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# On some Hadamard-type inequalities for $(h_1, h_2)$ -preinvex functions on the co-ordinates

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## Abstract

We introduce the class of  $(h_1, h_2)$ -preinvex functions on the co-ordinates, and we prove some new inequalities of Hermite-Hadamard and Fejér type for such mappings.

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## 1 Introduction

A function  $f : I \rightarrow R$ ,  $I \subseteq R$  is an interval, is said to be a convex function on  $I$  if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \quad (1.1)$$

holds for all  $x, y \in I$  and  $t \in [0, 1]$ . If the reversed inequality in (1.1) holds, then  $f$  is concave.

Many important inequalities have been established for the class of convex functions, but the most famous is the Hermite-Hadamard inequality. This double inequality is stated as follows:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a)+f(b)}{2}, \quad (1.2)$$

where  $f : [a, b] \rightarrow R$  is a convex function. The above inequalities are in reversed order if  $f$  is a concave function.

In 1978, Breckner introduced an  $s$ -convex function as a generalization of a convex function [1].

Such a function is defined in the following way: a function  $f : [0, \infty) \rightarrow R$  is said to be  $s$ -convex in the second sense if

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y) \quad (1.3)$$

holds for all  $x, y \in \infty$ ,  $t \in [0, 1]$  and for fixed  $s \in (0, 1]$ .

Of course,  $s$ -convexity means just convexity when  $s = 1$ .

In [2], Dragomir and Fitzpatrick proved the following variant of the Hermite-Hadamard inequality, which holds for  $s$ -convex functions in the second sense:

$$2^{s-1}f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a)+f(b)}{s+1}. \tag{1.4}$$

In the paper [3] a large class of non-negative functions, the so-called  $h$ -convex functions, is considered. This class contains several well-known classes of functions such as non-negative convex functions and  $s$ -convex in the second sense functions. This class is defined in the following way: a non-negative function  $f : I \rightarrow R$ ,  $I \subseteq R$  is an interval, is called  $h$ -convex if

$$f(tx + (1-t)y) \leq h(t)f(x) + h(1-t)f(y) \tag{1.5}$$

holds for all  $x, y \in I$ ,  $t \in (0, 1)$ , where  $h : J \rightarrow R$  is a non-negative function,  $h \neq 0$  and  $J$  is an interval,  $(0, 1) \subseteq J$ .

In the further text, functions  $h$  and  $f$  are considered without assumption of non-negativity.

In [4] Sarikaya, Saglam and Yildirim proved that for an  $h$ -convex function the following variant of the Hadamard inequality is fulfilled:

$$\frac{1}{2h(\frac{1}{2})}f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq [f(a)+f(b)] \cdot \int_0^1 h(t) dt. \tag{1.6}$$

In [5] Bombardelli and Varošanec proved that for an  $h$ -convex function the following variant of the Hermite-Hadamard-Fejér inequality holds:

$$\begin{aligned} \frac{\int_a^b w(x) dx}{2h(\frac{1}{2})}f\left(\frac{a+b}{2}\right) &\leq \int_a^b f(x)w(x) dx \\ &\leq (b-a)(f(a)+f(b)) \int_0^1 h(t)w(ta+(1-t)b) dt, \end{aligned} \tag{1.7}$$

where  $w : [a, b] \rightarrow R$ ,  $w \geq 0$  and symmetric with respect to  $\frac{a+b}{2}$ .

A modification for convex functions, which is also known as co-ordinated convex functions, was introduced by Dragomir [6] as follows.

Let us consider a bidimensional  $\Delta = [a, b] \times [c, d]$  in  $R^2$  with  $a < b$  and  $c < d$ . A mapping  $f : \Delta \rightarrow R$  is said to be convex on the co-ordinates on  $\Delta$  if the partial mappings  $f_y : [a, b] \rightarrow R$ ,  $f_y(u) = f(u, y)$  and  $f_x : [c, d] \rightarrow R$ ,  $f_x(v) = f(x, v)$  are convex for all  $x \in [a, b]$  and  $y \in [c, d]$ .

In the same article, Dragomir established the following Hadamard-type inequalities for convex functions on the co-ordinates:

$$\begin{aligned} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) &\leq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) dx dy \\ &\leq \frac{f(a, c) + f(b, c) + f(a, d) + f(b, d)}{4}. \end{aligned} \tag{1.8}$$

The concept of  $s$ -convex functions on the co-ordinates was introduced by Alomari and Darus [7]. Such a function is defined in following way: the mapping  $f : \Delta \rightarrow R$  is  $s$ -convex

in the second sense if the partial mappings  $f_y : [a, b] \rightarrow R$  and  $f_x : [c, d] \rightarrow R$  are  $s$ -convex in the second sense.

In the same paper, they proved the following inequality for an  $s$ -convex function:

$$4^{s-1}f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \leq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y) dx dy \leq \frac{f(a,c) + f(b,c) + f(a,d) + f(b,d)}{(s+1)^2}. \tag{1.9}$$

For refinements and counterparts of convex and  $s$ -convex functions on the co-ordinates, see [6–10].

The main purpose of this paper is to introduce the class of  $(h_1, h_2)$ -preinvex functions on the co-ordinates and establish new inequalities like those given by Dragomir in [6] and Bombardelli and Varošanec in [5].

Throughout this paper, we assume that considered integrals exist.

## 2 Main results

Let  $f : X \rightarrow R$  and  $\eta : X \times X \rightarrow R^n$ , where  $X$  is a nonempty closed set in  $R^n$ , be continuous functions. First, we recall the following well-known results and concepts; see [11–16] and the references therein.

**Definition 2.1** Let  $u \in X$ . Then the set  $X$  is said to be invex at  $u$  with respect to  $\eta$  if

$$u + t\eta(v, u) \in X$$

for all  $v \in X$  and  $t \in [0, 1]$ .

$X$  is said to be an invex set with respect to  $\eta$  if  $X$  is invex at each  $u \in X$ .

**Definition 2.2** The function  $f$  on the invex set  $X$  is said to be preinvex with respect to  $\eta$  if

$$f(u + t\eta(v, u)) \leq (1-t)f(u) + tf(v)$$

for all  $u, v \in X$  and  $t \in [0, 1]$ .

We also need the following assumption regarding the function  $\eta$  which is due to Mohan and Neogy [11].

**Condition C** Let  $X \subseteq R$  be an open invex subset with respect to  $\eta$ . For any  $x, y \in X$  and any  $t \in [0, 1]$ ,

$$\eta(y, y + t\eta(x, y)) = -t\eta(x, y),$$

$$\eta(x, y + t\eta(x, y)) = (1-t)\eta(x, y).$$

Note that for every  $x, y \in X$  and every  $t_1, t_2 \in [0, 1]$  from Condition C, we have

$$\eta(y + t_2\eta(x, y), y + t_1\eta(x, y)) = (t_2 - t_1)\eta(x, y).$$

In [12], Noor proved the Hermite-Hadamard inequality for preinvex functions

$$f\left(a + \frac{1}{2}\eta(b, a)\right) \leq \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} f(x) dx \leq \frac{f(a) + f(b)}{2}. \tag{2.1}$$

**Definition 2.3** Let  $h : [0, 1] \rightarrow R$  be a non-negative function,  $h \not\equiv 0$ . The non-negative function  $f$  on the invex set  $X$  is said to be  $h$ -preinvex with respect to  $\eta$  if

$$f(u + t\eta(v, u)) \leq h(1 - t)f(u) + h(t)f(v)$$

for each  $u, v \in X$  and  $t \in [0, 1]$ .

Let us note that:

- if  $\eta(v, u) = v - u$ , then we get the definition of an  $h$ -convex function introduced by Varošanec in [3];
- if  $h(t) = t$ , then our definition reduces to the definition of a preinvex function;
- if  $\eta(v, u) = v - u$  and  $h(t) = t$ , then we obtain the definition of a convex function.

Now let  $X_1$  and  $X_2$  be nonempty subsets of  $R^n$ , let  $\eta_1 : X_1 \times X_1 \rightarrow R^n$  and  $\eta_2 : X_2 \times X_2 \rightarrow R^n$ .

**Definition 2.4** Let  $(u, v) \in X_1 \times X_2$ . We say  $X_1 \times X_2$  is invex at  $(u, v)$  with respect to  $\eta_1$  and  $\eta_2$  if for each  $(x, y) \in X_1 \times X_2$  and  $t_1, t_2 \in [0, 1]$ ,

$$(u + t_1\eta_1(x, u), v + t_2\eta_2(y, v)) \in X_1 \times X_2.$$

$X_1 \times X_2$  is said to be an invex set with respect to  $\eta_1$  and  $\eta_2$  if  $X_1 \times X_2$  is invex at each  $(u, v) \in X_1 \times X_2$ .

**Definition 2.5** Let  $h_1$  and  $h_2$  be non-negative functions on  $[0, 1]$ ,  $h_1 \not\equiv 0$ ,  $h_2 \not\equiv 0$ . The non-negative function  $f$  on the invex set  $X_1 \times X_2$  is said to be co-ordinated  $(h_1, h_2)$ -preinvex with respect to  $\eta_1$  and  $\eta_2$  if the partial mappings  $f_y : X_1 \rightarrow R, f_y(x) = f(x, y)$  and  $f_x : X_2 \rightarrow R, f_x(y) = f(x, y)$  are  $h_1$ -preinvex with respect to  $\eta_1$  and  $h_2$ -preinvex with respect to  $\eta_2$ , respectively, for all  $y \in X_2$  and  $x \in X_1$ .

If  $\eta_1(x, u) = x - u$  and  $\eta_2(y, v) = y - v$ , then the function  $f$  is called  $(h_1, h_2)$ -convex on the co-ordinates.

**Remark 1** From the above definition it follows that if  $f$  is a co-ordinated  $(h_1, h_2)$ -preinvex function, then

$$\begin{aligned} & f(x + t_1\eta_1(b, x), y + t_2\eta_2(d, y)) \\ & \leq h_1(1 - t_1)f(x, y + t_2\eta_2(d, y)) + h_1(t_1)f(b, y + t_2\eta_2(d, y)) \\ & \leq h_1(1 - t_1)h_2(1 - t_2)f(x, y) + h_1(1 - t_1)h_2(t_2)f(x, d) \\ & \quad + h_1(t_1)h_2(1 - t_2)f(b, y) + h_1(t_1)h_2(t_2)f(b, d). \end{aligned}$$

**Remark 2** Let us note that if  $\eta_1(x, u) = x - u$ ,  $\eta_2(y, v) = y - v$ ,  $t_1 = t_2$  and  $h_1(t) = h_2(t) = t$ , then our definition of a co-ordinated  $(h_1, h_2)$ -preinvex function reduces to the definition

of a convex function on the co-ordinates proposed by Dragomir [6]. Moreover, if  $h_1(t) = h_2(t) = t^s$ , then our definition reduces to the definition of an  $s$ -convex function on the co-ordinates proposed by Alomari and Darus [7].

Now, we will prove the Hadamard inequality for the new class functions.

**Theorem 2.1** *Suppose that  $f : [a, a + \eta(b, a)] \rightarrow R$  is an  $h$ -preinvex function, Condition C for  $\eta$  holds and  $a < a + \eta(b, a)$ ,  $h(\frac{1}{2}) > 0$ . Then the following inequalities hold:*

$$\frac{1}{2h(\frac{1}{2})}f\left(a + \frac{1}{2}\eta(b, a)\right) \leq \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} f(x) dx \leq [f(a) + f(b)] \cdot \int_0^1 h(t) dt. \quad (2.2)$$

*Proof* From the definition of an  $h$ -preinvex function, we have that

$$f(a + t\eta(b, a)) \leq h(1 - t)f(a) + h(t)f(b).$$

Thus, by integrating, we obtain

$$\int_0^1 f(a + t\eta(b, a)) dt \leq f(a) \int_0^1 h(1 - t) dt + f(b) \int_0^1 h(t) dt = [f(a) + f(b)] \int_0^1 h(t) dt.$$

But

$$\int_0^1 f(a + t\eta(b, a)) dt = \frac{1}{\eta(b, a)} \cdot \int_a^{a+\eta(b, a)} f(x) dx.$$

So,

$$\frac{1}{\eta(b, a)} \cdot \int_a^{a+\eta(b, a)} f(x) dx \leq [f(a) + f(b)] \int_0^1 h(t) dt.$$

The proof of the second inequality follows by using the definition of an  $h$ -preinvex function, Condition C for  $\eta$  and integrating over  $[0, 1]$ .

That is,

$$\begin{aligned} f\left(a + \frac{1}{2}\eta(b, a)\right) &= f(a + t\eta(b, a) + \frac{1}{2}\eta(a + (1 - t)\eta(b, a), a + t\eta(b, a))) \\ &\leq h\left(\frac{1}{2}\right) [f(a + t\eta(b, a)) + f(a + (1 - t)\eta(b, a))], \\ f\left(a + \frac{1}{2}\eta(b, a)\right) &\leq h\left(\frac{1}{2}\right) \left[ \int_0^1 f(a + t\eta(b, a)) dt + \int_0^1 f(a + (1 - t)\eta(b, a)) dt \right], \\ f\left(a + \frac{1}{2}\eta(b, a)\right) &\leq 2 \cdot h\left(\frac{1}{2}\right) \frac{1}{\eta(b, a)} \cdot \int_a^{a+\eta(b, a)} f(x) dx. \end{aligned}$$

The proof is complete. □

**Theorem 2.2** *Suppose that  $f : [a, a + \eta_1(b, a)] \times [c, c + \eta_2(d, c)] \rightarrow R$  is an  $(h_1, h_2)$ -preinvex function on the co-ordinates with respect to  $\eta_1$  and  $\eta_2$ , Condition C for  $\eta_1$  and  $\eta_2$  is fulfilled,*

and  $a < a + \eta_1(b, a)$ ,  $c < c + \eta_2(d, c)$ , and  $h_1(\frac{1}{2}) > 0$ ,  $h_2(\frac{1}{2}) > 0$ . Then one has the following inequalities:

$$\begin{aligned}
 & \frac{1}{4h_1(\frac{1}{2})h_2(\frac{1}{2})}f\left(a + \frac{1}{2}\eta_1(b, a), c + \frac{1}{2}\eta_2(d, c)\right) \\
 & \leq \frac{1}{4 \cdot h_1(\frac{1}{2})\eta_2(d, c)} \int_c^{c+\eta_2(d, c)} f\left(a + \frac{1}{2}\eta_1(b, a), y\right) dy \\
 & \quad + \frac{1}{4 \cdot h_2(\frac{1}{2})\eta_1(b, a)} \int_a^{a+\eta_1(b, a)} f\left(x, c + \frac{1}{2}\eta_2(d, c)\right) dx \\
 & \leq \frac{1}{\eta_1(b, a)\eta_2(d, c)} \int_a^{a+\eta_1(b, a)} \int_c^{c+\eta_2(d, c)} f(x, y) dx dy \\
 & \leq \frac{1}{2\eta_1(b, a)} \int_0^1 h_2(t_2) dt_2 \left[ \int_a^{a+\eta_1(b, a)} f(x, c) dx + \int_a^{a+\eta_1(b, a)} f(x, d) dx \right] \\
 & \quad + \frac{1}{2\eta_2(d, c)} \int_0^1 h_1(t_1) dt_1 \left[ \int_c^{c+\eta_2(d, c)} f(a, y) dy + \int_c^{c+\eta_2(d, c)} f(b, y) dy \right] \\
 & \leq [f(a, c) + f(b, c) + f(a, d) + f(b, d)] \int_0^1 h_1(t_1) dt_1 \cdot \int_0^1 h_2(t_2) dt_2. \tag{2.3}
 \end{aligned}$$

*Proof* Since  $f$  is  $(h_1, h_2)$ -preinvex on the co-ordinates, it follows that the mapping  $f_x$  is  $h_2$ -preinvex and the mapping  $f_y$  is  $h_1$ -preinvex. Then, by the inequality (2.2), one has

$$\begin{aligned}
 \frac{1}{2h_2(\frac{1}{2})}f\left(x, c + \frac{1}{2}\eta_2(d, c)\right) & \leq \frac{1}{\eta_2(d, c)} \int_c^{c+\eta_2(d, c)} f(x, y) dy \\
 & \leq [f(x, c) + f(x, d)] \int_0^1 h_2(t) dt
 \end{aligned}$$

and

$$\begin{aligned}
 \frac{1}{2h_1(\frac{1}{2})}f\left(a + \frac{1}{2}\eta_1(b, a), y\right) & \leq \frac{1}{\eta_1(b, a)} \int_a^{a+\eta_1(b, a)} f(x, y) dx \\
 & \leq [f(a, y) + f(b, y)] \int_0^1 h_1(t) dt.
 \end{aligned}$$

Dividing the above inequalities for  $\eta_1(b, a)$  and  $\eta_2(d, c)$  and then integrating the resulting inequalities on  $[a, a + \eta_1(b, a)]$  and  $[c, c + \eta_2(d, c)]$ , respectively, we have

$$\begin{aligned}
 & \frac{1}{\eta_1(b, a) \cdot 2h_2(\frac{1}{2})} \int_a^{a+\eta_1(b, a)} f\left(x, c + \frac{1}{2}\eta_2(d, c)\right) dx \\
 & \leq \frac{1}{\eta_1(b, a)\eta_2(d, c)} \int_a^{a+\eta_1(b, a)} \int_c^{c+\eta_2(d, c)} f(x, y) dx dy \\
 & \leq \frac{1}{\eta_1(b, a)} \int_0^1 h_2(t) dt \left[ \int_a^{a+\eta_1(b, a)} f(x, c) dx + \int_a^{a+\eta_1(b, a)} f(x, d) dx \right]
 \end{aligned}$$

and

$$\begin{aligned} & \frac{1}{\eta_2(b, a) \cdot 2h_1(\frac{1}{2})} \int_c^{c+\eta_2(d, c)} f\left(a + \frac{1}{2}\eta_1(b, a), y\right) dy \\ & \leq \frac{1}{\eta_1(b, a)\eta_2(d, c)} \int_a^{a+\eta_1(b, a)} \int_c^{c+\eta_2(d, c)} f(x, y) dx dy \\ & \leq \frac{1}{\eta_2(d, c)} \int_0^1 h_1(t) dt \left[ \int_c^{c+\eta_2(d, c)} f(a, y) dy + \int_c^{c+\eta_2(d, c)} f(b, y) dy \right]. \end{aligned}$$

Summing the above inequalities, we get the second and the third inequalities in (2.3).

By the inequality (2.2), we also have

$$\frac{1}{2h_2(\frac{1}{2})} f\left(a + \frac{1}{2}\eta_1(b, a), c + \frac{1}{2}\eta_2(d, c)\right) \leq \frac{1}{\eta_2(d, c)} \int_c^{c+\eta_2(d, c)} f\left(a + \frac{1}{2}\eta_1(b, a), y\right) dy$$

and

$$\frac{1}{2h_1(\frac{1}{2})} f\left(a + \frac{1}{2}\eta_1(b, a), c + \frac{1}{2}\eta_2(d, c)\right) \leq \frac{1}{\eta_1(b, a)} \int_a^{a+\eta_1(b, a)} f\left(x, c + \frac{1}{2}\eta_2(d, c)\right) dx,$$

which give, by addition, the first inequality in (2.3).

Finally, by the same inequality (2.2), we can also state

$$\begin{aligned} & \frac{1}{\eta_2(d, c)} \int_c^{c+\eta_2(d, c)} f(a, y) dy \leq [f(a, c) + f(a, d)] \int_0^1 h_2(t) dt, \\ & \frac{1}{\eta_2(d, c)} \int_c^{c+\eta_2(d, c)} f(b, y) dy \leq [f(b, c) + f(b, d)] \int_0^1 h_2(t) dt, \\ & \frac{1}{\eta_1(b, a)} \int_a^{a+\eta_1(b, a)} f(x, c) dx \leq [f(a, c) + f(b, c)] \int_0^1 h_1(t) dt, \\ & \frac{1}{\eta_1(b, a)} \int_a^{a+\eta_1(b, a)} f(x, d) dx \leq [f(a, d) + f(b, d)] \int_0^1 h_1(t) dt, \end{aligned}$$

which give, by addition, the last inequality in (2.3). □

**Remark 3** In particular, for  $\eta_1(b, a) = b - a$ ,  $\eta_2(d, c) = d - c$ ,  $h_1(t_1) = h_2(t_2) = t$ , we get the inequalities obtained by Dragomir [6] for functions convex on the co-ordinates on the rectangle from the plane  $R^2$ .

**Remark 4** If  $\eta_1(b, a) = b - a$ ,  $\eta_2(d, c) = d - c$ , and  $h_1(t_1) = h_2(t_2) = t^s$ , then we get the inequalities obtained by Alomari and Darus in [7] for  $s$ -convex functions on the co-ordinates on the rectangle from the plane  $R^2$ .

**Theorem 2.3** Let  $f, g : [a, a + \eta_1(b, a)] \times [c, c + \eta_2(d, c)] \rightarrow R$  with  $a < a + \eta_1(b, a)$ ,  $c < c + \eta_2(d, c)$ . If  $f$  is  $(h_1, h_2)$ -preinvex on the co-ordinates and  $g$  is  $(k_1, k_2)$ -preinvex on the co-

ordinates with respect to  $\eta_1$  and  $\eta_2$ , then

$$\begin{aligned} & \frac{1}{\eta_1(b, a) \cdot \eta_2(d, c)} \int_a^{a+\eta_1(b, a)} \int_c^{c+\eta_2(d, c)} f(x, y)g(x, y) \, dx \, dy \\ & \leq M_1(a, b, c, d) \int_0^1 \int_0^1 h_1(t_1)h_2(t_2)k_1(t_1)k_2(t_2) \, dt_1 \, dt_2 \\ & \quad + M_2(a, b, c, d) \int_0^1 \int_0^1 h_1(t_1)h_2(t_2)k_1(t_1)k_2(1-t_2) \, dt_1 \, dt_2 \\ & \quad + M_3(a, b, c, d) \int_0^1 \int_0^1 h_1(t_1)h_2(t_2)k_1(1-t_1)k_2(t_2) \, dt_1 \, dt_2 \\ & \quad + M_4(a, b, c, d) \int_0^1 \int_0^1 h_1(t_1)h_2(t_2)k_1(1-t_1)k_2(1-t_2) \, dt_1 \, dt_2, \end{aligned}$$

where

$$\begin{aligned} M_1(a, b, c, d) &= f(a, c)g(a, c) + f(a, d)g(a, d) + f(b, c)g(b, c) + f(b, d)g(b, d), \\ M_2(a, b, c, d) &= f(a, c)g(a, d) + f(a, d)g(a, c) + f(b, c)g(b, d) + f(b, d)g(b, c), \\ M_3(a, b, c, d) &= f(a, c)g(b, c) + f(a, d)g(b, d) + f(b, c)g(a, c) + f(b, d)g(a, d), \\ M_4(a, b, c, d) &= f(a, c)g(b, d) + f(a, d)g(b, c) + f(b, c)g(a, d) + f(b, d)g(a, c). \end{aligned}$$

*Proof* Since  $f$  is  $(h_1, h_2)$ -preinvex on the co-ordinates and  $g$  is  $(k_1, k_2)$ -preinvex on the co-ordinates with respect to  $\eta_1$  and  $\eta_2$ , it follows that

$$\begin{aligned} & f(a + t_1\eta_1(b, a), c + t_2\eta_2(d, c)) \\ & \leq h_1(1-t_1)h_2(1-t_2)f(a, c) + h_1(1-t_1)h_2(t_2)f(a, d) \\ & \quad + h_1(t_1)h_2(1-t_2)f(b, c) + h_1(t_1)h_2(t_2)f(b, d) \end{aligned}$$

and

$$\begin{aligned} & g(a + t_1\eta_1(b, a), c + t_2\eta_2(d, c)) \\ & \leq k_1(1-t_1)k_2(1-t_2)g(a, c) + k_1(1-t_1)k_2(t_2)g(a, d) \\ & \quad + k_1(t_1)k_2(1-t_2)g(b, c) + k_1(t_1)k_2(t_2)g(b, d). \end{aligned}$$

Multiplying the above inequalities and integrating over  $[0, 1]^2$  and using the fact that

$$\begin{aligned} & \int_0^1 \int_0^1 f(a + t_1\eta_1(b, a), c + t_2\eta_2(d, c)) \cdot g(a + t_1\eta_1(b, a), c + t_2\eta_2(d, c)) \, dt_1 \, dt_2 \\ & = \frac{1}{\eta_1(b, a) \cdot \eta_2(d, c)} \int_a^{a+\eta_1(b, a)} \int_c^{c+\eta_2(d, c)} f(x, y)g(x, y) \, dx \, dy, \end{aligned}$$

we obtain our inequality. □

In the next two theorems, we will prove the so-called Hermite-Hadamard-Fejér inequalities for an  $(h_1, h_2)$ -preinvex function.



**Theorem 2.4** Let  $f : [a, a + \eta_1(b, a)] \times [c, c + \eta_2(d, c)] \rightarrow R$  be  $(h_1, h_2)$ -preinvex on the co-ordinates with respect to  $\eta_1$  and  $\eta_2$ ,  $a < a + \eta_1(b, a)$ ,  $c < c + \eta_2(d, c)$ , and  $w : [a, a + \eta_1(b, a)] \times [c, c + \eta_2(d, c)] \rightarrow R$ ,  $w \geq 0$ , symmetric with respect to

$$\left( a + \frac{1}{2}\eta_1(b, a), c + \frac{1}{2}\eta_2(d, c) \right).$$

Then

$$\begin{aligned} & \frac{1}{\eta_1(b, a) \cdot \eta_2(d, c)} \int_a^{a+\eta_1(b, a)} \int_c^{c+\eta_2(d, c)} f(x, y)w(x, y) dx dy \\ & \leq [f(a, c) + f(a, d) + f(b, c) + f(b, d)] \\ & \cdot \int_0^1 \int_0^1 h_1(t_1)h_2(t_2)w(a + t_1\eta_1(b, a), c + t_2\eta_2(d, c)) dt_1 dt_2. \end{aligned} \tag{2.4}$$

*Proof* From the definition of  $(h_1, h_2)$ -preinvex on the co-ordinates with respect to  $\eta_1$  and  $\eta_2$ , we have

(a)

$$\begin{aligned} & f(a + t_1\eta_1(b, a), c + t_2\eta_2(d, c)) \\ & \leq h_1(1 - t_1)h_2(1 - t_2)f(a, c) + h_1(1 - t_1)h_2(t_2)f(a, d) \\ & \quad + h_1(t_1)h_2(1 - t_2)f(b, c) + h_1(t_1)h_2(t_2)f(b, d), \end{aligned}$$

(b)

$$\begin{aligned} & f(a + (1 - t_1)\eta_1(b, a), c + (1 - t_2)\eta_2(d, c)) \\ & \leq h_1(t_1)h_2(t_2)f(a, c) + h_1(t_1)h_2(1 - t_2)f(a, d) \\ & \quad + h_1(1 - t_1)h_2(t_2)f(b, c) + h_1(1 - t_1)h_2(1 - t_2)f(b, d), \end{aligned}$$

(c)

$$\begin{aligned} & f(a + t_1\eta_1(b, a), c + (1 - t_2)\eta_2(d, c)) \\ & \leq h_1(1 - t_1)h_2(t_2)f(a, c) + h_1(1 - t_1)h_2(1 - t_2)f(a, d) \\ & \quad + h_1(t_1)h_2(t_2)f(b, c) + h_1(t_1)h_2(1 - t_2)f(b, d), \end{aligned}$$

(d)

$$\begin{aligned} & f(a + (1 - t_1)\eta_1(b, a), c + t_2\eta_2(d, c)) \\ & \leq h_1(t_1)h_2(1 - t_2)f(a, c) + h_1(t_1)h_2(t_2)f(a, d) \\ & \quad + h_1(1 - t_1)h_2(1 - t_2)f(b, c) + h_1(1 - t_1)h_2(t_2)f(b, d). \end{aligned}$$

Multiplying both sides of the above inequalities by  $w(a + t_1\eta_1(b, a), c + t_2\eta_2(d, c))$ ,  $w(a + (1 - t_1)\eta_1(b, a), c + (1 - t_2)\eta_2(d, c))$ ,  $w(a + t_1\eta_1(b, a), c + (1 - t_2)\eta_2(d, c))$ ,  $w(a + (1 - t_1)\eta_1(b, a), c +$

$t_2\eta_2(d, c)$ ), respectively, adding and integrating over  $[0, 1]^2$ , we obtain

$$\begin{aligned} & \frac{4}{\eta_1(b, a) \cdot \eta_2(d, c)} \int_a^{a+\eta_1(b, a)} \int_c^{c+\eta_2(d, c)} f(x, y)w(x, y) \, dx \, dy \\ & \leq [f(a, c) + f(a, d) + f(b, c) + f(b, d)] \\ & \quad \cdot 4 \int_0^1 \int_0^1 h_1(t_1)h_2(t_2)w(a + t_1\eta_1(b, a), c + t_2\eta_2(d, c)) \, dt_1 \, dt_2, \end{aligned}$$

where we use the symmetricity of the  $w$  with respect to  $(a + \frac{1}{2}\eta_1(b, a), c + \frac{1}{2}\eta_2(d, c))$ , which completes the proof.  $\square$

**Theorem 2.5** *Let  $f : [a, a + \eta_1(b, a)] \times [c, c + \eta_2(d, c)] \rightarrow R$  be  $(h_1, h_2)$ -preinvex on the co-ordinates with respect to  $\eta_1$  and  $\eta_2$ , and  $a < a + \eta_1(b, a)$ ,  $c < c + \eta_2(d, c)$ ,  $w : [a, a + \eta_1(b, a)] \times [c, c + \eta_2(d, c)] \rightarrow R$ ,  $w \geq 0$ , symmetric with respect to  $(a + \frac{1}{2}\eta_1(b, a), c + \frac{1}{2}\eta_2(d, c))$ . Then, if Condition C for  $\eta_1$  and  $\eta_2$  is fulfilled, we have*

$$\begin{aligned} & f\left(a + \frac{1}{2}\eta_1(b, a), c + \frac{1}{2}\eta_2(d, c)\right) \cdot \int_a^{a+\eta_1(b, a)} \int_c^{c+\eta_2(d, c)} w(x, y) \, dx \, dy \\ & \leq 4 \cdot h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right) \cdot \int_a^{a+\eta_1(b, a)} \int_c^{c+\eta_2(d, c)} f(x, y)w(x, y) \, dx \, dy. \end{aligned} \tag{2.5}$$

*Proof* Using the definition of an  $(h_1, h_2)$ -preinvex function on the co-ordinates and Condition C for  $\eta_1$  and  $\eta_2$ , we obtain

$$\begin{aligned} & f\left(a + \frac{1}{2}\eta_1(b, a), c + \frac{1}{2}\eta_2(d, c)\right) \\ & \leq h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right) \cdot [f(a + t_1\eta_1(b, a), c + t_2\eta_2(d, c)) \\ & \quad + f(a + t_1\eta_1(b, a), c + (1 - t_2)\eta_2(d, c)) + f(a + (1 - t_1)\eta_1(b, a), c + t_2\eta_2(d, c)) \\ & \quad + f(a + (1 - t_1)\eta_1(b, a), c + (1 - t_2)\eta_2(d, c))]. \end{aligned}$$

Now, we multiply it by  $w(a + t_1\eta_1(b, a), c + t_2\eta_2(d, c)) = w(a + t_1\eta_1(b, a), c + (1 - t_2)\eta_2(d, c)) = w(a + (1 - t_1)\eta_1(b, a), c + t_2\eta_2(d, c)) = w(a + (1 - t_1)\eta_1(b, a), c + (1 - t_2)\eta_2(d, c))$  and integrate over  $[0, 1]^2$  to obtain the inequality

$$\begin{aligned} & f\left(a + \frac{1}{2}\eta_1(b, a), c + \frac{1}{2}\eta_2(d, c)\right) \int_0^1 \int_0^1 w(a + t_1\eta_1(b, a), c + t_2\eta_2(d, c)) \, dt_1 \, dt_2 \\ & = f\left(a + \frac{1}{2}\eta_1(b, a), c + \frac{1}{2}\eta_2(d, c)\right) \frac{1}{\eta_1(b, a) \cdot \eta_2(d, c)} \int_a^{a+\eta_1(b, a)} \int_c^{c+\eta_2(d, c)} w(x, y) \, dx \, dy \\ & \leq 4 \cdot h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right) \frac{1}{\eta_1(b, a) \cdot \eta_2(d, c)} \int_a^{a+\eta_1(b, a)} \int_c^{c+\eta_2(d, c)} f(x, y)w(x, y) \, dx \, dy, \end{aligned}$$

which completes the proof.  $\square$

Now, for a mapping  $f : [a, b] \times [c, d] \rightarrow R$ , let us define a mapping  $H : [0, 1]^2 \rightarrow R$  in the following way:

$$H(t, r) = \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f\left(tx + (1-t)\frac{a+b}{2}, ry + (1-r)\frac{c+d}{2}\right) dx dy. \tag{2.6}$$

Some properties of this mapping for a convex on the co-ordinates function and an  $s$ -convex on the co-ordinates function are given in [6, 7], respectively. Here we investigate which of these properties can be generalized for  $(h_1, h_2)$ -convex on the co-ordinates functions.

**Theorem 2.6** *Suppose that  $f : [a, b] \times [c, d]$  is  $(h_1, h_2)$ -convex on the co-ordinates. Then:*

- (i) *The mapping  $H$  is  $(h_1, h_2)$ -convex on the co-ordinates on  $[0, 1]^2$ ,*
- (ii)  *$4h_1(\frac{1}{2})h_2(\frac{1}{2})H(t, r) \geq H(0, 0)$  for any  $(t, r) \in [0, 1]^2$ .*

*Proof* (i) The  $(h_1, h_2)$ -convexity on the co-ordinates of the mapping  $H$  is a consequence of the  $(h_1, h_2)$ -convexity on the co-ordinates of the function  $f$ . Namely, for  $r \in [0, 1]$  and for all  $\alpha, \beta \geq 0$  with  $\alpha + \beta = 1$  and  $t_1, t_2 \in [0, 1]$ , we have:

$$\begin{aligned} &H(\alpha t_1 + \beta t_2, r) \\ &= \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f\left((\alpha t_1 + \beta t_2)x + (1 - (\alpha t_1 + \beta t_2))\frac{a+b}{2}, ry + (1-r)\frac{c+d}{2}\right) dx dy \\ &= \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f\left(\alpha\left(t_1x + (1-t_1)\frac{a+b}{2}\right) + \beta\left(t_2x + (1-t_2)\frac{a+b}{2}\right), ry + (1-r)\frac{c+d}{2}\right) dx dy \\ &\leq h_1(\alpha) \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f\left(t_1x + (1-t_1)\frac{a+b}{2}, ry + (1-r)\frac{c+d}{2}\right) dx dy \\ &\quad + h_1(\beta) \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f\left(t_2x + (1-t_2)\frac{a+b}{2}, ry + (1-r)\frac{c+d}{2}\right) dx dy \\ &= h_1(\alpha)H(t_1, r) + h_1(\beta)H(t_2, r). \end{aligned}$$

Similarly, if  $t \in [0, 1]$  is fixed, then for all  $r_1, r_2 \in [0, 1]$  and  $\alpha, \beta \geq 0$  with  $\alpha + \beta = 1$ , we also have

$$H(t, \alpha r_1 + \beta r_2) \leq h_2(\alpha)H(t, r_1) + h_2(\beta)H(t, r_2),$$

which means that  $H$  is  $(h_1, h_2)$ -convex on the co-ordinates.

- (ii) After changing the variables  $u = tx + (1-t)\frac{a+b}{2}$  and  $v = ry + (1-r)\frac{c+d}{2}$ , we have

$$\begin{aligned} H(t, r) &= \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f\left(tx + (1-t)\frac{a+b}{2}, ry + (1-r)\frac{c+d}{2}\right) dx dy \\ &= \frac{1}{(b-a)(d-c)} \int_{u_L}^{u_U} \int_{v_L}^{v_U} f(u, v) \frac{b-a}{u_U - u_L} \cdot \frac{d-c}{v_U - v_L} du dv \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{(u_U - u_L)(v_U - v_L)} \int_{u_L}^{u_U} \int_{v_L}^{v_U} f(u, v) \, du \, dv \\
 &\geq \frac{1}{4h_1(\frac{1}{2})h_2(\frac{1}{2})} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right),
 \end{aligned}$$

where  $u_L = ta + (1-t)\frac{a+b}{2}$ ,  $u_U = tb + (1-t)\frac{a+b}{2}$ ,  $v_L = rc + (1-r)\frac{c+d}{2}$  and  $v_U = rd + (1-r)\frac{c+d}{2}$ , which completes the proof.  $\square$

**Remark 5** If  $f$  is convex on the co-ordinates, then we get  $H(t, r) \geq H(0, 0)$ . If  $f$  is  $s$ -convex on the co-ordinates in the second sense, then we have the inequality  $H(t, r) \geq 4^{s-1}H(0, 0)$ .

**Competing interests**

The author declares that he has no competing interests.

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**References**

1. Breckner, WW: Stetigkeitsanssagen für eine Klasse verallgemeinerter Konvexer Funktionen in topologischen linearen Räumen. *Publ. Inst. Math. (Belgr.)* **23**, 13-20 (1978)
2. Dragomir, SS, Fitzpatrick, S: The Hadamard's inequality for  $s$ -convex functions in the second sense. *Demonstr. Math.* **32**(4), 687-696 (1999)
3. Varošanec, S: On  $h$ -convexity. *J. Math. Anal. Appl.* **326**, 303-311 (2007)
4. Sarikaya, MZ, Saglam, A, Yildirim, H: On some Hadamard-type inequalities for  $h$ -convex functions. *J. Math. Inequal.* **2**, 335-341 (2008)
5. Bombardelli, M, Varošanec, S: Properties of  $h$ -convex functions related to the Hermite-Hadamard-Fejér inequalities. *Comput. Math. Appl.* **58**, 1869-1877 (2009)
6. Dragomir, SS: On the Hadamard's inequality for convex functions on the co-ordinates in a rectangle from the plane. *Taiwan. J. Math.* **5**(4), 775-788 (2001)
7. Alomari, M, Darus, M: The Hadamard's inequality for  $s$ -convex function of 2-variables on the co-ordinates. *Int. J. Math. Anal.* **2**(13), 629-638 (2008)
8. Latif, MA, Dragomir, SS: On some new inequalities for differentiable co-ordinated convex functions. *J. Inequal. Appl.* (2012). doi:10.1186/1029-242X-2012-28
9. Özdemir, ME, Latif, MA, Akademi, AO: On some Hadamard-type inequalities for product of two  $s$ -convex functions on the co-ordinates. *J. Inequal. Appl.* (2012). doi:10.1186/1029-242X-2012-21
10. Özdemir, ME, Kavurmaci, H, Akademi, AO, Avci, M: Inequalities for convex and  $s$ -convex functions on  $\Delta = [a, b] \times [c, d]$ . *J. Inequal. Appl.* (2012). doi:10.1186/1029-242X-2012-20
11. Mohan, SR, Neogy, SK: On invex sets and preinvex functions. *J. Math. Anal. Appl.* **189**, 901-908 (1995)
12. Noor, MS: Hadamard integral inequalities for product of two preinvex functions. *Nonlinear Anal. Forum* **14**, 167-173 (2009)
13. Noor, MS: Some new classes of non convex functions. *Nonlinear Funct. Anal. Appl.* **11**, 165-171 (2006)
14. Noor, MS: On Hadamard integral inequalities involving two log-preinvex functions. *J. Inequal. Pure Appl. Math.* **8**(3), 1-6 (2007)
15. Weir, T, Mond, B: Preinvex functions in multiobjective optimization. *J. Math. Anal. Appl.* **136**, 29-38 (1988)
16. Yang, XM, Li, D: On properties of preinvex functions. *J. Math. Anal. Appl.* **256**, 229-241 (2001)

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