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Mappings of type Orlicz and generalized Cesáro sequence space

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Abstract

We study the ideal of all bounded linear operators between any arbitrary Banach spaces whose sequence of approximation numbers belong to the generalized Cesáro sequence space and Orlicz sequence space ℓ_M, when $\mathbf{M}(t) = t^p$, $0 < p < \infty$; our results coincide with that known for the classical sequence space ℓ_p.

Keywords: approximation numbers; operator ideal; generalized Cesáro sequence space; Orlicz sequence space

1 Introduction

By $L(X,Y)$, we denote the space of all bounded linear operators from a normed space X into a normed space Y. The set of natural numbers will denote by $\mathbb{N} = \{0,1,2,\dots\}$ and the real numbers by $\mathbb{R}$. By ω, we denote the space of all real sequences. A map which assigns to every operator $T \in L(X,Y)$ a unique sequence $(s_n(T))_{n=0}^{\infty}$ is called an s-function and the number $s_n(T)$ is called the nth s-numbers of T if the following conditions are satisfied:

(a) $\|T\| = s_0(T) \geq s_1(T) \geq \cdots \geq 0$, for all $T \in L(X,Y)$.

(b) $s_{n+m}(T_1 + T_2) \leq s_n(T_1) + \|T_2\|$, for all $T_1, T_2 \in L(X,Y)$.

(c) $s_n(RST) \leq \|R\| s_n(S) \|T\|$, for all $T \in L(X_0,X)$, $S \in L(X,Y)$ and $R \in L(Y,Y_0)$.

(d) $s_n(\lambda T) = |\lambda| s_n(T)$, for all $T \in L(X,Y)$, $\lambda \in \mathbb{R}$.

(e) $\operatorname{rank}(T) \leq n$ If $s_n(T) = 0$, for all $T \in L(X,Y)$.

(f) $s_r(I_n) = \begin{cases} 1 & \text{for } r < n, \\ 0 & \text{for } r \geq n, \end{cases}$ where I_n is the identity operator on the Euclidean space ℓ_2^n.

Example of s-numbers, we mention approximation number $\alpha_r(T)$, Gelfand numbers $c_r(T)$, Kolmogorov numbers $d_r(T)$ and Tichomirov numbers $d_n^*(T)$ defined by:

(I) $\alpha_r(T) = \inf\{\|T - A\| : A \in L(X,Y) \text{ and } \operatorname{rank}(A) \leq r\}$.

(II) $c_r(T) = a_r(J_Y T)$, where J_Y is a metric injection (a metric injection is a one to one operator with closed range and with norm equal one) from the space Y into a higher space $\ell^\infty(\Lambda)$ for suitable index set Λ.

(III) $d_n(T) = \inf_{\dim Y \leq n} \sup_{\|x\| \leq 1} \inf_{y \in Y} \|Tx - y\|$.

(IV) $d_r^*(T) = d_r(J_Y T)$.

All of these numbers satisfy the following condition:

(g) $s_{n+m}(T_1 + T_2) \leq s_n(T_1) + s_m(T_2)$ for all $T_1, T_2 \in L(X,Y)$.

An operator ideal U is a subclass of $L = \{L(X,Y); X, Y \text{ are Banach spaces}\}$ such that its components $\{U(X,Y); X, Y \text{ are Banach spaces}\}$ satisfy the following conditions:

(i) $I_K \in U$, where K denotes the 1-dimensional Banach space, where $U \subset L$.

(ii) If $T_1, T_2 \in U(X,Y)$, then $\lambda_1 T_1 + \lambda_2 T_2 \in U(X,Y)$ for any scalars λ_1, λ_2.

(iii) If $V \in L(X_0,X)$, $T \in U(X,Y)$, $R \in L(Y,Y_0)$ then $RTV \in U(X_0,Y_0)$. See [1–3].

An Orlicz function is a function $M : [0,\infty[\to [0,\infty[$ which is continuous, non-decreasing and convex with $M(0) = 0$ and $M(x) > 0$ for $x > 0$, and $M(x) \to \infty$ as $x \to \infty$. See [4, 5].

If convexity of Orlicz function M is replaced by $M(x + y) \leq M(x) + M(y)$. Then this function is called modulus function, introduced by Nakano [6]; also, see [7, 8] and [9]. An Orlicz function M is said to satisfy Δ_2-condition for all values of u, if there exists a constant $k > 0$, such that $M(2u) \leq kM(u)$ $(u \geq 0)$. The Δ_2-condition is equivalent to $M(lu) \leq klM(u)$ for all values of u and for $l > 1$. Lindentrauss and Tzafriri [10] used the idea of Orlicz function to construct Orlicz sequence space

$$\ell_M = \left\{ x \in \omega : \sum_{n=0}^{\infty} M\left(\frac{|x_n|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\},$$

which is a Banach space with respect to the norm

$$\|x\| = \inf\left\{ \rho > 0 : \sum_{n=0}^{\infty} M\left(\frac{|x_n|}{\rho}\right) \leq 1 \right\}.$$

For $\mathrm{M}(t) = t^p$, $1 \leq p < \infty$ the space ℓ_M coincides with the classical sequence space ℓ_p. Recently, different classes of sequences have been introduced by using an Orlicz function. See [11] and [12].

Remark 1.1 Let M be an Orlicz function then $M(\lambda x) \leq \lambda M(x)$ for all λ with $0 < \lambda < 1$.

For a sequence $p = (p_n)$ of positive real numbers with $p_n \geq 1$, for all $n \in \mathbb{N}$ the generalized Cesáro sequence space is defined by

$$Ces(p_n) = \left\{ x = (x_k) \in \omega : \rho(\lambda x) < \infty \text{ for some } \lambda > 0 \right\},$$

where

$$\rho(x) = \sum_{n=0}^{\infty} \left(\frac{1}{n+1} \sum_{k=0}^{n} |x_k| \right)^{p_n}.$$

The space $Ces(p_n)$ is a Banach space with the norm

$$\|x\| = \inf\left\{ \lambda > 0 : \rho\left(\frac{x}{\lambda}\right) \leq 1 \right\}.$$

If $p = (p_n)$ is bounded, we can simply write

$$Ces(p_n) = \left\{ x \in \omega : \sum_{n=0}^{\infty} \left(\frac{1}{n+1} \sum_{k=0}^{n} |x_k| \right)^{p_n} < \infty \right\}.$$

Also, some geometric properties of $Ces(p_n)$ are studied by Sanhan and Suantai [13].

Throughout this paper, the sequence (p_n) is a bounded sequence of positive real numbers, we denote $e_i = (0, 0, \ldots, 1, 0, 0, \ldots)$ where 1 appears at ith place for all $i \in \mathbb{N}$. Different classes of paranormed sequence spaces have been introduced and their different properties have been investigated. See [14–18] and [19].

For any bounded sequence of positive numbers (p_k), we have the following well-known inequality $|a_k + b_k|^{p_k} \leq 2^{h-1}(|a_k|^{p_k} + |b_k|^{p_k})$, $h = \sup_n p_n$, and $p_k \geq 1$ for all $k \in \mathbb{N}$. See [20].

2 Preliminary and notation

Definition 2.1 A class of linear sequence spaces E, called a special space of sequences (sss) having the following conditions:

(1) E is a linear space and $e_n \in E$, for each $n \in \mathbb{N}$.
(2) If $x \in \omega$, $y \in E$ and $|x_n| \leq |y_n|$, for all $n \in \mathbb{N}$, then $x \in E$ '*i.e.* E is solid',
(3) if $(x_n)_{n=0}^{\infty} \in E$, then $(x_{[\frac{n}{2}]})_{n=0}^{\infty} = (x_0, x_0, x_1, x_1, x_2, x_2, \ldots) \in E$, where $[\frac{n}{2}]$ denotes the integral part of $\frac{n}{2}$.

We call such space E_ρ a pre modular special space of sequences if there exists a function $\rho : E \to [o, \infty[$, satisfies the following conditions:

(i) $\rho(x) \geq 0$ $\forall x \in E_\rho$ and $\rho(\theta) = 0$, where θ is the zero element of E,
(ii) there exists a constant $l \geq 1$ such that $\rho(\lambda x) \leq l|\lambda|\rho(x)$ for all values of $x \in E$ and for any scalar λ,
(iii) for some numbers $k \geq 1$, we have the inequality $\rho(x + y) \leq k(\rho(x) + \rho(y))$, for all $x, y \in E$,
(iv) if $|x_n| \leq |y_n|$, for all $n \in \mathbb{N}$ then $\rho((x_n)) \leq \rho((y_n))$,
(v) for some numbers $k_0 \geq 1$ we have the inequality $\rho((x_n)) \leq \rho((x_{[\frac{n}{2}]})) \leq k_0\rho((x_n))$,
(vi) for each $x = (x(i))_{i=0}^{\infty} \in E$ there exists $s \in \mathbb{N}$ such that $\rho(x(i))_{i=s}^{\infty} < \infty$. This means the set of all finite sequences is ρ-dense in E.
(vii) for any $\lambda > 0$ there exists a constant $\zeta > 0$ such that $\rho(\lambda, 0, 0, 0, \ldots) \geq \zeta\lambda\rho(1, 0, 0, 0, \ldots)$.

It is clear that from condition (ii) that ρ is continuous at θ. The function ρ defines a metrizable topology in E endowed with this topology is denoted by E_ρ.

Example 2.2 ℓ_p is a pre-modular special space of sequences for $0 < p < \infty$, with $\rho(x) = \sum_{n=0}^{\infty} |x_n|^p$.

Example 2.3 ces_p is a pre-modular special space of sequences for $1 < p < \infty$, with $\rho(x) = \sum_{n=0}^{\infty}(\frac{1}{n+1}\sum_{k=0}^{n} |x_n|)^p$.

Definition 2.4

$$U_E^{\mathrm{app}} := \{U_E^{\mathrm{app}}(X, Y); X, Y \text{ are Banach spaces}\},$$

where

$$U_E^{\mathrm{app}}(X, Y) := \{T \in L(X, Y) : (\alpha_n(T))_{n=0}^{\infty} \in E\}.$$

3 Main results

Theorem 3.1 *U_E^{app} is an operator ideal if E is a special space of sequences* (*sss*).

Proof To prove U_E^{app} is an operator ideal:

(i) let $A \in F(X, Y)$ and $\mathrm{rank}(A) = m$ for all $m \in \mathbb{N}$, since E is a linear space and $e_n \in E$ for each $n \in \mathbb{N}$, then $(\alpha_n(A))_{n=0}^{\infty} = (\alpha_0(A), \alpha_1(A), \ldots, \alpha_{m-1}(A), 0, 0, 0, \ldots) = \sum_{i=0}^{m-1} \alpha_i(A)e_i \in E$; for that $A \in U_E^{\mathrm{app}}(X, Y)$, which implies $F(X, Y) \subset U_E^{\mathrm{app}}(X, Y)$.

(ii) Let $T_1, T_2 \in U_E^{\mathrm{app}}(X,Y)$ and $\lambda_1, \lambda_2 \in \mathbb{R}$ then from Definition 2.1 condition (3) we get $(\alpha_{[\frac{n}{2}]}(T_1))_{n=0}^\infty \in E$ and $(\alpha_{[\frac{n}{2}]}(T_2))_{n=0}^\infty \in E$, since $n \geq 2[\frac{n}{2}], \alpha_n(T)$ is a decreasing sequence and from the definition of approximation numbers we get

$$\begin{aligned}\alpha_n(\lambda_1 T_1 + \lambda_2 T_2) &\leq \alpha_{2[\frac{n}{2}]}(\lambda_1 T_1 + \lambda_2 T_2) \leq \alpha_{[\frac{n}{2}]}(\lambda_1 T_1) + \alpha_{[\frac{n}{2}]}(\lambda_2 T_2) \\ &\leq |\lambda_1|\alpha_{[\frac{n}{2}]}(T_1) + |\lambda_2|\alpha_{[\frac{n}{2}]}(T_2) \quad \text{for each } n \in \mathbb{N}.\end{aligned}$$

Since E is a linear space and from Definition 2.1 condition (2) we get $(\alpha_n(\lambda_1 T_1 + \lambda_2 T_2))_{n=0}^\infty \in E$, hence $\lambda_1 T_1 + \lambda_2 T_2 \in U_E^{\mathrm{app}}(X,Y)$.

(iii) If $V \in L(X_0, X)$, $T \in U_E^{\mathrm{app}}(X,Y)$ and $R \in L(Y, Y_0)$, then we get $(\alpha_n(T))_{n=0}^\infty \in E$ and since $\alpha_n(RTV) \leq \|R\|\alpha_n(T)\|V\|$, from Definition 2.1 conditions (1) and (2) we get $(\alpha_n(RTV))_{n=0}^\infty \in E$, then $RTV \in U_E^{\mathrm{app}}(X_0, Y_0)$. □

Theorem 3.2 *$U_{\ell_M}^{\mathrm{app}}$ is an operator ideal, if M is an Orlicz function satisfying Δ_2-condition and there exists a constant $l \geq 1$ such that $M(x+y) \leq l(M(x) + M(y))$.*

Proof

(1-i) Let $x, y \in \ell_M$, since M is non-decreasing, we get $\sum_{n=0}^\infty M(|x_n + y_n|) \leq l[\sum_{n=0}^\infty M(|x_n|) + \sum_{n=0}^\infty M(|y_n|)] < \infty$, then $x + y \in \ell_M$.

(1-ii) $\lambda \in \mathbb{R}$, $x \in \ell_M$ since M satisfies Δ_2-condition, we get $\sum_{n=0}^\infty M(|\lambda x_n|) \leq |\lambda| l \sum_{n=0}^\infty M(|x_n|) < \infty$, for that $\lambda x \in \ell_M$, then from (1-i) and (1-ii) ℓ_M is a linear space over the field of numbers. Also $e_n \in \ell_M$ for each $n \in \mathbb{N}$ since $\sum_{i=0}^\infty M(|e_n(i)|) = M(1) < \infty$.

(2) Let $|x_n| \leq |y_n|$ for each $n \in \mathbb{N}$, $(y_n)_{n=0}^\infty \in \ell_M$, since M is none decreasing, then we get $\sum_{n=0}^\infty M(|x_n|) \leq \sum_{n=0}^\infty M(|y_n|) < \infty$, then $(x_n)_{n=0}^\infty \in \ell_M$.

(3) Let $(x_n)_{n=0}^\infty \in \ell_M$, $\sum_{n=0}^\infty M(|x_{[\frac{n}{2}]}|) \leq 2\sum_{n=0}^\infty M(|x_n|) < \infty$, then $(x_{[\frac{n}{2}]})_{n=0}^\infty \in \ell_M$. Hence, from Theorem 3.1, it follows that $U_{\ell_M}^{\mathrm{app}}$ is an operator ideal. □

Theorem 3.3 *$U_{ces(p_n)}^{\mathrm{app}}$ is an operator ideal, if (p_n) is an increasing sequence of positive real numbers*, $\lim_{n\to\infty} \sup p_n < \infty$ *and* $\lim_{n\to\infty} \inf p_n > 1$.

Proof

(1-i) Let $x, y \in ces(p_n)$ since

$$\begin{aligned}&\sum_{n=0}^\infty \left(\frac{1}{n+1}\sum_{k=0}^n |x_k + y_k|\right)^{p_n} \\ &\quad\leq 2^{h-1}\left(\sum_{n=0}^\infty \left(\frac{1}{n+1}\sum_{k=0}^n |x_k|\right)^{p_n} + \sum_{n=0}^\infty \left(\frac{1}{n+1}\sum_{k=0}^n |y_k|\right)^{p_n}\right), \\ &h = \sup_n p_n,\end{aligned}$$

then $x + y \in ces(p_n)$.

(1-ii) Let $\lambda \in \mathbb{R}$, $x \in ces(p_n)$, then

$$\sum_{n=0}^\infty \left(\frac{1}{n+1}\sum_{k=0}^n |\lambda x_k|\right)^{p_n} \leq \sup_n |\lambda|^{p_n} \sum_{n=0}^\infty \left(\frac{1}{n+1}\sum_{k=0}^n |x_k|\right)^{p_n} < \infty,$$

we get $\lambda x \in ces(p_n)$, from (1-i) and (1-ii) $ces(p_n)$ is a linear space.

To show that $e_m \in ces(p_n)$ for each $m \in \mathbb{N}$, since $\lim_{n\to\infty} \inf p_n > 1$ we have $\sum_{n=0}^{\infty}(\frac{1}{n+1})^{p_n} < \infty$. Thus, we get

$$\rho(e_m) = \sum_{n=m}^{\infty}\left(\frac{1}{n+1}\sum_{k=0}^{n}\left|e_m(k)\right|\right)^{p_n} = \sum_{n=m}^{\infty}\left(\frac{1}{n+1}\right)^{p_n} < \infty.$$

Hence $e_m \in ces(p_n)$.

(2) Let $|x_n| \le |y_n|$ for each $n \in \mathbb{N}$, then

$$\sum_{n=0}^{\infty}\left(\frac{1}{n+1}\sum_{k=0}^{n}|\lambda x_k|\right)^{p_n} \le \sup_n |\lambda|^{p_n}\sum_{n=0}^{\infty}\left(\frac{1}{n+1}\sum_{k=0}^{n}|y_k|\right)^{p_n} < \infty,$$

since $y \in ces(p_n)$. Thus, $x \in ces(p_n)$.

(3) Let $(x_n) \in ces(p_n)$, then we have

$$\begin{aligned}
&\sum_{n=0}^{\infty}\left(\frac{1}{n+1}\sum_{k=0}^{n}|x_{[\frac{k}{2}]}|\right)^{p_n} \\
&\quad= \sum_{n=0}^{\infty}\left(\frac{1}{2n+1}\sum_{k=0}^{2n}|x_{[\frac{k}{2}]}|\right)^{p_{2n}} + \sum_{n=0}^{\infty}\left(\frac{1}{2n+2}\sum_{k=0}^{2n+1}|x_{[\frac{k}{2}]}|\right)^{p_{2n+1}} \\
&\quad= \sum_{n=0}^{\infty}\left(\frac{1}{2n+1}\left(\left(\sum_{k=0}^{n}2|x_k|\right)+|x_n|\right)\right)^{p_n} + \sum_{n=0}^{\infty}\left(\frac{1}{2n+2}\left(\sum_{k=0}^{n}2|x_k|\right)\right)^{p_n} \\
&\quad\le 2^{h-1}\left(\sum_{n=0}^{\infty}\left(\frac{1}{n+1}\left(2\sum_{k=0}^{n}|x_k|\right)\right)^{p_n} + \sum_{n=0}^{\infty}\left(\frac{1}{n+1}\sum_{k=0}^{n}|x_k|\right)^{p_n}\right) \\
&\qquad+ \sum_{n=0}^{\infty}\left(\frac{1}{n+1}\sum_{k=0}^{n}|x_k|\right)^{p_n} \\
&\quad\le 2^{h-1}\left(2^h+1\right)\sum_{n=0}^{\infty}\left(\frac{1}{n+1}\sum_{k=0}^{n}|x_k|\right)^{p_n} + \sum_{n=0}^{\infty}\left(\frac{1}{n+1}\sum_{k=0}^{n}|x_k|\right)^{p_n} \\
&\quad\le \left(2^{2h-1}+2^{h-1}+1\right)\sum_{n=0}^{\infty}\left(\frac{1}{n+1}\sum_{k=0}^{n}|x_k|\right)^{p_n} < \infty.
\end{aligned}$$

Hence, $(x_{[\frac{n}{2}]})_{n=0}^{\infty} \in ces(p_n)$. Hence, from Theorem 3.1 it follows that $U_{ces(p_n)}^{\text{app}}$ is an operator ideal.

□

Theorem 3.4 *Let M be an Orlicz function. Then the linear space $F(X,Y)$ is dense in $U_{\ell_M}^{\text{app}}(X,Y)$.*

Proof Define $\rho(x) = \sum_{n=0}^{\infty} M(|x_n|)$ on ℓ_M. First we prove that every finite mapping $T \in F(X,Y)$ belongs to $U_{\ell_M}^{\text{app}}(X,Y)$. Since $e_m \in \ell_M$ for each $m \in \mathbb{N}$ and ℓ_M is a linear space then for every finite mapping $T \in F(X,Y)$ the sequence $(\alpha_n(T))_{n=0}^{\infty}$ contains only finitely many numbers different from zero. To prove that $U_{\ell_M}^{\text{app}}(X,Y) \subseteq \overline{F(X,Y)}$, let $T \in U_{\ell_M}^{\text{app}}(X,Y)$, we get $(\alpha_n(T))_{n=0}^{\infty} \in \ell_M$, and since $\sum_{n=0}^{\infty} M(\alpha_n(T)) < \infty$, let $\varepsilon \in]0,1]$ then there exists a natural number $s > 0$ such that $\sum_{n=s}^{\infty} M(\alpha_n(T)) < \frac{\varepsilon}{4}$, since ρ is none decreasing and $\alpha_n(T)$ is decreasing for each $n \in \mathbb{N}$, we get

$$sM\big(\alpha_{2s}(T)\big) \le \sum_{n=s+1}^{2s} M\big(\alpha_n(T)\big) \le \sum_{n=s}^{\infty} M\big(\alpha_n(T)\big) < \frac{\varepsilon}{4},$$

then there exists $A \in F_{2s}(X,Y)$, $\operatorname{rank}(A) \le 2s$ with $M(\|T-A\|) < \frac{\varepsilon}{4s}$, and by using the conditions of M we get

$$\begin{aligned} d(T,A) &= \rho\big(\alpha_n(T-A)\big)_{n=0}^{\infty} = \sum_{n=0}^{\infty} M\big(\alpha_n(T-A)\big) \\ &= \sum_{n=0}^{3s-1} M\big(\alpha_n(T-A)\big) + \sum_{n=3s}^{\infty} M\big(\alpha_n(T-A)\big) \\ &\le \sum_{n=0}^{3s-1} M\big(\|T-A\|\big) + \sum_{n=3s}^{\infty} M\big(\alpha_n(T-A)\big) \\ &\le 3sM\big(\|T-A\|\big) + \sum_{n=s}^{\infty} M\big(\alpha_{n+2s}(T-A)\big) \\ &\le 3sM\big(\|T-A\|\big) + \sum_{n=s}^{\infty} M\big(\alpha_n(T)\big) < \varepsilon. \end{aligned}$$

□

Corollary 3.5 *If* $0 < p < \infty$ *and* $\mathrm{M}(t) = t^p$, *we get* $U_{\ell^p}^{\mathrm{app}}(X,Y) = \overline{F(X,Y)}$. *See* [3].

Theorem 3.6 *The linear space* $F(X,Y)$ *is dense in* $U_{ces_{(p_n)}}^{\mathrm{app}}(X,Y)$, *if* (p_n) *is an increasing sequence of positive real numbers with* $\lim_{n\to\infty}\sup p_n < \infty$ *and* $\lim_{n\to\infty}\inf p_n > 1$.

Proof First we prove that every finite mapping $T \in F(X,Y)$ belongs to $U_{ces_{(p_n)}}^{\mathrm{app}}(X,Y)$. Since $e_m \in ces(p_n)$ for each $m \in \mathbb{N}$ and $ces(p_n)$ is a linear space, then for every finite mapping $T \in F(X,Y)$ *i.e.* the sequence $(\alpha_n(T))_{n=0}^{\infty}$ contains only finitely many numbers different from zero. Now we prove that $U_{ces_{(p_n)}}^{\mathrm{app}}(X,Y) \subseteq \overline{F(X,Y)}$. Since $\lim_{n\to\infty}\inf p_n > 1$, we have $\sum_{n=0}^{\infty}(\frac{1}{n+1})^{p_n} < \infty$, let $T \in U_{ces_{(p_n)}}^{\mathrm{app}}(X,Y)$ we get $(\alpha_n(T))_{n=0}^{\infty} \in ces(p_n)$, and since $\rho((\alpha_n(T))_{n=0}^{\infty}) < \infty$, let $\varepsilon \in]0,1]$ then there exists a natural number $s > 0$ such that $\rho((\alpha_n(T))_{n=s}^{\infty}) < \frac{\varepsilon}{2^{h+3}\delta c}$ for some $c \ge 1$, where $\delta = \max\{1, \sum_{n=s}^{\infty}(\frac{1}{n+1})^{p_n}\}$, since $\alpha_n(T)$ is decreasing for each $n \in \mathbb{N}$, we get

$$\begin{aligned} \sum_{n=s+1}^{2s}\left(\frac{1}{n+1}\sum_{k=0}^{n}\alpha_{2s}(T)\right)^{p_n} &\le \sum_{n=s+1}^{2s}\left(\frac{1}{n+1}\sum_{k=0}^{n}\alpha_n(T)\right)^{p_n} \\ &\le \sum_{n=s}^{\infty}\left(\frac{1}{n+1}\sum_{k=0}^{n}\alpha_k(T)\right)^{p_n} < \frac{\varepsilon}{2^{h+3}\delta c}, \end{aligned} \tag{1}$$

then there exists $A \in F_{2s}(X,Y)$,

$$\begin{aligned} \operatorname{rank}(A) \le 2s \quad \text{with} \sum_{n=2s+1}^{3s}\left(\frac{1}{n+1}\sum_{k=0}^{n}\|T-A\|\right)^{p_n} &\le \sum_{n=s+1}^{2s}\left(\frac{1}{n+1}\sum_{k=0}^{n}\|T-A\|\right)^{p_n} \\ &< \frac{\varepsilon}{2^{h+3}\delta c}, \end{aligned} \tag{2}$$

and

$$\sup_{n=s}^{\infty}\left(\sum_{k=0}^{s}\|T-A\|\right)^{p_n} < \frac{\varepsilon}{2^{2h+2}\delta}, \tag{3}$$

since $\alpha_n(T) = \inf\{\|T-A\| : A \in L(X,Y) \text{ and } \operatorname{rank}(A) \leq n\}$. Then there exists a natural number $N > 0$, A_N with $\operatorname{rank}(A_N) \leq N$ and $\|T - A_N\| \leq 2\alpha_N(T)$. Since $\alpha_n(T) \overset{n\to\infty}{\longrightarrow} 0$, then $\|T - A_N\| \overset{N\to\infty}{\longrightarrow} 0$, so we can take

$$\sum_{n=0}^{s}\left(\frac{1}{n+1}\sum_{k=0}^{n}\|T-A\|\right)^{p_n} < \frac{\varepsilon}{2^{h+3}\delta c}, \tag{4}$$

since (p_n) is an increasing sequence and by using (1), (2), (3) and (4), we get

$$\begin{aligned}
d(T,A) &= \rho\big(\alpha_n(T-A)\big)_{n=0}^{\infty} \\
&= \sum_{n=0}^{3s-1}\left(\frac{1}{n+1}\sum_{k=0}^{n}\alpha_k(T-A)\right)^{p_n} + \sum_{n=3s}^{\infty}\left(\frac{1}{n+1}\sum_{k=0}^{n}\alpha_k(T-A)\right)^{p_n} \\
&\leq \sum_{n=0}^{3s}\left(\frac{1}{n+1}\sum_{k=0}^{n}\|T-A\|\right)^{p_n} + \sum_{n=s}^{\infty}\left(\frac{1}{n+1}\sum_{k=0}^{n+2s}\alpha_k(T-A)\right)^{p_{n+2s}} \\
&\leq 3\sum_{n=0}^{s}\left(\frac{1}{n+1}\sum_{k=0}^{n}\|T-A\|\right)^{p_n} \\
&\quad + \sum_{n=s}^{\infty}\left(\frac{1}{n+1}\sum_{k=0}^{2s-1}\alpha_k(T-A) + \frac{1}{n+1}\sum_{k=2s}^{n+2s}\alpha_k(T-A)\right)^{p_n} \\
&\leq 3\sum_{n=0}^{s}\left(\frac{1}{n+1}\sum_{k=0}^{n}\|T-A\|\right)^{p_n} \\
&\quad + 2^{h-1}\left(\sum_{n=s}^{\infty}\left(\frac{1}{n+1}\sum_{k=0}^{2s-1}\alpha_k(T-A)\right)^{p_n} + \sum_{n=s}^{\infty}\left(\frac{1}{n+1}\sum_{k=2s}^{n+2s}\alpha_k(T-A)\right)^{p_n}\right) \\
&\leq 3\sum_{n=0}^{s}\left(\frac{1}{n+1}\sum_{k=0}^{n}\|T-A\|\right)^{p_n} \\
&\quad + 2^{h-1}\left(\sum_{n=s}^{\infty}\left(\frac{1}{n+1}\sum_{k=0}^{2s-1}\|T-A\|\right)^{p_n} + \sum_{n=s}^{\infty}\left(\frac{1}{n+1}\sum_{k=0}^{n}\alpha_{k+2s}(T-A)\right)^{p_n}\right) \\
&\leq 3\sum_{n=0}^{s}\left(\frac{1}{n+1}\sum_{k=0}^{n}\|T-A\|\right)^{p_n} + 2^{2h-1}\left(\sup_{n=s}^{\infty}\left(\sum_{k=0}^{s}\|T-A\|\right)^{p_n}\right)\sum_{n=s}^{\infty}\left(\frac{1}{n+1}\right)^{p_n} \\
&\quad + 2^{h-1}\sum_{n=s}^{\infty}\left(\frac{1}{n+1}\sum_{k=0}^{n}\alpha_k(T)\right)^{p_n} < \varepsilon.
\end{aligned}$$

□

Theorem 3.7 *Let X be a normed space, Y a Banach space and E_ρ be a pre modular special space of sequences (sss), then $U_{E_\rho}^{\mathrm{app}}(X,Y)$ is complete.*

Proof Let (T_m) be a Cauchy sequence in $U^{\mathrm{app}}_{E_\rho}(X,Y)$, then by using Definition 2.1 condition (vii) and since $U^{\mathrm{app}}_{E_\rho}(X,Y) \subseteq L(X,Y)$, we have

$$\begin{aligned}\rho\big(\big(\alpha_n(T_i - T_j)\big)_{n=0}^{\infty}\big) &\geq \rho\big(\alpha_0(T_i - T_j), 0, 0, 0, \ldots\big)\\ &= \rho\big(\|T_i - T_j\|, 0, 0, 0, \ldots\big) \geq \zeta \|T_i - T_j\| \rho(1, 0, 0, 0, \ldots),\end{aligned}$$

then (T_m) is also Cauchy sequence in $L(X,Y)$. Since the space $L(X,Y)$ is a Banach space, then there exists $T \in L(X,Y)$ such that $\|T_m - T\| \overset{m\to\infty}{\longrightarrow} 0$ and since $(\alpha_n(T_m))_{n=0}^{\infty} \in E$ for all $m \in \mathbb{N}$, ρ is continuous at θ and using Definition 2.1(iii), we have

$$\begin{aligned}\rho\big(\alpha_n(T)\big)_{n=0}^{\infty} &= \rho\big(\alpha_n(T - T_m + T_m)\big)_{n=0}^{\infty} \leq k\rho\big(\alpha_{[\frac{n}{2}]}(T_m - T)\big)_{n=0}^{\infty} + k\rho\big(\alpha_{[\frac{n}{2}]}(T_m)\big)_{n=0}^{\infty}\\ &\leq k\rho\big(\big(\|T_m - T\|\big)_{n=0}^{\infty}\big) + k\rho\big(\alpha_n(T_m)\big)_{n=0}^{\infty} < \varepsilon, \quad \text{for some } k \geq 1.\end{aligned}$$

Hence $(\alpha_n(T))_{n=0}^{\infty} \in E$ as such $T \in U^{\mathrm{app}}_{E_\rho}(X,Y)$. □

Corollary 3.8 *Let X be a normed space, Y a Banach space and M be an Orlicz function such that M satisfies Δ_2-condition. Then M is continuous at $\theta = (0,0,0,\ldots)$ and $U^{\mathrm{app}}_{\ell_M}(X,Y)$ is complete.*

Corollary 3.9 *Let X be a normed space, Y a Banach space and (p_n) be an increasing sequence of positive real numbers with $\lim_{n\to\infty} \sup p_n < \infty$ and $\lim_{n\to\infty} \inf p_n > 1$, then $U^{\mathrm{app}}_{ces_{(p_n)}}(X,Y)$ is complete.*

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
NFM gave the idea of the article. AAB carried out the proofs and its application. All authors read and approved the final manuscript.

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