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On the harmonic number expansion by Ramanujan

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Abstract

Let $\gamma = 0.577215664\ldots$ denote the Euler-Mascheroni constant, and let the sequences

$$u_n = \sum_{k=1}^{n} \frac{1}{k} - \frac{1}{2}\ln\left(n^2 + n + \frac{1}{3}\right) - \frac{1}{r(n^2+n+\frac{1}{3}) + s(n^2+n+\frac{1}{3})^2 + t} \quad \text{and}$$

$$v_n = \sum_{k=1}^{n} \frac{1}{k} - \frac{1}{2}\ln\left(n^2 + n + \frac{1}{3}\right)$$
$$- \left(\frac{a}{(n^2+n+\frac{1}{3})^2} + \frac{b}{(n^2+n+\frac{1}{3})^3} + \frac{c}{(n^2+n+\frac{1}{3})^4} + \frac{d}{(n^2+n+\frac{1}{3})^5}\right).$$

The main aim of this paper is to find the values r, s, t, a, b, c and d which provide the fastest sequences $(u_n)_{n\geq 1}$ and $(v_n)_{n\geq 1}$ approximating the Euler-Mascheroni constant. Also, we give the upper and lower bounds for $\sum_{k=1}^{n} \frac{1}{k} - \frac{1}{2}\ln(n^2+n+\frac{1}{3}) - \gamma$ in terms of $n^2+n+\frac{1}{3}$.

MSC: 11Y60; 40A05; 33B15

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1 Introduction

The Euler-Mascheroni constant $\gamma = 0.577215664\ldots$ is defined as the limit of the sequence

$$D_n = H_n - \ln n, \tag{1.1}$$

where H_n denotes the nth harmonic number defined for $n \in \mathbb{N} := \{1, 2, 3, \ldots\}$ by

$$H_n = \sum_{k=1}^{n} \frac{1}{k}.$$

Several bounds for $D_n - \gamma$ have been given in the literature [1–7]. For example, the following bounds for $D_n - \gamma$ were established in [3, 7]:

$$\frac{1}{2(n+1)} < D_n - \gamma < \frac{1}{2n} \quad (n \in \mathbb{N}).$$

The convergence of the sequence D_n to γ is very slow. Some quicker approximations to the Euler-Mascheroni constant were established in [8–21]. For example, Cesàro [8] proved that for every positive integer $n \geq 1$, there exists a number $c_n \in (0,1)$ such that the following approximation is valid:

$$\sum_{k=1}^{n} \frac{1}{k} - \frac{1}{2}\ln\left(n^2 + n\right) - \gamma = \frac{c_n}{6n(n+1)}.$$

Entry 9 of Chapter 38 of Berndt's edition of *Ramanujan's Notebooks* [22, p.521] reads,
'Let $m := \frac{n(n+1)}{2}$, where n is a positive integer. Then, as n approaches infinity,

$$\sum_{k=1}^{\infty} \frac{1}{k} \sim \frac{1}{2}\ln(2m) + \gamma + \frac{1}{12m} - \frac{1}{120m^2} + \frac{1}{630m^3} - \frac{1}{1{,}680m^4} + \frac{1}{2{,}310m^5} - \frac{191}{360{,}360m^6} + \frac{1}{30{,}030m^7} - \frac{2{,}833}{1{,}166{,}880m^8} + \frac{140{,}051}{17{,}459{,}442m^9} - [\cdots].'$$

For the history and the development of Ramanujan's formula, see [20].

Recently, by changing the logarithmic term in (1.1), DeTemple [15], Negoi [18] and Chen *et al.* [14] have presented, respectively, *faster* and *faster* asymptotic formulas as follows:

$$\sum_{k=1}^{n} \frac{1}{k} - \ln\left(n + \frac{1}{2}\right) = \gamma + O\left(n^{-2}\right) \quad (n \to \infty); \tag{1.2}$$

$$\sum_{k=1}^{n} \frac{1}{k} - \ln\left(n + \frac{1}{2} + \frac{1}{24n}\right) = \gamma + O\left(n^{-3}\right) \quad (n \to \infty); \tag{1.3}$$

$$\sum_{k=1}^{n} \frac{1}{k} - \ln\left(n + \frac{1}{2} + \frac{1}{24n} - \frac{1}{48n^2}\right) = \gamma + O\left(n^{-4}\right) \quad (n \to \infty). \tag{1.4}$$

Chen and Mortici [13] provided a *faster* asymptotic formula than those in (1.2) to (1.4),

$$\sum_{k=1}^{n} \frac{1}{k} - \ln\left(n + \frac{1}{2} + \frac{1}{24n} - \frac{1}{48n^2} + \frac{23}{5{,}760n^3}\right) = \gamma + O\left(n^{-5}\right) \quad (n \to \infty), \tag{1.5}$$

and posed the following natural question.

Open problem For a given positive integer p, find the constants a_j ($j = 0, 1, 2, \ldots, p$) such that

$$\sum_{k=1}^{n} \frac{1}{k} - \ln\left(n + \sum_{j=0}^{p} \frac{a_j}{n^j}\right) \tag{1.6}$$

is the sequence which would converge to γ in the *fastest* way.

Very recently, Yang [21] published the solution of the open problem (1.6) by using logarithmic-type Bell polynomials.

For all $n \in \mathbb{N}$, let

$$P_n = \sum_{k=1}^{n} \frac{1}{k} - \frac{1}{2} \ln\left(n^2 + n + \frac{1}{3}\right) \tag{1.7}$$

and

$$Q_n = \sum_{k=1}^{n} \frac{1}{k} - \frac{1}{4} \ln\left[\left(n^2 + n + \frac{1}{3}\right)^2 - \frac{1}{45}\right].$$

Chen and Li [12] proved that for all integers $n \geq 1$,

$$\frac{1}{180(n+1)^4} < \gamma - P_n < \frac{1}{180n^4} \tag{1.8}$$

and

$$\frac{8}{2{,}835(n+1)^6} < Q_n - \gamma < \frac{8}{2{,}835n^6}.$$

Now we define the sequences

$$u_n = \sum_{k=1}^{n} \frac{1}{k} - \frac{1}{2} \ln\left(n^2 + n + \frac{1}{3}\right) - \frac{1}{r(n^2+n+\frac{1}{3}) + s(n^2+n+\frac{1}{3})^2 + t} \tag{1.9}$$

and

$$\begin{aligned} v_n = {} & \sum_{k=1}^{n} \frac{1}{k} - \frac{1}{2} \ln\left(n^2 + n + \frac{1}{3}\right) \\ & - \left(\frac{a}{(n^2+n+\frac{1}{3})^2} + \frac{b}{(n^2+n+\frac{1}{3})^3} + \frac{c}{(n^2+n+\frac{1}{3})^4} + \frac{d}{(n^2+n+\frac{1}{3})^5}\right), \end{aligned} \tag{1.10}$$

respectively. Our Theorems 1 and 2 are to find the values r, s, t, a, b, c and d which provide the fastest sequences $(u_n)_{n\geq 1}$ and $(v_n)_{n\geq 1}$ approximating the Euler-Mascheroni constant.

Theorem 1 *Let* $(u_n)_{n\geq 1}$ *be defined by* (1.9). *For*

$$r = -\frac{640}{7}, \qquad s = -180, \qquad t = \frac{26{,}770}{441},$$

we have

$$\lim_{n\to\infty} n^{11}(u_n - u_{n+1}) = \frac{457{,}528}{123{,}773{,}265} \tag{1.11}$$

and

$$\lim_{n\to\infty} n^{10}(u_n - \gamma) = \frac{457{,}528}{123{,}773{,}265}. \tag{1.12}$$

The speed of convergence of the sequence $(u_n)_{n\geq 1}$ *is* n^{-10}.

Theorem 2 *Let* $(v_n)_{n\geq 1}$ *be defined by* (1.10). *For*

$$a = -\frac{1}{180}, \qquad b = \frac{8}{2{,}835}, \qquad c = -\frac{5}{1{,}512}, \qquad d = \frac{592}{93{,}555},$$

we have

$$\lim_{n\to\infty} n^{13}(v_n - v_{n+1}) = -\frac{796{,}801}{3{,}648{,}645} \quad \textit{and} \quad \lim_{n\to\infty} n^{12}(v_n - \gamma) = -\frac{796{,}801}{43{,}783{,}740}.$$

The speed of convergence of the sequence $(v_n)_{n\geq 1}$ *is* n^{-12}.

Our Theorems 3 and 4 establish the bounds for $\gamma - P_n$ in terms of $n^2 + n + \frac{1}{3}$.

Theorem 3 *Let* P_n *be defined by* (1.7). *Then*

$$\begin{aligned} &\frac{1}{\frac{640}{7}(n^2+n+\frac{1}{3}) + 180(n^2+n+\frac{1}{3})^2} \\ &\quad < \gamma - P_n < \frac{1}{\frac{640}{7}(n^2+n+\frac{1}{3}) + 180(n^2+n+\frac{1}{3})^2 - \frac{26{,}770}{441}}. \end{aligned} \tag{1.13}$$

Theorem 4 *Let* P_n *be defined by* (1.7). *Then*

$$\begin{aligned} &\frac{\frac{1}{180}}{(n^2+n+\frac{1}{3})^2} - \frac{\frac{8}{2{,}835}}{(n^2+n+\frac{1}{3})^3} + \frac{\frac{5}{1{,}512}}{(n^2+n+\frac{1}{3})^4} - \frac{\frac{592}{93{,}555}}{(n^2+n+\frac{1}{3})^5} \\ &\quad < \gamma - P_n < \frac{\frac{1}{180}}{(n^2+n+\frac{1}{3})^2} - \frac{\frac{8}{2{,}835}}{(n^2+n+\frac{1}{3})^3} + \frac{\frac{5}{1{,}512}}{(n^2+n+\frac{1}{3})^4}. \end{aligned} \tag{1.14}$$

Remark 1 The inequality (1.14) is sharper than (1.8), while the inequality (1.13) is sharper than (1.14).

2 Lemmas

Before we prove the main theorems, let us give some preliminary results.

The constant γ is deeply related to the gamma function $\Gamma(z)$ thanks to the Weierstrass formula:

$$\Gamma(z) = \frac{e^{-\gamma z}}{z} \prod_{k=1}^{\infty} \left\{ \left(1 + \frac{z}{k}\right)^{-1} e^{z/k} \right\} \quad \left(z \in \mathbb{C} \setminus Z_0^-; Z_0^- := \{-1, -2, -3, \ldots\}\right).$$

The logarithmic derivative of the gamma function

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} \quad \text{or} \quad \ln\Gamma(z) = \int_1^z \psi(t)\,\mathrm{d}t$$

is known as the psi (or digamma) function. The successive derivatives of the psi function $\psi(z)$

$$\psi^{(n)}(z) := \frac{\mathrm{d}^n}{\mathrm{d}z^n}\{\psi(z)\} \quad (n \in \mathbb{N})$$

are called the polygamma functions.

The following recurrence and asymptotic formulas are well known for the psi function:

$$\psi(z+1) = \psi(z) + \frac{1}{z} \tag{2.1}$$

(see [23, p.258]), and

$$\psi(z) \sim \ln z - \frac{1}{2z} - \frac{1}{12z^2} + \frac{1}{120z^4} - \frac{1}{252z^6} + \cdots \quad \bigl(z \to \infty \text{ in } |\arg z| < \pi\bigr) \tag{2.2}$$

(see [23, p.259]). From (2.1) and (2.2), we get

$$\psi(n+1) \sim \ln n + \frac{1}{2n} - \frac{1}{12n^2} + \frac{1}{120n^4} - \frac{1}{252n^6} + \cdots \quad (n \to \infty). \tag{2.3}$$

It is also known [23, p.258] that

$$\psi(n+1) = -\gamma + \sum_{k=1}^{n} \frac{1}{k}.$$

Lemma 1 [24, 25] *If* $(\lambda_n)_{n\geq 1}$ *is convergent to zero and there exists the limit*

$$\lim_{n\to\infty} n^k(\lambda_n - \lambda_{n+1}) = l \in \mathbb{R},$$

with $k > 1$, *then there exists the limit*

$$\lim_{n\to\infty} n^{k-1}\lambda_n = \frac{l}{k-1}.$$

Lemma 1 gives a method for measuring the speed of convergence.

Lemma 2 [26, Theorem 9] *Let* $k \geq 1$ *and* $n \geq 0$ *be integers. Then, for all real numbers* $x > 0$,

$$S_k(2n;x) < (-1)^{k+1}\psi^{(k)}(x) < S_k(2n+1;x), \tag{2.4}$$

where

$$S_k(p;x) = \frac{(k-1)!}{x^k} + \frac{k!}{2x^{k+1}} + \sum_{i=1}^{p}\left[B_{2i}\prod_{j=1}^{k-1}(2i+j)\right]\frac{1}{x^{2i+k}},$$

and B_i $(i = 0,1,2,\ldots)$ *are Bernoulli numbers defined by*

$$\frac{t}{e^t - 1} = \sum_{i=0}^{\infty} B_i \frac{t^i}{i!}.$$

It follows from (2.4) that for $x > 0$,

$$\begin{aligned}
&\frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} + \frac{1}{42x^7} - \frac{1}{30x^9} + \frac{5}{66x^{11}} - \frac{691}{2{,}730x^{13}} \\
&\quad < \psi'(x) < \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} + \frac{1}{42x^7} - \frac{1}{30x^9} + \frac{5}{66x^{11}} - \frac{691}{2{,}730x^{13}} + \frac{7}{6x^{15}},
\end{aligned}$$

from which we imply that for $x > 0$,

$$\begin{aligned}&\frac{1}{x}-\frac{1}{2x^2}+\frac{1}{6x^3}-\frac{1}{30x^5}+\frac{1}{42x^7}-\frac{1}{30x^9}+\frac{5}{66x^{11}}-\frac{691}{2{,}730x^{13}}\\&\quad<\psi'(x+1)\\&\quad<\frac{1}{x}-\frac{1}{2x^2}+\frac{1}{6x^3}-\frac{1}{30x^5}+\frac{1}{42x^7}-\frac{1}{30x^9}+\frac{5}{66x^{11}}-\frac{691}{2{,}730x^{13}}+\frac{7}{6x^{15}}.\end{aligned}\tag{2.5}$$

3 Proofs of Theorems 1-4

Proof of Theorem 1 By using the Maple software, we write the difference $u_n - u_{n+1}$ as a power series in n^{-1}:

$$\begin{aligned}u_n-u_{n+1}={}&\left(-\frac{s+180}{45s}\right)\frac{1}{n^5}+\left(\frac{s+180}{9s}\right)\frac{1}{n^6}\\&+\left(\frac{2(-6{,}048s+567r-32s^2)}{189s^2}\right)\frac{1}{n^7}\\&+\left(\frac{2(-567r+2{,}268s+11s^2)}{27s^2}\right)\frac{1}{n^8}\\&+\left(\frac{2(-23s^3+2{,}430sr-5{,}310s^2+108st-108r^2)}{27s^3}\right)\frac{1}{n^9}\\&+\left(\frac{2(-13{,}770sr+19{,}170s^2-1{,}620st+1{,}620r^2+73s^3)}{45s^3}\right)\frac{1}{n^{10}}\\&+\frac{1}{2{,}673s^4}\bigl(-15{,}443s^4+4{,}834{,}566s^2r-4{,}650{,}624s^3+1{,}033{,}560s^2t\\&-1{,}033{,}560sr^2-53{,}460srt+26{,}730r^3\bigr)\frac{1}{n^{11}}\\&+O\left(\frac{1}{n^{12}}\right).\end{aligned}\tag{3.1}$$

According to Lemma 1, we have three parameters r, s and t which produce the fastest convergence of the sequence from (3.1)

$$\begin{cases}s+180=0,\\-6{,}048s+567r-32s^2=0,\\-23s^3+2{,}430sr-5{,}310s^2+108st-108r^2=0,\end{cases}$$

namely if

$$r=-\frac{640}{7},\qquad s=-180,\qquad t=\frac{26{,}770}{441}.$$

Thus, we have

$$u_n-u_{n+1}=\frac{457{,}528}{123{,}773{,}265n^{11}}+O\left(\frac{1}{n^{12}}\right).$$

By using Lemma 1, we obtain the assertion of Theorem 1. □

Proof of Theorem 2 By using the Maple software, we write the difference $v_n - v_{n+1}$ as a power series in n^{-1}:

$$\begin{aligned} v_n - v_{n+1} = {}& \left(-\frac{1}{45} - 4a\right)\frac{1}{n^5} + \left(\frac{1}{9} + 20a\right)\frac{1}{n^6} + \left(-64a - 6b - \frac{64}{189}\right)\frac{1}{n^7} \\ &+ \left(\frac{22}{27} + 168a + 42b\right)\frac{1}{n^8} + \left(-\frac{1{,}180}{3}a - 8c - \frac{46}{27} - 180b\right)\frac{1}{n^9} \\ &+ \left(72c + \frac{146}{45} + 852a + 612b\right)\frac{1}{n^{10}} \\ &+ \left(-\frac{1{,}160}{3}c - \frac{15{,}443}{2{,}673} - \frac{5{,}426}{3}b - 10d - \frac{46{,}976}{27}a\right)\frac{1}{n^{11}} \\ &+ \left(\frac{2{,}375}{243} + \frac{14{,}542}{3}b + \frac{4{,}840}{3}c + \frac{91{,}432}{27}a + 110d\right)\frac{1}{n^{12}} + O\left(\frac{1}{n^{13}}\right). \end{aligned} \tag{3.2}$$

According to Lemma 1, we have four parameters a, b, c and d which produce the fastest convergence of the sequence from (3.2)

$$\begin{cases} -\frac{1}{45} - 4a = 0, \\ -64a - 6b - \frac{64}{189} = 0, \\ -\frac{1{,}180}{3}a - 8c - \frac{46}{27} - 180b = 0, \\ -\frac{1{,}160}{3}c - \frac{15{,}443}{2{,}673} - \frac{5{,}426}{3}b - 10d - \frac{46{,}976}{27}a = 0, \end{cases}$$

namely if

$$a = -\frac{1}{180}, \qquad b = \frac{8}{2{,}835}, \qquad c = -\frac{5}{1{,}512}, \qquad d = \frac{592}{93{,}555}.$$

Thus, we have

$$v_n - v_{n+1} = -\frac{796{,}801}{3{,}648{,}645n^{13}} + O\left(\frac{1}{n^{14}}\right).$$

By using Lemma 1, we obtain the assertion of Theorem 2. □

Proof of Theorem 3 Here we only prove the second inequality in (1.13). The proof of the first inequality in (1.13) is similar. The upper bound of (1.13) is obtained by considering the function F for $x \geq 1$ defined by

$$F(x) = \frac{1}{2}\ln\left(x^2 + x + \frac{1}{3}\right) - \psi(x+1) - \frac{1}{\frac{640}{7}(n^2 + n + \frac{1}{3}) + 180(n^2 + n + \frac{1}{3})^2 - \frac{26{,}770}{441}}.$$

Differentiation and applying the right-hand inequality of (2.5) yield

$$\begin{aligned} F'(x) = {}& -\psi'(x+1) + \frac{2x+1}{2(x^2 + x + \frac{1}{3})} \\ &+ \frac{55{,}566(126x^3 + 189x^2 + 137x + 37)}{5(7{,}938x^4 + 15{,}876x^3 + 17{,}262x^2 + 9{,}324x - 451)^2} \end{aligned}$$

$$
\begin{aligned}
&> -\left(\frac{1}{x} - \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} + \frac{1}{42x^7} - \frac{1}{30x^9} + \frac{5}{66x^{11}} - \frac{691}{2{,}730x^{13}} + \frac{7}{6x^{15}}\right) \\
&\quad + \frac{2x+1}{2(x^2+x+\frac{1}{3})} + \frac{55{,}566(126x^3+189x^2+137x+37)}{5(7{,}938x^4+15{,}876x^3+17{,}262x^2+9{,}324x-451)^2} \\
&= \frac{P(x)}{30{,}030x^{13}(3x^2+3x+1)^6},
\end{aligned}
$$

where

$$
\begin{aligned}
P(x) &= 35{,}471{,}898{,}974{,}548{,}627{,}145 + 138{,}773{,}138{,}144{,}376{,}345{,}519(x-4) \\
&\quad + 241{,}909{,}257{,}272{,}859{,}643{,}240(x-4)^2 \\
&\quad + 253{,}899{,}751{,}881{,}744{,}791{,}655(x-4)^3 \\
&\quad + 181{,}059{,}030{,}163{,}487{,}870{,}836(x-4)^4 \\
&\quad + 93{,}303{,}260{,}620{,}236{,}720{,}571(x-4)^5 \\
&\quad + 35{,}932{,}291{,}146{,}874{,}735{,}228(x-4)^6 + 10{,}519{,}794{,}292{,}714{,}982{,}599(x-4)^7 \\
&\quad + 2{,}353{,}926{,}972{,}956{,}528{,}576(x-4)^8 + 400{,}626{,}844{,}002{,}342{,}775(x-4)^9 \\
&\quad + 51{,}041{,}813{,}866{,}867{,}916(x-4)^{10} + 4{,}719{,}218{,}347{,}433{,}667(x-4)^{11} \\
&\quad + 299{,}247{,}577{,}164{,}158(x-4)^{12} + 11{,}646{,}155{,}626{,}560(x-4)^{13} \\
&\quad + 209{,}840{,}641{,}920(x-4)^{14} > 0 \quad \text{for } x \geq 4.
\end{aligned}
$$

Therefore, $F'(x) > 0$ for $x \geq 4$.

For $x = 1, 2, 3, 4$, we compute directly:

$$
\begin{aligned}
&F(1) = -0.000018306, \qquad F(2) = -2.171 \times 10^{-7}, \\
&F(3) = -1.0 \times 10^{-8}, \qquad F(4) = -1.0 \times 10^{-9}.
\end{aligned}
$$

Hence, the sequence $(F(n))_{n\geq 1}$ is strictly increasing. This leads to

$$
F(n) < \lim_{n\to\infty} F(n) = 0
$$

by using the asymptotic formula (2.3). This completes the proof of the second inequality in (1.13). □

Proof of Theorem 4 Here we only prove the first inequality in (1.14). The proof of the second inequality in (1.14) is similar. The lower bound of (1.14) is obtained by considering the function G for $x \geq 1$ defined by

$$
\begin{aligned}
G(x) &= \psi(x+1) - \frac{1}{2}\ln\left(x^2 + x + \frac{1}{3}\right) \\
&\quad + \left(\frac{\frac{1}{180}}{(x^2+x+\frac{1}{3})^2} + \frac{-\frac{8}{2{,}835}}{(x^2+x+\frac{1}{3})^3} + \frac{\frac{5}{1{,}512}}{(x^2+x+\frac{1}{3})^4} + \frac{-\frac{592}{93{,}555}}{(x^2+x+\frac{1}{3})^5}\right).
\end{aligned}
$$

Differentiation and applying the left-hand inequality of (2.5) yield

$$\begin{aligned}
G'(x) &= \psi'(x+1) - \frac{2x+1}{2(x^2+x+\frac{1}{3})} \\
&\quad - \frac{3(4{,}158x^7 + 14{,}553x^6 + 19{,}701x^5 + 12{,}870x^4 + 8{,}283x^3 + 6{,}831x^2 - 8{,}276x - 5{,}194)}{770(3x^2+3x+1)^6} \\
&> \left(\frac{1}{x} - \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} + \frac{1}{42x^7} - \frac{1}{30x^9} + \frac{5}{66x^{11}} - \frac{691}{2{,}730x^{13}}\right) - \frac{2x+1}{2(x^2+x+\frac{1}{3})} \\
&\quad - \frac{3(41{,}58x^7 + 14{,}553x^6 + 19{,}701x^5 + 12{,}870x^4 + 8{,}283x^3 + 6{,}831x^2 - 8{,}276x - 5{,}194)}{770(3x^2+3x+1)^6} \\
&= \frac{Q(x)}{30{,}030x^{13}(3x^2+3x+1)^6},
\end{aligned}$$

where

$$\begin{aligned}
Q(x) &= 274{,}317{,}996{,}839{,}484 + 1{,}074{,}684{,}262{,}984{,}527(x-5) \\
&\quad + 1{,}571{,}352{,}927{,}565{,}772(x-5)^2 + 1{,}266{,}557{,}271{,}610{,}345(x-5)^3 \\
&\quad + 652{,}427{,}951{,}634{,}329(x-5)^4 + 230{,}639{,}944{,}842{,}034(x-5)^5 \\
&\quad + 57{,}987{,}546{,}990{,}473(x-5)^6 + 10{,}515{,}845{,}175{,}406(x-5)^7 \\
&\quad + 1{,}371{,}027{,}303{,}124(x-5)^8 + 125{,}702{,}024{,}549(x-5)^9 \\
&\quad + 7{,}709{,}579{,}845(x-5)^{10} + 284{,}457{,}957(x-5)^{11} \\
&\quad + 4{,}780{,}806(x-5)^{12} > 0 \quad \text{for } x \geq 5.
\end{aligned}$$

Therefore, $G'(x) > 0$ for $x \geq 5$.

For $x = 1, 2, 3, 4, 5$, we compute directly:

$$\begin{aligned}
&G(1) = -0.000046245\ldots, \qquad G(2) = -1.799 \times 10^{-7}, \qquad G(3) = -4 \times 10^{-9}, \\
&G(4) = -1 \times 10^{-9}, \qquad G(5) = -1 \times 10^{-10}.
\end{aligned}$$

Hence, the sequence $(G(n))_{n\geq 1}$ is strictly increasing. This leads to

$$G(n) < \lim_{n\to\infty} G(n) = 0$$

by using the asymptotic formula (2.3). This completes the proof of the first inequality in (1.14). □

Remark 2 Some calculations in this work were performed by using the Maple software for symbolic calculations.

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
CM proposed the sequence u_n. CPC proposed the sequence v_n. CM proposed to solve the problems using Lemma 1, while CPC used Lemma 2 in evaluations. Both authors made the computations and verified their corectedness. The authors read and approved the final manuscript.

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