be an E-convex set with respect to an operator $E : R^n \to R^n$. A function $f : M \to R$ is said to be a pseudo E-convex function if for each $x_1, x_2 \in M$ with $\nabla(f \circ E)(x_1)(x_2 - x_1) \ge 0$ implies $f(Ex_2) \ge f(Ex_1)$ or for all $x_1, x_2 \in M$ and $f(Ex_2) < f(Ex_1)$ implies $\nabla(f \circ E)(x_1)(x_2 - x_1) < 0$.

Definition 9 [1] Let $M \subseteq R^n$ be an E-convex set with respect to an operator $E : R^n \to R^n$. A function $f : M \to R$ is said to be a quasi-E-convex function if and only if

$$f\bigl(\lambda Ex + (1-\lambda)Ey\bigr) \le \max\bigl\{(f \circ E)x, (f \circ E)y\bigr\}$$

for each $x, y \in M$ and $\lambda \in [0,1]$.

3 *E*-differentiable function

Definition 10 Let $f : M \subseteq R^n \to R$ be a non-differentiable function at $\overline{x}$ and let $E : R^n \to R^n$ be an operator. A function f is said to be E-differentiable at $\overline{x}$ if and only if $(f \circ E)$ is a differentiable function at $\overline{x}$ and

$$(f \circ E)(x) = (f \circ E)(\overline{x}) + \nabla(f \circ E)(\overline{x})(x - \overline{x}) + \|x - \overline{x}\|\alpha(\overline{x}, x - \overline{x}),$$
$$\alpha(\overline{x}, x - \overline{x}) \to 0 \quad \text{as } x \to \overline{x}.$$

Example 11 Let $f(x) = |x|$ be a non-differentiable function at the point $x = 0$ and let $E : R \to R$ be an operator such that $E(x) = x^2$, then the function $(f \circ E)(x) = f(Ex) = x^2$ is a differentiable function at the point $x = 0$, and hence f is an E-differentiable function.

3.1 Problem formulation

Now, we formulate problems P and P_E, which have a non-differentiable function and an E-differentiable function, respectively.

Let $E : R^n \to R^n$ be an operator, M be an E-convex set and f be an E-differentiable function. The problem P is defined as

$$P \begin{cases} \text{Min} f(x), \\ \text{subject to } M = \{x : g_i(x) \leq 0, i = 1, 2, \dots, m\}, \end{cases}$$

where f is a non-differentiable function, and the problem P_E is defined as

$$P_E \begin{cases} \text{Min}(f \circ E)(x), \\ \text{subject to } M' = \{x : (g_i \circ E)(x) \leq 0, i = 1, 2, \dots, m\}, \end{cases}$$

where f is an E-differentiable function.

Now, we will discuss the relationship between the solutions of problems P and P_E.

Lemma 12 [11] *Let $E : R^n \to R^n$ be a one-to-one and onto operator and let $M' = \{x : (g_i \circ E)(x) \leq 0, i = 1, 2, \dots, m\}$. Then $E(M') = M$, where M and M' are feasible regions of problems P and P_E, respectively.*

Theorem 13 *Let $E : R^n \to R^n$ be a one-to-one and onto operator and let f be an E-differentiable function. If f is non-differentiable at $\overline{x}$, and $\overline{x}$ is an optimal solution of the problem P, then there exists $\overline{y} \in M'$ such that $\overline{x} = E(\overline{y})$ and $\overline{y}$ is an optimal solution of the problem P_E.*

Proof Let $\overline{x}$ be an optimal solution of the problem P. From Lemma 12 there exists $\overline{y} \in M'$ such that $\overline{x} = E(\overline{y})$. Let $\overline{y}$ be a not optimal solution of the problem P_E, then there is $\widehat{y} \in M'$ such that $(f \circ E)(\widehat{y}) \leq (f \circ E)(\overline{y})$. Also, there exists $\widehat{x} \in M$ such that $\widehat{x} = E(\widehat{y})$. Then $f(\widehat{x}) < f(\overline{x})$ contradicts the optimality of $\overline{x}$ for the problem P. Hence the proof is complete. □

Theorem 14 *Let $E : R^n \to R^n$ be a one-to-one and onto operator, and let f be an E-differentiable function and strictly quasi-E-convex. If $\overline{x}$ is an optimal solution of the problem P, then there exists $\overline{y} \in M'$ such that $\overline{x} = E(\overline{y})$ and $\overline{y}$ is an optimal solution of the problem P_E.*

Proof Let $\overline{x}$ be an optimal solution of the problem P. Then from Lemma 12 there is $\overline{y} \in M'$ such that $\overline{x} = E(\overline{y})$. Let $\overline{y}$ be a not optimal solution of the problem P_E, then there is $\widehat{y} \in M'$ and also $\widehat{x} \in M$, $\widehat{x} = E(\widehat{y})$ such that $(f \circ E)(\widehat{y}) \leq (f \circ E)(\overline{y})$. Since f is strictly quasi-E-convex function, then

$$\begin{aligned} f\big(\lambda E(\overline{y}) + (1-\lambda)E(\widehat{y})\big) &< \max\big\{(f \circ E)(\overline{y}), (f \circ E)(\widehat{y})\big\} \\ &< \max\big\{f(\overline{x}), f(\widehat{x})\big\} \\ &< f(\overline{x}). \end{aligned}$$

Since M is an E-convex set and $E(M) \subset M$, then $\lambda E(\overline{y}) + (1-\lambda)E(\widehat{y}) \in M$ contradicts the assumption that $\overline{x}$ is a solution of the problem P, then there exists $\overline{y} \in M'$, a solution of the problem P_E, such that $\overline{x} = E(\overline{y})$. □

Theorem 15 *Let M be an E-convex set, $E : R^n \to R^n$ be a one-to-one and onto operator and $f : M \subseteq R^n \to R$ be an E-differentiable function at $\overline{x}$. If there is a vector $d \subset R^n$ such that $\nabla(f \circ E)(\overline{x})d < 0$, then there exists $\delta > 0$ such that*

$$(f \circ E)(\overline{x} + \lambda d) < (f \circ E)(\overline{x}) \quad \textit{for each } \lambda \in (0, \delta).$$

Proof Since f is an E-differentiable function at $\overline{x}$, then

$$\begin{aligned} &(f \circ E)(\overline{x} + \lambda d) = (f \circ E)(\overline{x}) + \lambda \nabla(f \circ E)(\overline{x}) + \lambda \|d\| \alpha(\overline{x}, \lambda d), \\ &\quad \alpha(\overline{x}, \lambda d) \to 0 \quad \text{as } \lambda \to 0. \end{aligned}$$

Since $\nabla(f \circ E)(\overline{x})d < 0$ and $\alpha(\overline{x}, \lambda d) \to 0$ as $\lambda \to 0$, then there exists $\delta > 0$ such that

$$\nabla(f \circ E)(\overline{x}) + \|d\| \alpha(\overline{x}, \lambda d) < 0 \quad \text{for each } \lambda \in (0, \delta)$$

and thus $(f \circ E)(\overline{x} + \lambda d) < (f \circ E)(\overline{x})$. □

Corollary 16 *Let M be an E-convex set, let $E : R^n \to R^n$ be a one-to-one and onto operator, and let $f : M \subseteq R^n \to R$ be an E-differentiable and strictly E-convex function at $\overline{x}$. If $\overline{x}$ is a local minimum of the function $(f \circ E)$, then $\nabla(f \circ E)(\overline{x}) = 0$.*

Proof Suppose that $\nabla(f \circ E)(\overline{x}) \neq 0$ and let $d = -\nabla(f \circ E)(\overline{x})$, then $\nabla(f \circ E)(\overline{x})d = -\|\nabla(f \circ E)(\overline{x})\|^2 < 0$. By Theorem 15 there exists $\delta > 0$ such that

$$(f \circ E)(\overline{x} + \lambda d) < (f \circ E)(\overline{x}) \quad \text{for each } \lambda \in (0, \delta)$$

contradicting the assumption that $\overline{x}$ is a local minimum of $(f \circ E)(x)$, and thus $\nabla(f \circ E)(\overline{x}) = 0$. □

Theorem 17 *Let M be an E-convex set, $E : R^n \to R^n$ be a one-to-one and onto operator, and $f : M \subseteq R^n \to R$ be twice E-differentiable and strictly E-convex function at $\overline{x}$. If $\overline{x}$ is a local minimum of $(f \circ E)$, then $\nabla(f \circ E)(\overline{x}) = 0$ and the Hessian matrix $H(\overline{x}) = \nabla^2(f \circ E)(\overline{x})$ is positive semidefinite.*

Proof Suppose that d is an arbitrary direction. Since f is a twice E-differentiable function at $\bar{x}$, then

$$(f \circ E)(\bar{x} + \lambda d) = (f \circ E)(\bar{x}) + \lambda \nabla (f \circ E)(\bar{x}) d + \frac{1}{2}\lambda^2 d^t \nabla^2 (f \circ E)(\bar{x}) d + \lambda^2 \|d\|^2 \alpha(\bar{x}, \lambda d),$$

where $\alpha(\bar{x}, \lambda d) \to 0$ as $\lambda \to 0$.

From Corollary 16 we have $\nabla(f \circ E)(\bar{x}) = 0$, and

$$\frac{(f \circ E)(\bar{x} + \lambda d) - (f \circ E)(\bar{x})}{\lambda^2} = \frac{1}{2} d^t \nabla^2 (f \circ E)(\bar{x}) d.$$

Since $\bar{x}$ is a local minimum of $(f \circ E)$, then $(f \circ E)(\bar{x}) < (f \circ E)(\bar{x} + \lambda d)$, and

$$d^t \nabla^2 (f \circ E)(\bar{x}) d \geq 0, \quad \textit{i.e.}, \quad H(\bar{x}) = \nabla^2 (f \circ E)(\bar{x}) \quad \text{is positive semidefinite.}$$ □

Example 18 Let $f(x, y) = x + 2y^2 - 2x^{\frac{1}{3}}$ be a non-differentiable function at $(0, y)$, and let $E(x, y) = (x^3, y)$, then $(f \circ E)(x, y) = x^3 + 2y^2 - 2x$, and

$$\frac{\partial (f \circ E)}{\partial x} = 3x^2 - 2 = 0 \quad \text{implies} \quad x = \pm\sqrt{\frac{2}{3}},$$

$$\frac{\partial (f \circ E)}{\partial y} = 4y = 0 \quad \text{implies} \quad y = 0,$$

$$\frac{\partial^2 (f \circ E)}{\partial x^2} = 6x, \qquad \frac{\partial^2 (f \circ E)}{\partial y \, \partial x} = 0, \qquad \frac{\partial^2 (f \circ E)}{\partial y^2} = 4, \qquad \frac{\partial^2 (f \circ E)}{\partial x \, \partial y} = 0.$$

Then $(x_1, y_1) = (\sqrt{\frac{2}{3}}, 0)$ and $(x_2, y_2) = (-\sqrt{\frac{2}{3}}, 0)$ are extremum points of $(f \circ E)(x, y)$, and the Hessian matrix $H(\sqrt{\frac{2}{3}}, 0) = \begin{bmatrix} 6\sqrt{\frac{2}{3}} & 0 \\ 0 & 4 \end{bmatrix}$ is positive definite. And thus the point $(\sqrt{\frac{2}{3}}, 0)$ is a local minimum of the function $(f \circ E)(x, y)$, but the Hessian matrix $H(-\sqrt{\frac{2}{3}}, 0) = \begin{bmatrix} -6\sqrt{\frac{2}{3}} & 0 \\ 0 & 4 \end{bmatrix}$ is indefinite.

Theorem 19 *Let M be an E-convex set, let $E : R^n \to R^n$ be a one-to-one and onto operator, and let $f : M \subseteq R^n \to R$ be a twice E-differentiable and strictly E-convex function at $\bar{x}$. If $\nabla(f \circ E)(\bar{x}) = 0$ and the Hessian matrix $H(\bar{x}) = \nabla^2(f \circ E)(\bar{x})$ is positive definite, then $\bar{x}$ is a local minimum of $(f \circ E)$.*

Proof Suppose that $\bar{x}$ is not a local minimum of $(f \circ E)(x)$, and there exists a sequence $\{x_k\}$ is converging to $\bar{x}$ such that $(f \circ E)(x_k) < (f \circ E)(\bar{x})$ for each k. Since $\nabla(f \circ E)(\bar{x}) = 0$, and f is twice E-differentiable at $\bar{x}$, then

$$(f \circ E)(x_k) = (f \circ E)(\bar{x}) + \lambda \nabla (f \circ E)(\bar{x})(x_k - \bar{x}) + \frac{1}{2}(x_k - \bar{x})^t \nabla^2 (f \circ E)(\bar{x})(x_k - \bar{x}) + \left\|(x_k - \bar{x})\right\|^2 \alpha\big(\bar{x}, (x_k - \bar{x})\big),$$

where $\alpha(\bar{x}, (x_k - \bar{x})) \to 0$ as $k \to \infty$, and

$$\frac{1}{2}(x_k - \bar{x})^t \nabla^2 (f \circ E)(\bar{x})(x_k - \bar{x}) + \left\|(x_k - \bar{x})\right\|^2 \alpha\big(\bar{x}, (x_k - \bar{x})\big) < 0 \quad \text{for each } k.$$

By dividing on $\|(x_k-\overline{x})\|^2$, and letting $d_k = \frac{(x_k-\overline{x})}{\|(x_k-\overline{x})\|}$, we get

$$\frac{1}{2}d_k^t\nabla^2(f\circ E)(\overline{x})d_k + \alpha\bigl(\overline{x},(x_k-\overline{x})\bigr) < 0 \quad \text{for each } k.$$

But $\|d_k\| = 1$ for each k, and hence there exists an index set K such that $\{d_k\}_K \to d$, where $\|d\| = 1$. Considering this subsequence and the fact that $\alpha(\overline{x},(x_k-\overline{x})) \to 0$ as $k \to \infty$, then $d^t\nabla^2(f\circ E)(\overline{x})d < 0$. This contradicts the assumption that $H(\overline{x})$ is positive definite. Therefore $\overline{x}$ is indeed a local minimum. □

Example 20 Let $f(x,y) = x^{\frac{2}{3}} + y^2 - 1$ be a non-differentiable at the point $(0,y)$, and let $E(x,y) = (x^3,y)$, then $(f\circ E)(x,y) = x^2 + y^2 - 1$

$$\frac{\partial(f\circ E)}{\partial x} = 2x, \qquad \frac{\partial^2(f\circ E)}{\partial y\,\partial x} = 0, \qquad \frac{\partial^2(f\circ E)}{\partial x^2} = 2,$$
$$\frac{\partial(f\circ E)}{\partial y} = 2y, \qquad \frac{\partial^2(f\circ E)}{\partial x\,\partial y} = 0, \qquad \frac{\partial^2(f\circ E)}{\partial y^2} = 2.$$

The necessary condition for $\overline{x}$ is a local minimum of $(f\circ E)$ is $\nabla(f\circ E)(\overline{x}) = 0$, then $\overline{x} = (0,0)$, and the Hessian matrix $H(\overline{x})$

$$H = \begin{bmatrix} \frac{\partial^2(f\circ E)}{\partial x^2} & \frac{\partial^2(f\circ E)}{\partial y\,\partial x} \\ \frac{\partial^2(f\circ E)}{\partial x\,\partial y} & \frac{\partial^2(f\circ E)}{\partial y^2} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

is positive definite.

Example 21 Let $f(x,y) = x^{\frac{1}{3}} + y - 1$ be non-differentiable at the point $(0,y)$, and let $E(x,y) = (x^3,y)$, then $(f\circ E)(x,y) = x + y - 1$.

Now, let $M = \{\lambda_1(0,0) + \lambda_2(0,3) + \lambda_3(1,2) + \lambda_4(1,0)\} \cup \{\lambda_1(0,0) + \lambda_2(0,-3) + \lambda_3(1,-2) + \lambda_4(1,0)\}$, $\sum_{i=1}^4 \lambda_i = 1$, $\lambda_i \geq 0$ be an E-convex set with respect to operator E (the feasible region is shown in Figure 1) and

$$f(0,0) = -1, \qquad (f\circ E)(0,0) = -1, \qquad f(0,-3) = -4, \qquad (f\circ E)(0,3) = -4,$$
$$f(1,2) = 2, \qquad (f\circ E)(1,2) = 2, \qquad f(1,0) = 0, \qquad (f\circ E)(1,0) = 0,$$
$$f(0,3) = 2, \qquad (f\circ E)(0,3) = 2, \qquad f(1,-2) = -2, \qquad (f\circ E)(1,2) = -2.$$

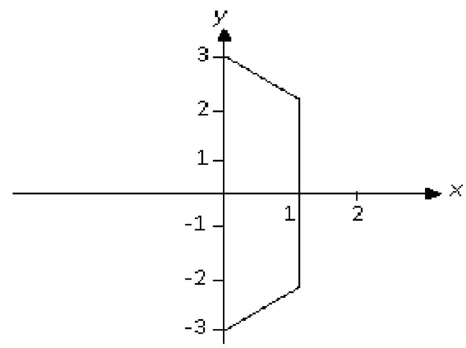
Then $\overline{x} = (0,-3)$ is a solution of the problem P_E and $E(\overline{x}) = E(0,-3) = (0,-3)$ is a solution of the problem P.

Definition 22 Let M be a nonempty E-convex set in R^n and let $E(\overline{x}) \in clM$. The cone of feasible direction of $E(M)$ at $E(\overline{x})$ denoted by D is given by

$$D = \bigl\{d : d \neq 0, E(\overline{x}) + \lambda d \in M \text{ for each } \lambda \in [0,\delta], \delta > 0\bigr\}.$$

Lemma 23 *Let M be an E-convex set with respect to an operator $E : R^n \to R^n$, and let $f : M \subseteq R^n \to R$ be E-differentiable at $\overline{x}$. If $\overline{x}$ is a local minimum of the problem P_E, then*

Figure 1 The feasible region M.



$F_0 \cap D = \phi$, where $F_0 = \{d : \nabla(f \circ E)(\overline{x})d < 0\}$, and D is the cone of feasible direction of M at $\overline{x}$.

Proof Suppose that there exists a vector $d \in F_0 \cap D$. Then by Theorem 15, there exists δ_1 such that

$$(f \circ E)(\overline{x} + \lambda d) < (f \circ E)(\overline{x}) \quad \text{for each } \lambda \in (0, \delta_1). \tag{3.1}$$

By the definition of the cone of feasible direction, there exists δ_2 such that

$$E(\overline{x}) + \lambda d \in M \quad \text{for each } \lambda \in (0, \delta_2). \tag{3.2}$$

From 3.1 and 3.2 we have $(f \circ E)(\overline{x} + \lambda d) < (f \circ E)(\overline{x})$ for each $\lambda \in (0, \delta)$, where $\delta = \min\{\delta_1, \delta_2\}$, which contradicts the assumption that $\overline{x}$ is a local optimal solution, then $F_0 \cap D = \phi$. □

Lemma 24 *Let M be an open E-convex set with respect to an operator $E : R^n \to R^n$, let $f : M \subseteq R^n \to R$ be E-differentiable at $\overline{x}$ and let $g_i : R^n \to R$ for $i = 1, 2, \dots, m$. Let $\overline{x}$ be a feasible solution of the problem P_E and let $I = \{i : (g_i \circ E)(\overline{x}) = 0\}$. Furthermore, suppose that g_i for $i \in I$ is E-differentiable at $\overline{x}$ and that g_i for $i \notin I$ is continuous at $\overline{x}$. If $\overline{x}$ is a local optimal solution, then $F_0 \cap G_0 = \phi$, where*

$$F_0 = \{d : \nabla(f \circ E)(\overline{x})d < 0\},$$
$$G_0 = \{d : \nabla(g_i \circ E)(\overline{x})d < 0, \text{ for each } i \in I\}$$

and E is one-to-one and onto.

Proof Let $d \in G_0$. Since $E(\overline{x}) \in M$ and M is an open E-convex set, there exists a $\delta_1 > 0$ such that

$$E(\overline{x}) + \lambda d \in M \quad \text{for } \lambda \in (0, \delta_1). \tag{3.3}$$

Also, since $(g_i \circ E)(\overline{x}) < 0$ and since g_i is continuous at $\overline{x}$ for $i \notin I$, there exists a $\delta_2 > 0$ such that

$$(g_i \circ E)(\overline{x} + \lambda d) < 0 \quad \text{for } \lambda \in (0, \delta_2) \text{ and for } i \notin I. \tag{3.4}$$

Finally, since $d \in G_0$, $\nabla(g_i \circ E)(\overline{x})d < 0$ for each $i \in I$ and by Theorem 15, there exists $\delta_3 > 0$ such that

$$(g_i \circ E)(\overline{x} + \lambda d) < (g_i \circ E)(\overline{x}) \quad \text{for } \lambda \in (0, \delta_3) \text{ and } i \in I. \tag{3.5}$$

From 3.3, 3.4 and 3.5, it is clear that points of the form $E(\overline{x}) + \lambda d$ are feasible to the problem P_E for each $\lambda \in (0, \delta)$, where $\delta = \min(\delta_1, \delta_2, \delta_3)$. Thus $d \in D$, where D is the cone of feasible direction of the feasible region at $\overline{x}$. We have shown that for $d \in G_0$ implies that $d \in D$, and hence $G_0 \subset D$. By Lemma 23, since $\overline{x}$ is a local solution of the problem $P_E, F_0 \cap D = \phi$. It follows that $F_0 \cap G_0 = \phi$. □

Theorem 25 (Fritz-John optimality conditions) *Let M be an open E-convex set with respect to the one-to-one and onto operator $E : R^n \to R^n$, let $f : M \subseteq R^n \to R$ be E-differentiable at $\overline{x}$ and let $g_i : R^n \to R$ for $i = 1, 2, \ldots, m$. Let $\overline{x}$ be feasible solution of the problem P_E and let $I = \{i : (g_i \circ E)(\overline{x}) = 0\}$. Furthermore, suppose that g_i for $i \in I$ is differentiable at $\overline{x}$ and that g_i for $i \notin I$ is continuous at $\overline{x}$. If $\overline{x}$ is a local optimal solution, then there exist scalars u_0 and u_i for $i \in I$ such that*

$$u_\circ \nabla(f \circ E)(\overline{x}) + \sum_{i \in I} u_i \nabla(g_i \circ E)(\overline{x}) = 0, \quad u_\circ, u_i \geq 0 \textit{ for } i \in I,$$

$$(u_\circ, u_i) \neq (0,0) \quad \textit{for } i \in I$$

and $E(\overline{x})$ is a local solution of the problem P.

Proof Let $\overline{x}$ be a local solution of the problem P_E, then there is no vector d such that $\nabla(f \circ E)(\overline{x})d < 0$ and $\nabla(g_i \circ E)(\overline{x})d < 0$. Let A be a matrix with rows $\nabla(f \circ E)(\overline{x})$ and $\nabla(g_i \circ E)(\overline{x})$. From Gordon's theorem [10], we have the system $Ad < 0$ is inconsistent, then there exists a vector $b \geq 0$ such that $Ab = 0$, where $b = (u_0 \cdot u_i)$ for each $i \in I$. And thus

$$u_\circ \nabla(f \circ E)(\overline{x}) + \sum_{i \in I} u_i \nabla(g_i \circ E)(\overline{x}) = 0$$

holds and $E(\overline{x})$ is a local solution of the problem P. □

Theorem 26 *Let $E : R^n \to R^n$ be a one-to-one and onto operator and let $f : M \subseteq R^n \to R$ be an E-differentiable function. If $\overline{x}$ is an optimal solution of the problem P, then there exists $\overline{y} \in M'$ such that $\overline{x} = E(\overline{y})$ is an optimal solution of the problem P_E and the Fritz-John optimality condition of the problem P_E is satisfied.*

Proof Let $\overline{x}$ be an optimal solution of the problem P. Since E is one-to-one and onto, according to Theorem 13, there exists $\overline{y} \in M', \overline{x} = E(\overline{y})$ is an optimal solution of the problem P_E. Hence there exist scalars $u_0.u_i$ satisfying the Fritz-John optimality conditions of the problem P_E

$$u_\circ \nabla(f \circ E)(\overline{x}) + \sum_{i \in I} u_i \nabla(g_i \circ E)(\overline{x}) = 0,$$

$$(u_0, u_i) = 0,$$

$$u_0, u_i \geq 0.$$

□

Theorem 27 (Kuhn-Tucker necessary condition) *Let M be an open E-convex set with respect to the one-to-one and onto operator $E : R^n \to R^n$, let $f : M \subseteq R^n \to R$ be E-differentiable and strictly E-convex at $\bar{x}$ and let $g_i : R^n \to R$ for $i = 1, 2, \dots, m$. Let $\bar{y}$ be a feasible solution of the problem P_E and let $I = \{i : (g_i \circ E)(\bar{y}) = 0\}$. Furthermore, suppose that $(g_i \circ E)$ is continuous at $\bar{y}$ for $i \notin I$ and $\nabla(g_i \circ E)(\bar{y})$ for $i \in I$ are linearly independent. If $\bar{x}$ is a solution of the problem P, $\bar{x} = E(\bar{y})$ and $\bar{y}$ is a local solution of the problem P_E, then there exist scalars u_i for $i \in I$ such that*

$$\nabla(f \circ E)(\bar{y}) + \sum_{i\in I} u_i \nabla(g_i \circ E)(\bar{y}) = 0, \quad u_i \geq 0 \text{ for each } i \in I.$$

Proof From the Fritz-John optimality condition theorem, there exist scalars u_0 and u_i for each $i \in I$ such that

$$u_0 \nabla(f \circ E)(\bar{y}) + \sum_{i\in I} \widehat{u}_i \nabla(g_i \circ E)(\bar{y}) = 0, \quad u_0, \widehat{u}_i \geq 0 \text{ for each } i \in I.$$

If $u_0 = 0$, the assumption of linear independence of $\nabla(g_i \circ E)(\bar{y})$ does not hold, then $u_0 > 0$. By taking $u_i = \frac{\widehat{u}_i}{u_0}$, then $\nabla(f \circ E)(\bar{y}) + \sum_{i\in I} u_i \nabla(g_i \circ E)(\bar{y}) = 0$, $u_i \geq 0$ holds for each $i \in I$. From Theorem 26, $\bar{y}$ is a local solution of the problem P_E. □

Theorem 28 *Let M be an open E-convex set with respect to the one-to-one and onto operator $E : R^n \to R^n$, $g_i : R^n \to R$ for $i = 1, 2, \dots, m$, and let $f : M \subseteq R^n \to R$ be E-differentiable at $\bar{x}$ and strictly E-convex at $\bar{x}$. Let $\bar{x} = E(\bar{y})$ be a feasible solution of the problem P_E and $I = \{i : (g_i \circ E)(\bar{y}) = 0\}$. Suppose that f is pseudo-E-convex at $\bar{y}$ and that g_i is quasi-E-convex and differentiable at $\bar{y}$ for each $i \in I$. Furthermore, suppose that the Kuhn-Tucker conditions hold at $\bar{y}$. Then $\bar{y}$ is a global optimal solution of the problem P_E and hence $\bar{x} = E(\bar{y})$ is a solution of the problem P.*

Proof Let $\widehat{y}$ be a feasible solution of the problem P_E, then $(g_i \circ E)(\widehat{y}) \leq (g_i \circ E)(\bar{y})$ for each $i \in I$. Since $(g_i \circ E)(\widehat{y}) \leq 0$, $(g_i \circ E)(\bar{y}) = 0$ and g_i is quasi-E-convex at $\bar{y}$, then

$$\begin{aligned}
(g_i \circ E)\big(\bar{y} + \lambda(\widehat{y} - \bar{y})\big) &= (g_i \circ E)\big(\lambda\widehat{y} + (1-\lambda)\bar{y}\big) \\
&\leq \max\big\{(g_i \circ E)(\widehat{y}), (g_i \circ E)(\bar{y})\big\} \\
&= (g_i \circ E)(\bar{y}).
\end{aligned}$$

This means that $(g_i \circ E)$ does not increase by moving from $\bar{y}$ along the direction $\widehat{y} - \bar{y}$. Then we must have from Theorem 15 that $\nabla(g_i \circ E)(y - \bar{y}) \leq 0$. Multiplying by u_i and summing over I, we get

$$\left[\sum_{i\in I} u_i \nabla(g_i \circ E)(\bar{y})\right](y - \bar{y}) \leq 0.$$

But since

$$\nabla(f \circ E)(\bar{y}) + \sum_{i\in I} u_i \nabla(g_i \circ E)(\bar{y}) = 0,$$

it follows that $\nabla(f \circ E)(\overline{y})(y - \overline{y}) \geq 0$. Since f is pseudo E-convex at $\overline{y}$, we get

$$(f \circ E)(y) \geq (f \circ E)(\overline{y}).$$

Then $\overline{y}$ is a global solution of the problem P_E and from Theorem 13 $\overline{x} = E(\overline{y})$ is a global solution of the problem P. □

Example 29 Consider the following problem (problem P):

$$\begin{aligned}
&\operatorname{Min} f(x,y) = x^{\frac{2}{3}} + y^2, \\
&\text{subject to } x^2 + y^2 \leq 5, \\
&x + 2y \leq 4, \\
&x, y \geq 0.
\end{aligned}$$

The feasible region of this problem is shown in Figure 2.

Let $E(x,y) = (\frac{1}{8}x^3, \frac{1}{3}y)$, then the problem P_E is as follows:

$$\begin{aligned}
&\min(f \circ E)(x,y) = \frac{1}{4}x^2 + \frac{1}{9}y^2, \\
&\text{subject to } \frac{x^6}{64} + \frac{y^2}{9} \leq 5, \\
&\frac{1}{8}x^3 + \frac{2}{3}y \leq 4, \\
&x, y \geq 0.
\end{aligned}$$

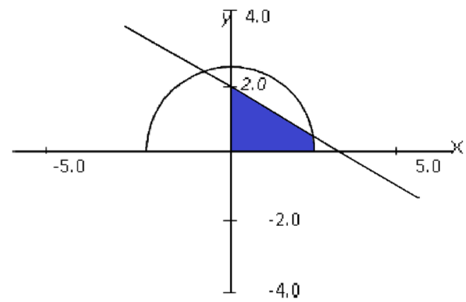
We note that $E(M) \subset M$, where

$$\begin{aligned}
&(\sqrt{5}, 0) \in M \quad \text{implies} \quad E(\sqrt{5}, 0) = \left(5\frac{\sqrt{5}}{8}, 0\right) \in M, \\
&(0,2) \in M \quad \text{implies} \quad E(0,2) = \left(0, \frac{2}{3}\right) \in M, \\
&(0,0) \in M \quad \text{implies} \quad E(0,0) = (0,0) \in M, \\
&(2,1) \in M \quad \text{implies} \quad E(2,1) = \left(1, \frac{1}{3}\right) \in M.
\end{aligned}$$

The Kuhn-Tucker conditions are as follows:

$$\begin{aligned}
&\nabla(f \circ E)(x,y) + u_1 \nabla(g_1 \circ E)(x,y) + u_2 \nabla(g_2 \circ E)(x,y) = 0, \\
&\begin{bmatrix} \frac{1}{2}x \\ \frac{2}{9}y \end{bmatrix} + u_1 \begin{bmatrix} \frac{6}{64}x^5 \\ \frac{2}{9}y \end{bmatrix} + u_2 \begin{bmatrix} \frac{3}{8}x^2 \\ \frac{2}{3} \end{bmatrix} = 0, \\
&u_1 \left[\frac{x^6}{64} + \frac{y^2}{9} - 5\right] = 0, \\
&u_2 \left[\frac{1}{8}x^3 + \frac{2}{3}y - 4\right] = 0.
\end{aligned}$$

Figure 2 The feasible region M.



The solution is $\{[x = 0.0, u_1 = 0.0, u_2 = 0.0, y = 0.0]\}$, $\bar{z} = (0, 0)$, and $\bar{x} = E(\bar{z}) = (0, 0)$ is a solution of the problem P.

4 Conclusion

In this paper we introduced a new definition of an E-differentiable convex function, which transforms a non-differentiable function to a differentiable function under an operator $E : R^n \to R^n$, and we studied Kuhn-Tucker and Fritz-John conditions for obtaining an optimal solution of mathematical programming with a non-differentiable function. At the end, some examples have been presented to clarify the results.

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