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A new characterization of *Mathieu*-groups by the order and one irreducible character degree

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Abstract

The main aim of this article is to characterize the finite simple groups by less character quantity. In fact, we show that each *Mathieu*-group G can be determined by their largest and second largest irreducible character degrees.

MSC: 20C15

Keywords: finite group; simple group; character degree

1 Introduction and preliminary results

Classifying finite groups by the properties of their characters is an interesting problem in group theory. In 2000, Huppert conjectured that each finite non-abelian simple group G is characterized by the set $cd(G)$ of degrees of its complex irreducible characters. In [1–4], it was shown that many non-abelian simple groups such as $L_2(q)$ and $S_z(q)$ satisfy the conjecture. In this paper, we manage to characterize the finite simple groups by less character quantity. Let G be a finite group; $L(G)$ denotes the largest irreducible character degree of G and $S(G)$ denotes the second largest irreducible character degree of G. We characterize the five Mathieu groups G by the order of G and its largest and second largest irreducible character degrees. Our main results are the following theorems.

Theorem A *Let G be a finite group and let M be one of the following Mathieu groups: M_{11}, M_{12} and M_{23}. Then $G \cong M$ if and only if the following conditions are fulfilled:*

(1) $|G| = |M|$;
(2) $L(G) = L(M)$.

Theorem B *Let G be a finite group. Then $G \cong M_{24}$ if and only if $|G| = |M_{24}|$ and $S(G) = S(M_{24})$.*

Theorem C *Let G be a finite group. If $|G| = |M_{22}|$ and $L(G) = L(M_{22})$, then either G is isomorphic to M_{22} or $H \times M_{11}$, where H is a Frobenius group with an elementary kernel of order 8 and a cyclic complement of order 7.*

We need the following lemmas.

Lemma 1 *Let G be a non-solvable group. Then G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a direct product of isomorphic non-abelian simple groups and $|G/K| \mid |\mathrm{Out}(K/H)|$.*

Proof Let G be a non-solvable group. Then G has a chief factor M/N such that M/N is a direct product of isomorphic non-abelian simple groups. Hence $C_{G/N}(M/N) \cap M/N = Z(M/N) = 1$, and so

$$M/N \cong \frac{C_{G/N}(M/N) \times M/N}{C_{G/N}(M/N)} \leq \frac{G/N}{C_{G/N}(M/N)} \lesssim \mathrm{Aut}(M/N).$$

Let $K/N = C_{G/N}(M/N) \times M/N$ and $H/N = C_{G/N}(M/N)$. Then $G/K \leq \mathrm{Out}(M/N)$ and $K/H \cong M/N$ is a direct product of isomorphic non-abelian simple groups. Thus $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ is a normal series, as desired. □

Lemma 2 *Let G be a finite solvable group of order $p_1^{a_1} p_2^{a_2} \cdots p_n^{a_n}$, where $p_1, p_2, \ldots, p_n$ are distinct primes. If $kp_n + 1 \nmid p_i^{a_i}$ for each $i \leq n - 1$ and $k > 0$, then the Sylow p_n-subgroup is normal in G.*

Proof Let N be a minimal normal subgroup of G. Then $|N| = p^m$ for G is solvable. If $p = p_n$, by induction on G/N, we see that normality of the Sylow p_n-subgroup in G. Now suppose that $p = p_i$ for some $i < n$. Now consider G/N. By induction, the Sylow p_n-subgroup P/N of G/N is normal in G/N. Thus $P \trianglelefteq G$. Let Q be a Sylow p_n-subgroup of P. Then $P = NQ$. By Sylow's theorem, $|P : N_P(Q)| = p_i^l$ ($l \leq m \leq a_i$) and $p_n \mid p_i^l - 1$. But this means that $kp_n + 1 \mid p^{a_i}$, and then $k = 0$ by assumption. Hence $Q \trianglelefteq P$ and $Q \trianglelefteq G$. □

2 Proof of theorems

Proof of Theorem A We only need to prove the sufficiency. We divide the proof into three cases.

Case 1.1 $M = M_{11}$

In this case, we have $|G| = 2^4 \cdot 3^2 \cdot 5 \cdot 11$ and $L(G) = 55$. We first show that G is non-solvable. Assume the contrary. By Lemma 2, we know that the Sylow 11-subgroup of G is normal in G. Let N be the 11-Sylow subgroup of G. Since N is abelian, we have $\chi(1) \mid |G/N|$ for all $\chi \in \mathrm{Irr}(G)$. But $L(G) = 55$ and $55 \nmid |G/N|$, a contradiction. Therefore, G is non-solvable.

Since G is non-solvable, by Lemma 1, we get that G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a direct product of isomorphic non-abelian simple groups and $|G/K| \mid |\mathrm{Out}(K/H)|$. As $|G| = 2^4 \cdot 3^2 \cdot 5 \cdot 11$, we have $K/H \cong A_5, A_6, L_2(11)$ or M_{11}.

We first assume that $K/H \cong A_5$. Since $|\mathrm{Out}(A_5)| = 2$, we have $|G/K| \mid 2$ and $|H| = 2^t \cdot 3 \cdot 11$, where $t = 1$ or 2. Let $\chi \in \mathrm{Irr}(G)$ such that $\chi(1) = L(G) = 55$ and $\theta \in \mathrm{Irr}(H)$ such that $[\chi_H, \theta] \neq 0$. Then $\theta(1) = 11$ by the Clifford theorem (see Theorem 6.2 in [5]). On the other hand, since $|H| = 2^t \cdot 3 \cdot 11$, we have H is solvable. Let N be a Sylow 11-subgroup of H. Then $N \trianglelefteq H$ by Lemma 2. Hence $\theta(1) \mid |H/N| = 2^t \cdot 3$, a contradiction.

By the same reason as above, one has that $K/H \ncong A_6$.

Suppose that $K/H \cong L_2(11)$. Since $|\mathrm{Out}(L_2(11))| = 2$, we have $|G/K| \mid 2$ and so $|H| = 2^a \cdot 3$, where $a = 1$ or 2. Let $\theta \in \mathrm{Irr}(H)$ such that $e = [\chi_H, \theta] \neq 0$ and let $t = |G : I_G(\theta)|$. Then $\theta(1) = 1$

and $et = \chi(1)/\theta(1) = 55$. Since $|H| = 2^t \cdot 3$, where $a = 1$ or 2, we have that $55 \nmid |\operatorname{Aut}(H/H')|$. Hence $t = 1$, $e = 55$. But $(55)^2 = e^2t = [\chi_H, \chi_H] > |G : H| = 2^b \cdot 3 \cdot 5 \cdot 11$, where $2 \leq b \leq 3$, a contradiction.

If $K/H \cong M_{11}$, by comparing the orders of G and M_{11}, we have $|H| = 1$. Therefore $G = K \cong M_{11}$.

Case 1.2 $M = M_{23}$

In this case, we have $|G| = 2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$ and $L(G) = 2^3 \cdot 11 \cdot 23$. Then $O_{23}(G) = 1$. If not, then $|O_{23}(G)| = 23$ and $O_{23}(G)$ is abelian. Hence $L(G) = 2^3 \cdot 11 \cdot 23 \mid |G/O_{23}(G)|$, a contradiction.

If G is solvable, then the Sylow 23-subgroup of G is normal in G by Lemma 2, which leads to a contradiction as above. Therefore G is non-solvable.

Since G is non-solvable, by Lemma 1, we get that G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a direct product of isomorphic non-abelian simple groups and $|G/K| \mid |\operatorname{Out}(K/H)|$. As $|G| = 2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$, we have that K/H can be isomorphic to one of the simple groups: A_5, $L_2(7)$, A_6, $L_2(8)$, $L_2(11)$, A_7, M_{11}, $L_3(4)$, A_8, M_{22} and M_{23}.

We first assume that $K/H \cong A_5$. Since $|\operatorname{Out}(A_5)| = 2$, we have $|G/K| \mid 2$ and $|H| = 2^m \cdot 3 \cdot 7 \cdot 11 \cdot 23$, where $m = 4$ or 5. Suppose that H is non-solvable. By Lemma 1, H has a normal series $1 \trianglelefteq A \trianglelefteq B \trianglelefteq H$ such that B/A is a direct product of isomorphic non-abelian simple groups and $|H/B| \mid |\operatorname{Out}(B/A)|$. Since $|H| = 2^m \cdot 3 \cdot 7 \cdot 11 \cdot 23$, we have $B/A \cong L_2(7)$ and $|H/B| \mid 2$. Thus $|A| = 2^a \cdot 11 \cdot 23$, where $0 \leq a \leq 2$. Let N be a Sylow 23-subgroup of A. Then $N \trianglelefteq A$ by Lemma 2. Hence we get a subnormal series of G, $N char A \trianglelefteq B \trianglelefteq H \trianglelefteq K \trianglelefteq G$, which implies that $N \trianglelefteq G$. But $O_{23}(G) = 1$, a contradiction. If H is solvable, then the Sylow 23-subgroup of H is normal in H by Lemma 2, which leads to a contradiction as before.

By the same arguments as the proofs of $K/H \cong A_5$, we show that K/H cannot be isomorphic to one of the simple groups: A_6, $L_2(7)$, $L_2(8)$, $L_2(11)$, A_7, M_{11}, $L_3(4)$, A_8 and M_{22}.

If $K/H \cong M_{23}$, since $|G| = 2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$, we have that $|H| = 1$ and $G = K \cong M_{23}$.

Case 1.3 $M = M_{12}$

In this case, $|G| = 2^6 \cdot 3^3 \cdot 5 \cdot 11$ and $L(G) = 2^4 \cdot 11$. Since $11 \mid L(G)$, by the same arguments as the proofs of Case 1.2, we have that $O_{11}(G) = 1$.

We will show that G is non-solvable. If G is solvable, then the Sylow 11-subgroup of G is normal in G by Lemma 2, a contradiction. Therefore, G is non-solvable.

By Lemma 1, we get that G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a direct product of isomorphic non-abelian simple groups and $|G/K| \mid |\operatorname{Out}(K/H)|$. As $|G| = 2^6 \cdot 3^3 \cdot 5 \cdot 11$, we have $K/H \cong A_5, A_6, L_2(11), M_{11}$ or M_{12}.

By the same arguments as the proofs of Case 1.2, we can prove that K/H cannot be isomorphic to A_5 or A_6.

Assume that $K/H \cong L_2(11)$. Since $|\operatorname{Out}(L_2(11))| = 2$, we have $|G/K| \mid 2$ and $|H| = 2^a \cdot 3$, where $a = 3$ or 4. Suppose that $|G/K| = 1$. Then $|H| = 2^4 \cdot 3^2$. Let $\chi \in \operatorname{Irr}(G)$ such that $\chi(1) = L(G) = 2^4 \cdot 11$ and $\theta \in \operatorname{Irr}(H)$ such that $e = [\chi_H, \theta] \neq 0$. Then $\chi(1) = et\theta(1) = 176$, where $t = |G : I_G(\theta)|$. Since $\chi(1)/\theta(1) \mid |G/H|$, we have that $\theta(1) = 4$ or 8. If $\theta(1) = 4$, then $et = 44$. Since $|H| = 2^4 \cdot 3^2$, we have that H has at most eight irreducible characters of degree 4. Hence $t \leq 4$. We assert that $I_G(\theta) = G$. If not, then $I_G(\theta) < G$. Let U containing $I_G(\theta)$ be a maximal subgroup of G. Then $1 \leq |G : U| \mid |G : I_G(\theta)| = 4$. By checking the maximal subgroups of $L_2(11)$ (see ATLAS table in [6]), it is easy to get a contradiction. Hence $I_G(\theta) = G$, and so $t = 1$ and $e = 44$. But $e^2 \cdot t = [\chi_H, \chi_H] > |G : H|$, a contradiction. If $\theta(1) = 8$, then $|O_3(H)| = 9$ and $I_G(\theta) = G$. Since $H \trianglelefteq G$, we have that $O_3(H) \trianglelefteq G$. Let $\lambda \in$

$\mathrm{Irr}(O_3(H))$ such that $[\theta_{O_3(H)}, \lambda] \neq 0$. Since $\theta(1) = 8$, we have $4 \leq |H : I_H(\lambda)| \leq 8$. But $I_G(\theta) = G$, which implies that $4 \leq |G : I_G(\lambda)| = |H : I_H(\lambda)| \leq 8$. Let $S = \bigcap_{g\in G} I_G(\lambda)^g$. Then $S \trianglelefteq G$ and $G/S \lesssim S_8$. By the Jordan-Hölder theorem, S has a normal series $1 \trianglelefteq O_3(H) \trianglelefteq C \trianglelefteq D \trianglelefteq S$ such that $D/C \cong L_2(11)$ and $|C/O_3(H)| = 1, 2$ or 4. Let $\alpha \in \mathrm{Irr}(S)$ such that $[\chi_S, \alpha] \neq 0$. Since $\chi(1)/\alpha(1) \mid |G/S|$, we have that $22 \mid \alpha(1)$. Since λ^g is invariant in S, for each $g \in G$ and $4 \leq |G : I_G(\lambda)|$, we have that each irreducible character is invariant in S and $O_3(H) \leq Z(S)$. Therefore, the following conclusions hold:

(a) $S \cong L_2(11) \times O_3(H)$ if $|C/O_3(H)| = 1$;

(b) $S \cong (2 \cdot L_2(11)) \times O_3(H)$ or $(Z_2 \times L_2(11)) \times O_3(H)$ if $|C/O_3(H)| = 2$.

By checking the character table of $2 \cdot L_2(11)$ and $L_2(11)$, we see that both conclusions (a) and (b) are not satisfied with the above conditions. Now, we suppose that $|C/O_3(H)| = 4$. Then $44 \mid \alpha(1)$. Since $O_3(H) \leq Z(S)$, one has that $C \cong O_3(H) \times B$, where B is a group of order 4. Let β be an irreducible component of α_C and $t_1 = |S : I_S(\beta)|$. Then $\beta(1) = 1$ and $t_1 \mid \alpha(1)/\beta(1) \mid 44$. Since the indexes of the maximal subgroups of S containing $I_S(\beta)$ divide t_1 and $t_1 \mid |\mathrm{Aut}(C)|$, we have that $t_1 = 1$. Hence $[\alpha_C, \alpha_C] > |S : C| = 2^2 \cdot 3 \cdot 5 \cdot 11$, a contradiction.

Similarly, we can show that $|G/K| \neq 2$.

Suppose that $K/H \cong M_{11}$. Since $|\mathrm{Out}(M_{11})| = \mathrm{Mult}(M_{11}) = 1$, we have $G \cong H \times M_{11}$, where $|H| = 2^2 \cdot 3$. By checking the character table of M_{11}, we see that G has no irreducible character of degree $L(G) = 2^4 \cdot 11$, a contradiction.

If $K/H \cong M_{12}$, since $|G| = 2^6 \cdot 3^3 \cdot 5 \cdot 11$, we conclude that $|H| = 1$ and $G = K \cong M_{12}$, which completes the proof of Theorem A. □

Proof of Theorem B We only need to prove the sufficiency.

In this case, we have $|G| = 2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$ and $S(G) = 2^2 \cdot 3^2 \cdot 7 \cdot 23$. Let $\chi \in \mathrm{Irr}(G)$ such that $\chi(1) = S(G)$. If $O_{23}(G) \neq 1$, then $|O_{23}(G)| = 23$, which implies that $\chi(1) \mid |G : N|$, a contradiction. Hence $O_{23}(G) = 1$.

We have to show that G is non-solvable. Assume the contrary, by Lemma 2, we have that the Sylow 23-subgroup is normal in G, a contradiction. Therefore, G is non-solvable.

Since G is non-solvable, by Lemma 1, one has that G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a direct product of isomorphic non-abelian simple groups and $|G/K| \mid |\mathrm{Out}(K/H)|$. As $|G| = 2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$, then K/H can be isomorphic to one of the following simple groups: A_5, $L_2(7)$, A_6, $L_2(8)$, $L_2(11)$, A_7, $U_3(3)$, M_{11}, $L_3(4)$, A_8, M_{12}, M_{22}, M_{23} and M_{24}.

We first assume that $K/H \cong A_5$. Since $|\mathrm{Out}(A_5)| = 2$, we have $|G/K| \mid 2$ and $|H| = 2^t \cdot 3^2 \cdot 7 \cdot 11 \cdot 23$, where $t = 7$ or 8. Let $\theta \in \mathrm{Irr}(H)$ such that $[\chi_H, \theta] \neq 0$. Since $\chi(1)/\theta(1) \mid |G/H|$, it implies that $23 \mid \theta(1)$. If H is solvable, then $O_{23}(H) \neq 1$ by Lemma 2, which implies that $O_{23}(H) = O_{23}(G) \neq 1$, a contradiction. Thus H is non-solvable. Then there exists a normal series of H: $1 \trianglelefteq N \trianglelefteq M \trianglelefteq H$ such that M/N is a direct product of isomorphic non-abelian simple groups and $|H/M| \mid |\mathrm{Out}(M/N)|$. As $|H| = 2^t \cdot 3^2 \cdot 7 \cdot 11 \cdot 23$, we have $M/N \cong L_2(7)$ or $L_2(8)$, which implies that $23 \mid |N|$. Hence $O_{23}(N) \neq 1$ by Lemma 2, which implies that $O_{23}(N) = O_{23}(G) \neq 1$, a contradiction.

By the same arguments as the proof of $K/H \cong A_5$, we show that K/H cannot be isomorphic to one of the simple groups: $L_2(7)$, A_6, $L_2(8)$, $L_2(11)$, A_7, $U_3(3)$, M_{11}, $L_3(4)$, A_8, M_{12} and M_{22}.

Suppose that $K/H \cong M_{23}$. Since $|\operatorname{Out}(M_{23})| = \operatorname{Mult}(M_{23}) = 1$, we have that $G \cong H \times M_{23}$, where $|H| = 2^3 \cdot 3$. By checking the character table of M_{23}, it is easy to see that there exists no irreducible character of degree $2^2 \cdot 3^2 \cdot 7 \cdot 23$ in G, a contradiction.

If $K/H \cong M_{24}$, since $|G| = 2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$, one has that $|H| = 1$ and $G = K \cong M_{24}$, which completes the proof of Theorem B. □

Proof of Theorem C We only need to prove the sufficiency.

In this case, $|G| = 2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$ and $L(G) = 385$. Let $\chi \in \operatorname{Irr}(G)$ such that $\chi(1) = L(G) = 5 \cdot 7 \cdot 11$. We assert that $O_{11}(G) = 1$. Otherwise, we have that $|O_{11}(G)| = 11$ and $O_{11}(G)$ is abelian. Hence $\chi(1) \mid |G/O_{11}(G)|$, a contradiction. Similarly, $O_5(G) = O_7(G) = 1$.

If G is solvable, then the Sylow 11-subgroup of G is normal in G by Lemma 2. But $O_{11}(G) = 1$, a contradiction. Therefore, G is non-solvable.

Since G is non-solvable, by Lemma 1, we get that G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a direct product of isomorphic non-abelian simple groups and $|G/K| \mid |\operatorname{Out}(K/H)|$. As $|G| = 2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$, we see that K/H is isomorphic to one of the simple groups: A_5, $L_2(7)$, A_6, $L_2(8)$, $L_2(11)$, A_7, M_{11}, $L_3(4)$, A_8 and M_{22}.

We first assume that $K/H \cong A_5$. Since $|\operatorname{Out}(A_5)| = 2$, we have $|G/K| \mid 2$ and $|H| = 2^t \cdot 3 \cdot 7 \cdot 11$, where $t = 4$ or 5. If H is solvable, then $O_{11}(H) = O_{11}(G) \neq 1$ by Lemma 2, a contradiction. Hence H is non-solvable and H has a normal series $1 \trianglelefteq N \trianglelefteq M \trianglelefteq H$ such that M/N is a direct product of isomorphic non-abelian simple groups and $|H/M| \mid |\operatorname{Out}(M/N)|$. As $|H| = 2^t \cdot 3 \cdot 7 \cdot 11$, one has that $M/N \cong L_3(2)$ and $|N| = 2^s \cdot 11$, where $0 \leq s \leq 2$. Let P be the Sylow 11-subgroup of N. Then P is normal in N by Sylow theorem. Since P is also a Sylow 11-subgroup in G and N is subnormal in G, we have $P \trianglelefteq G$, a contradiction.

Similarly, K/H cannot be isomorphic to the simple groups: $L_2(7)$, A_6, $L_2(8)$, $L_2(11)$, A_7, $L_3(4)$ or A_8.

Assume that $K/H \cong L_2(11)$. Since $|\operatorname{Out}(L_2(11))| = 2$, we have $|G/K| \mid 2$ and $|H| = 2^\alpha \cdot 3 \cdot 7$, where $\alpha = 4$ or 5. Suppose that $H = H'$. Then H has a normal subgroup S such that $H/S \cong L_2(7)$, where $|S| = 2$ or 4. Obviously, we know that $S \leq Z(H)$, and then $S \trianglelefteq G$. Let $\theta \in \operatorname{Irr}(S)$ such that $[\chi_S, \theta] \neq 0$. Then $\theta(1) = 1$ since S is abelian. Let $e = [\chi_S, \theta]$ and $t = |G : I_G(\theta)|$. Then $t = 1$ and $e = \chi(1) = 385$ by the Clifford theorem (see Theorem 6.2 in [5]). But $e^2 \cdot t = [\chi_H, \chi_H] > |G : H|$, a contradiction. Hence $H' < H$. Suppose that $|H/H'| = 2$. Then H/H' is central in G/H. Let β be an irreducible component of χ_H, and let θ be an irreducible component of $\beta_{H'}$. Then $\theta(1) = \beta(1) = 7$ and θ is extendible to β. Hence $\lambda\beta$ is invariant in G for every $\lambda \in \operatorname{Irr}(H/H')$ if β is invariant in G. Since $|H| = 2^\alpha \cdot 3 \cdot 7 \cdot 11$, where $\alpha = 4$ or 5, H has at most 12 irreducible characters of degree 7. Let $t = |G : I_G(\beta)|$. Then $t \leq 12$. Since the index of the maximal subgroup of U containing $I_G(\theta)$ divides t, we have that $t = 1$ or 11 by checking maximal subgroups of $L_2(11)$ (see ATLAS table in [6]). If $t = 11$, then H has exactly 12 irreducible characters of degree 7, and one of them, say δ, is invariant in G. Hence, $\lambda\delta$ is also invariant in G for $\lambda \in \operatorname{Irr}(H/H')$, which forces $t \leq 10$, a contradiction. Therefore $t = 1$ and $e = 55$. But $(55)^2 = [\chi_H, \chi_H] > |G : H|$, a contradiction. By the same reasoning as before, we can prove that $|H/H'| \neq 2^m \cdot 3^n$, where $1 \leq m \leq 3$ and $0 \leq n \leq 1$. If $|H/H'| = 2^m \cdot 3^n$, where $m = 4$ or 5, then the Sylow 7 subgroup of H' is normal in H', and so it is normal in G, a contradiction. Now we assume that $7 \mid |H/H'|$. Let $H' \leq A \trianglelefteq H$ such that $|H/A| = 7$, then $H/A \leq Z(G/A)$. Since $\operatorname{Mult}(L_2(11)) = 2$, we have $G/A \cong H/A \times L_2(11)$. Hence G has a normal series $1 \trianglelefteq N \trianglelefteq M \trianglelefteq G$ such that $M/N \cong L_2(11)$ and $|N| = 2^\alpha \cdot 3$, where $\alpha = 4$ or 5. Let φ be an irreducible component of χ_M, and let η be an irreducible component of φ_N. Then $\varphi(1) = 55$ and $\eta(1) = 1$ by the Clifford theorem (see Theorem 6.2 in [5]). Since $|\operatorname{Aut}(N)|$ is not divided by 5 and 11, one has that $t = |M : I_M(\eta)| = 1$. Therefore $(55)^2 = [\varphi_N, \varphi_N] > |M : N|$, a contradiction.

Suppose that $K/H \cong M_{11}$. Since $|\operatorname{Out}(M_{11})| = \operatorname{Mult}(M_{11}) = 1$, we have $G \cong H \times M_{11}$, where $|H| = 2^3 \cdot 7$. Let $\theta \in \operatorname{Irr}(H)$ such that $[\chi_H, \theta] \neq 0$. Since $\theta(1) \mid \chi(1)$ and $\theta(1) \mid |H|$, one has that $\theta(1) = 7$, which implies that H is a Frobenius group with an elementary kernel of order 8 and a cyclic complement of order 7.

Suppose that $K/H \cong M_{22}$. Since $|G| = 2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$, we have that $|H| = 1$ and $G = K \cong M_{22}$, which completes the proof of Theorem C. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
HX carried out the study of the Mathieu groups M_{11}, M_{12} and M_{23}. YY carried out the study of the Mathieu group M_{24}. GC carried out the study of the Mathieu group M_{22}. All authors read and approved the final manuscript.

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