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Hermite-Hadamard type and Fejér type inequalities for general weights (I)

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Abstract

In this paper, we establish some weighted versions of the Hermite-Hadamard type and Fejér type inequalities and from which generalize Hermite-Hadamard inequality, Fejér inequality and several results in (Dragomir in J. Math. Anal. Appl. 167:49-56, 1992; Yang and Hong in Tamkang. J. Math. 28(1):33-37, 1997; Yang and Tseng in J. Math. Anal. Appl. 239:180-187, 1999; Yang and Tseng in Taiwan. J. Math. 7(3):433-440, 2003).

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1 Introduction

Throughout this paper, let $a < b$ in $\mathbb{R}$, $c < d$ in $\mathbb{R}$, $f : [a,b] \to \mathbb{R}$ be convex, the weight function $p : [a,b] \to [0,\infty)$ be integrable and symmetric about the line $s = \frac{a+b}{2}$, the weight function $p_1 : [c,d] \to [0,\infty)$ be integrable and symmetric about the line $s = \frac{c+d}{2}$ and let the weight function $g : [c,d] \to [a,b]$ be continuous and symmetric about the point $(\frac{c+d}{2}, g(\frac{c+d}{2}))$, that is, $\frac{1}{2}[g(s) + g(c+d-s)] = g(\frac{c+d}{2})$ $(s \in [c,d])$. Define the following functions on $[0,1]$:

$$H(t) = \frac{1}{b-a}\int_a^b f\left(ts + (1-t)\frac{a+b}{2}\right) ds;$$

$$H_g(t) = \frac{1}{d-c}\int_c^d f\left(tg(s) + (1-t)g\left(\frac{c+d}{2}\right)\right) ds;$$

$$WH(t) = \int_a^b f\left(ts + (1-t)\frac{a+b}{2}\right)p(s)\, ds;$$

$$WH_g(t) = \int_c^d f\left(tg(s) + (1-t)g\left(\frac{c+d}{2}\right)\right)p_1(s)\, ds;$$

$$F(t) = \frac{1}{(b-a)^2}\int_a^b\int_a^b f\big(ts + (1-t)u\big)\, ds\, du;$$

$$F_g(t) = \frac{1}{(d-c)^2}\int_c^d\int_c^d f\big(tg(s) + (1-t)g(u)\big)\, ds\, du;$$

$$WF(t) = \int_a^b\int_a^b f\big(ts + (1-t)u\big)p(s)p(u)\, ds\, du;$$

$$WF_g(t) = \int_c^d\int_c^d f\big(tg(s) + (1-t)g(u)\big)p_1(s)p_1(u)\, ds\, du;$$

$$P(t) = \frac{1}{2(b-a)} \int_a^b \left[f\left(\left(\frac{1+t}{2} \right) a + \left(\frac{1-t}{2} \right) s \right) + f\left(\left(\frac{1+t}{2} \right) b + \left(\frac{1-t}{2} \right) s \right) \right] ds;$$

$$P_g(t) = \frac{1}{2(d-c)} \int_c^d \left[f\left((1-t) g\left(\frac{s+c}{2} \right) + t g(c) \right) + f\left((1-t) g\left(\frac{s+d}{2} \right) + t g(d) \right) \right] ds;$$

$$WP(t) = \frac{1}{2} \int_a^b \left[f\left(\left(\frac{1+t}{2} \right) a + \left(\frac{1-t}{2} \right) s \right) p\left(\frac{s+a}{2} \right) + f\left(\left(\frac{1+t}{2} \right) b + \left(\frac{1-t}{2} \right) s \right) p\left(\frac{s+b}{2} \right) \right] ds$$

and

$$WP_g(t) = \frac{1}{2} \int_c^d \left[f\left((1-t) g\left(\frac{s+c}{2} \right) + t g(c) \right) p_1\left(\frac{s+c}{2} \right) + f\left((1-t) g\left(\frac{s+d}{2} \right) + t g(d) \right) p_1\left(\frac{s+d}{2} \right) \right] ds.$$

Remark 1

(1) Let $c = a$, $d = b$ and the function $g(s) = s$ on $[a, b]$. Then the functions $H_g(t) = H(t)$, $F_g(t) = F(t)$ and $P_g(t) = P(t)$ on $[0, 1]$.
(2) Let $c = a$, $d = b$ and let the functions $g(s) = s$ and $p_1(s) = p(s)$ on $[a, b]$. Then the functions $WH_g(t) = WH(t)$, $WF_g(t) = WF(t)$ and $WP_g(t) = WP(t)$ on $[0, 1]$.

In 1893, Hadamard [1] established the following inequality.

If the function f is defined as above, then

$$f\left(\frac{a+b}{2} \right) \le \frac{1}{b-a} \int_a^b f(s)\, ds \le \frac{f(a) + f(b)}{2} \tag{1.1}$$

is known as Hermite-Hadamard inequality.

See [2–8] and [9–16] for some results in which this famous integral inequality (1.1) is generalized, improved and extended.

Dragomir [2] established the following Hermite-Hadamard type inequalities related to the functions H, F, which refine the first inequality of (1.1).

Theorem A *Let the functions f, H be defined as in the first page. Then the function H is convex, increasing on $[0, 1]$, and for all $t \in [0, 1]$, we have*

$$f\left(\frac{a+b}{2} \right) = H(0) \le H(t) \le H(1) = \frac{1}{b-a} \int_a^b f(s)\, ds. \tag{1.2}$$

Theorem B *Let the functions f, F be defined as in the first page. Then*:

(1) *The function F is convex on $[0, 1]$, symmetric about $\frac{1}{2}$, F is decreasing on $[0, \frac{1}{2}]$ and*

increasing on $[\frac{1}{2},1]$, *and we have*

$$\sup_{t\in[0,1]} F(t) = F(0) = F(1) = \frac{1}{b-a}\int_a^b f(s)\,ds$$

and

$$\inf_{t\in[0,1]} F(t) = F\left(\frac{1}{2}\right) = \frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\frac{s+u}{2}\right)ds\,du.$$

(2) *We have*

$$f\left(\frac{a+b}{2}\right) \le F\left(\frac{1}{2}\right); \qquad H(t) \le F(t), \quad t\in[0,1]. \tag{1.3}$$

Yang and Hong [12] established the following Hermite-Hadamard type inequality related to the function P, which refines the second inequality of (1.1).

Theorem C *Let the functions* f, P *be defined as in the first and second pages. Then the function* P *is convex, increasing on* $[0,1]$, *and for all* $t\in[0,1]$, *we have*

$$\frac{1}{b-a}\int_a^b f(s)\,ds = P(0) \le P(t) \le P(1) = \frac{f(a)+f(b)}{2}. \tag{1.4}$$

In 1906, Fejér [8] established the following weighted generalization of Hermite-Hadamard inequality (1.1).

Theorem D *Let the functions* f, p *be defined as in the first page. Then*

$$f\left(\frac{a+b}{2}\right)\int_a^b p(s)\,ds \le \int_a^b f(s)p(s)\,ds \le \frac{f(a)+f(b)}{2}\int_a^b p(s)\,ds \tag{1.5}$$

is known as the Fejér inequality.

Yang and Tseng [13, 16] established the following Fejér type inequalities related to the functions WH, WP, WF and which generalize Theorems A-C and refine Fejér inequality (1.5).

Theorem E [13] *Let the functions* f, p, WH, WP *be defined as in the first and second pages. Then the functions* Hg, Pg *are convex and increasing on* $[0,1]$, *and for all* $t\in[0,1]$, *we have*

$$\begin{aligned} f\left(\frac{a+b}{2}\right)\int_a^b g(s)\,ds &= WH(0) \le WH(t) \le WH(1) \\ &= \int_a^b f(s)p(s)\,ds \\ &= WP(0) \le WP(t) \le WP(1) \\ &= \frac{f(a)+f(b)}{2}\int_a^b p(s)\,ds. \end{aligned} \tag{1.6}$$

Theorem F [16] *Let the functions f, p, WH, WF be defined as in the first and second pages. Then we have the following results*:

(1) *The function WF is convex on $[0,1]$ and symmetric about $\frac{1}{2}$.*

(2) *The function WF is decreasing on $[0,\frac{1}{2}]$ and increasing on $[\frac{1}{2},1]$,*

$$\sup_{t\in[0,1]} WF(t) = WF(0) = WF(1) = \int_a^b f(s)p(s)\,ds \tag{1.7}$$

and

$$\inf_{t\in[0,1]} WF(t) = WF\left(\frac{1}{2}\right) = \int_a^b \int_a^b f\left(\frac{s+u}{2}\right)p(s)p(u)\,ds\,du. \tag{1.8}$$

(3) *We have*:

$$f\left(\frac{a+b}{2}\right)\left(\int_a^b p(s)\,ds\right)^2 \le WF\left(\frac{1}{2}\right) \tag{1.9}$$

and

$$WH(t)\int_a^b p(s)\,ds \le WF(t) \tag{1.10}$$

for all $t \in [0,1]$.

In this paper, we establish some weighted versions of the Hermite-Hadamard type and Fejér type inequalities related to the functions H_g, F_g, P_g, WH_g, WF_g, WP_g, which generalize the inequality (1.1) and Theorems A-F.

2 Hermite-Hadamard type inequalities for general weights

In this section, we establish some Hermite-Hadamard type inequalities for general weights, which generalize the Hermite-Hadamard inequality (1.1) and Theorems A-C.

In order to prove the results in this paper, we need the following lemmas.

Lemma 1 (see [9]) *Let the function f be defined as in the first page and let $a \le A \le C \le D \le B \le b$ with $A+B = C+D$. Then*

$$f(C) + f(D) \le f(A) + f(B).$$

The assumptions in Lemma 1 can be weakened as in the following lemma.

Lemma 2 *Let the function f be defined as in the first page and let $A, B, C, D \in [a,b]$ with $A+B = C+D$ and $|C-D| \le |A-B|$. Then*

$$f(C) + f(D) \le f(A) + f(B).$$

Proof Without loss of generalization, we can assume that $a \le A \le B \le b$ and $a \le C \le D \le b$. For $|C-D| \le |A-B|$, we have $A-B \le C-D$ and $D-C \le B-A$. Hence, by the above inequalities and $A+B=C+D$, we get $a \leq A \leq C \leq D \leq B \leq b$. Thus, the proof is completed by Lemma 1. □

Now, we are ready to state and prove our new results.

Theorem 1 *Let the functions f, g be defined as in the first page. Then*:

(1) *We have*

$$f\left(g\left(\frac{c+d}{2}\right)\right) \leq \frac{1}{d-c}\int_c^d f\big(g(s)\big)\,ds. \tag{2.1}$$

(2) *As the function g is monotonic on $[c,d]$, we obtain*

$$\frac{1}{d-c}\int_c^d f\big(g(s)\big)\,ds \leq \frac{f(g(c))+f(g(d))}{2}. \tag{2.2}$$

Proof

(1) Using simple techniques of integration, we have the identity

$$\frac{1}{d-c}\int_c^d f\big(g(s)\big)\,ds = \frac{1}{d-c}\int_c^{\frac{c+d}{2}} \big[f\big(g(s)\big)+g(c+d-s)\big]\,ds. \tag{2.3}$$

Next, using $g(s)+g(c+d-s)=2g(\frac{c+d}{2})$ and

$$\left|g\left(\frac{c+d}{2}\right)-g\left(\frac{c+d}{2}\right)\right| \leq \big|g(s)-g(c+d-s)\big|$$

in Lemma 2, we obtain

$$2f\left(g\left(\frac{c+d}{2}\right)\right) \leq f\big(g(s)\big)+f\big(g(c+d-s)\big), \tag{2.4}$$

where $s \in [c,d]$. Integrating the above inequality over s on $[c, \frac{c+d}{2}]$, dividing both sides by $d-c$ and using the above identity, we obtain the inequality (2.1).

(2) For the monotonicity of g, we have $|g(s)-g(c+d-s)| \leq |g(c)-g(d)|$ for all $s \in [c,d]$. Using the above inequality and $g(s)+g(c+d-s)=g(c)+g(d)$ in Lemma 2, we obtain

$$f\big(g(s)\big)+f\big(g(c+d-s)\big) \leq f\big(g(c)\big)+f\big(g(d)\big), \tag{2.5}$$

where $s \in [c,d]$. Integrating the above inequality over s on $[c, \frac{c+d}{2}]$, dividing both sides by $d-c$ and using the inequality (2.3), we obtain the inequality (2.2). This completes the proof. □

Remark 2 In Theorem 1, let $c=a$, $d=b$ and the function $g(s)=s$ on $[a,b]$. Then Theorem 1 reduces to the Hermite-Hadamard inequality (1.1).

Theorem 2 *Let the functions f, g, H_g be defined as in the first page. Then*:

(1) *The function H_g is convex on $[0,1]$.*

(2) *The function H_g is increasing on* $[0,1]$ *and for all* $t \in [0,1]$, *we have*

$$f\left(g\left(\frac{c+d}{2}\right)\right) = H_g(0) \le H_g(t) \le H_g(1) = \frac{1}{d-c}\int_c^d f\big(g(s)\big)\,ds. \tag{2.6}$$

Proof

(1) It is easily observed from the convexity of f that the function H_g is convex on $[0,1]$.

(2) Using simple techniques of integration, we have the following identity:

$$\begin{aligned} H_g(t) = \frac{1}{d-c}\int_c^{\frac{c+d}{2}} &\left[f\left(tg(s) + (1-t)g\left(\frac{c+d}{2}\right)\right)\right. \\ &\left. + f\left(tg(c+d-s) + (1-t)g\left(\frac{c+d}{2}\right)\right)\right] ds \end{aligned}$$

for all $t \in [0,1]$. Let $t_1 < t_2$ in $[0,1]$. Since $g(s) + g(c+d-s) = 2g(\frac{c+d}{2})$ $(s \in [c,d])$, we obtain

$$\begin{aligned} &\left[t_1g(s) + (1-t_1)g\left(\frac{c+d}{2}\right)\right] + \left[t_1g(c+d-s) + (1-t_1)g\left(\frac{c+d}{2}\right)\right] \\ &\quad = \left[t_2g(s) + (1-t_2)g\left(\frac{c+d}{2}\right)\right] + \left[t_2g(c+d-s) + (1-t_2)g\left(\frac{c+d}{2}\right)\right] \end{aligned}$$

and

$$\begin{aligned} &\left|\left[t_1g(s) + (1-t_1)g\left(\frac{c+d}{2}\right)\right] - \left[t_1g(c+d-s) + (1-t_1)g\left(\frac{c+d}{2}\right)\right]\right| \\ &\quad = t_1\big|g(s) - g(c+d-s)\big| \\ &\quad \le t_2\big|g(s) - g(c+d-s)\big| \\ &\quad = \left|\left[t_2g(s) + (1-t_2)g\left(\frac{c+d}{2}\right)\right] - \left[t_2g(c+d-s) + (1-t_2)g\left(\frac{c+d}{2}\right)\right]\right| \end{aligned}$$

for all $s \in [c, \frac{c+d}{2}]$. Therefore, by Lemma 2, the following inequality holds for all $s \in [c, \frac{c+d}{2}]$:

$$\begin{aligned} &f\left(t_1g(s) + (1-t_1)g\left(\frac{c+d}{2}\right)\right) + f\left(t_1g(c+d-s) + (1-t_1)g\left(\frac{c+d}{2}\right)\right) \\ &\quad \le f\left(t_2g(s) + (1-t_2)g\left(\frac{c+d}{2}\right)\right) + f\left(t_2g(c+d-s) + (1-t_2)g\left(\frac{c+d}{2}\right)\right), \end{aligned} \tag{2.7}$$

where $A = t_2g(s) + (1-t_2)g(\frac{c+d}{2})$, $B = t_2g(c+d-s) + (1-t_2)g(\frac{c+d}{2})$, $C = t_1g(s) + (1-t_1)g(\frac{c+d}{2})$ and $t_1g(c+d-s) + (1-t_1)g(\frac{c+d}{2})$. Integrating the above inequality over s on $[c, \frac{c+d}{2}]$, dividing both sides by $d-c$ and using the above identity, we have

$$H_g(t_1) \le H_g(t_2).$$

Thus, the function H_g is increasing on $[0,1]$ and from which the inequality (2.6) holds. This completes the proof. □

Remark 3

(1) In Theorem 2, the inequality (2.6) refines the inequality (2.1).

(2) In Theorem 2, let $c = a$, $d = b$and the function $g(s) = s$ on $[a,b]$. Then the functions $H_g(t) = H(t)$ $(t \in [0,1])$ and Theorem 1 reduces to Theorem A.

Theorem 3 *Let the functions f, g, P_g be defined as in the first and second pages. Then*:

(1) *The function P_g is convex on* $[0,1]$.

(2) *The function P_g is increasing on* $[0,1]$ *and, for all* $t \in [0,1]$, *we have*

$$\frac{1}{d-c}\int_c^d f\big(g(s)\big)\,ds = P_g(0) \le P_g(t) \le P_g(1) = \frac{f(g(c)) + f(g(d))}{2} \tag{2.8}$$

as the function g is monotonic on $[c,d]$.

Proof

(1) It is easily observed from the convexity of f that the function P_g is convex on $[0,1]$.

(2) Using simple techniques of integration, we have the following identity:

$$\begin{aligned} P_g(t) = \frac{1}{d-c}\int_c^{\frac{c+d}{2}} &\big[f\big(tg(c) + (1-t)g(s)\big) \\ &+ f\big(tg(d) + (1-t)g(c+d-s)\big)\big]\,ds \end{aligned}$$

for all $t \in [0,1]$. Let $t_1 < t_2$ in $[0,1]$. Since $g(s) + g(c+d-s) = 2g(\frac{c+d}{2})$ $(s \in [c,d])$ and the monotonicity of g on $[c,d]$, we obtain

$$\begin{aligned} &\big|g(s) - g(c+d-s)\big| \le \big|g(c) - g(d)\big|, \\ &\big[t_1g(c) + (1-t_1)g(s)\big] + \big[t_1g(d) + (1-t_1)g(c+d-s)\big] \\ &\quad = \big[t_2g(c) + (1-t_2)g(s)\big] + \big[t_2g(d) + (1-t_2)g(c+d-s)\big] \end{aligned}$$

and

$$\begin{aligned} &\big|\big[t_1g(c) + (1-t_1)g(s)\big] - \big[t_1g(d) + (1-t_1)g(c+d-s)\big]\big| \\ &\quad = \big|t_1\big[g(c) - g(d)\big] + (1-t_1)\big[g(s) - g(c+d-s)\big]\big| \\ &\quad = t_1\big|g(c) - g(d)\big| + (1-t_1)\big|g(s) - g(c+d-s)\big| \\ &\quad \le t_1\big|g(c) - g(d)\big| + (1-t_1)\big|g(s) - g(c+d-s)\big| \\ &\quad = \big|\big[t_2g(c) + (1-t_2)g(s)\big] - \big[t_2g(d) + (1-t_2)g(c+d-s)\big]\big| \end{aligned}$$

for all $s \in [c, \frac{c+d}{2}]$. Therefore, by Lemma 2, the following inequality holds for all $s \in [c, \frac{c+d}{2}]$:

$$\begin{aligned} &f\big(t_1g(c) + (1-t_1)g(s)\big) + f\big(t_1g(d) + (1-t_1)g(c+d-s)\big) \\ &\quad \le f\big(t_2g(c) + (1-t_2)g(s)\big) + f\big(t_2g(d) + (1-t_2)g(c+d-s)\big) \end{aligned} \tag{2.9}$$

where $A = t_2g(c) + (1-t_2)g(s)$, $B = t_2g(d) + (1-t_2)g(c+d-s)$, $C = t_1g(c) + (1-t_1)g(s)$ and $t_1g(d) + (1-t_1)g(c+d-s)$. Integrating the above inequality over s on $[c, \frac{c+d}{2}]$, dividing both sides by $d - c$ and using the above identity, we have

$$P_g(t_1) \leq P_g(t_2).$$

Thus, the function P_g is increasing on $[0,1]$ and from which the inequality (2.8) holds. This completes the proof. □

Remark 4

(1) In Theorem 3, the inequality (2.8) refines the inequality (2.2).

(2) In Theorem 3, let $c = a$, $d = b$and the function $g(s) = s$ on $[a,b]$. Then the functions $P_g(t) = P(t)$ $(t \in [0,1])$ and Theorem 3 reduces to Theorem C.

Theorem 4 *Let the functions f, g, H_g, F_g be defined as in the first page. Then we have the following results*:

(1) *The function F_g is convex on* $[0,1]$ *and symmetric about* $\frac{1}{2}$.

(2) *The function F_g is decreasing on* $[0,\frac{1}{2}]$ *and increasing on* $[\frac{1}{2},1]$,

$$\sup_{t\in[0,1]} F_g(t) = F_g(0) = F_g(1) = \frac{1}{d-c}\int_c^d f\bigl(g(s)\bigr)\,ds \tag{2.10}$$

and

$$\begin{aligned}\inf_{t\in[0,1]} F_g(t) &= F_g\left(\frac{1}{2}\right)\\ &= \frac{1}{(d-c)^2}\int_c^d\int_c^d f\left(\frac{g(s)+g(u)}{2}\right)ds\,du.\end{aligned} \tag{2.11}$$

(3) *We have*:

$$H_g(t) \leq F_g(t) \quad \bigl(t \in [0,1]\bigr) \tag{2.12}$$

and

$$f\left(g\left(\frac{c+d}{2}\right)\right) \leq F_g\left(\frac{1}{2}\right). \tag{2.13}$$

Proof

(1) It is easily observed from the convexity of f that the function F_g is convex on $[0,1]$. By changing variables, we have

$$F_g(t) = F_g(1-t), \quad t \in [0,1]$$

from which we get that the function F_g is symmetric about $\frac{1}{2}$.

(2) Let $t_1 < t_2$ in $[0,\frac{1}{2}]$. Then $t_2 + (1-t_2) = t_1 + (1-t_1)$, $|t_2 - (1-t_2)| \leq |t_1 - (1-t_1)|$ and by Lemma 2, we obtain

$$\frac{1}{2}\bigl[F_g(t_2) + F_g(1-t_2)\bigr] \leq \frac{1}{2}\bigl[F_g(t_1) + F_g(1-t_1)\bigr]. \tag{2.14}$$

Using the symmetry of F_g, we have

$$F_g(t_1) = \frac{1}{2}\big[F_g(t_1) + F_g(1-t_1)\big], \tag{2.15}$$

$$F_g(t_2) = \frac{1}{2}\big[F_g(t_2) + F_g(1-t_2)\big] \tag{2.16}$$

From (2.14)-(2.16), we obtain that the function F_g is decreasing on $[0, \frac{1}{2}]$. Since the function F_g is symmetric about $\frac{1}{2}$ and the function F_g is decreasing on $[0, \frac{1}{2}]$, we obtain that the function F_g is increasing on $[\frac{1}{2}, 1]$. Using the symmetry and monotonicity of F_g, we derive the inequalities (2.10) and (2.11).

(3) Using the substitution rules for integration, we have the identity

$$\begin{aligned} F_g(t) = {}& \frac{1}{(d-c)^2}\int_c^d \int_c^{\frac{c+d}{2}} \Big[f\big(tg(s) + (1-t)g(u)\big) \\ & + f\big(tg(s) + (1-t)g(c+d-u)\big)\Big]\,du\,ds \end{aligned}$$

for all $t \in [0,1]$. Let $t \in [0,1]$. Since $g(u) + g(c+d-u) = 2g(\frac{c+d}{2})$ $(u \in [c,d])$, we obtain

$$\begin{aligned} & 2\left[tg(s) + (1-t)g\left(\frac{c+d}{2}\right)\right] \\ & \quad = \big[tg(s) + (1-t)g(u)\big] + \big[tg(s) + (1-t)g(c+d-u)\big] \end{aligned}$$

and

$$\begin{aligned} & \left|\left[tg(s) + (1-t)g\left(\frac{c+d}{2}\right)\right] - \left[tg(s) + (1-t)g\left(\frac{c+d}{2}\right)\right]\right| \\ & \quad \le \Big|\big[tg(s) + (1-t)g(u)\big] - \big[tg(s) + (1-t)g(c+d-u)\big]\Big| \end{aligned}$$

for all $s \in [c,d]$ and $u \in [c, \frac{c+d}{2}]$. Therefore, by Lemma 2, the following inequality holds for all $s \in [c,d]$ and $u \in [c, \frac{c+d}{2}]$:

$$\begin{aligned} & 2f\left(tg(s) + (1-t)g\left(\frac{c+d}{2}\right)\right) \\ & \quad \le f\big(tg(s) + (1-t)g(u)\big) + f\big(tg(s) + (1-t)g(c+d-u)\big), \end{aligned} \tag{2.17}$$

where $A = tg(s) + (1-t)g(u)$, $B = tg(s) + (1-t)g(c+d-u)$ and $C = D = tg(s) + (1-t)g(\frac{c+d}{2})$. Dividing the above inequality by $(d-c)^2$, integrating it over s on $[c,d]$, over u on $[c, \frac{c+d}{2}]$ and using the above identity, we derive the inequality (2.12).

From the inequalities (2.6), (2.12) and the monotonicity of H_g, we derive the inequality (2.13).

This completes the proof. □

Remark 5 In Theorem 4, let $c = a$, $d = b$ and the function $g(s) = s$ on $[a,b]$. Then the functions $F_g(t) = F(t)$ $(t \in [0,1])$ and Theorem 4 reduces to Theorem B.

3 Fejér type inequalities for general weights

In this section, we establish some Fejér type inequalities for general weights which generalize Theorems D-F.

Theorem 5 *Let the functions f, g, p_1 be defined as in the first page. Then*:

(1) *We have*

$$f\left(g\left(\frac{c+d}{2}\right)\right)\int_c^d p_1(s)\,ds \le \int_c^d f\big(g(s)\big)p_1(s)\,ds. \tag{3.1}$$

(2) *As the function g is monotonic on $[c,d]$, we obtain*

$$\int_c^d f\big(g(s)\big)p_1(s)\,ds \le \frac{f(g(c))+f(g(d))}{2}\int_c^d p_1(s)\,ds. \tag{3.2}$$

Proof

(1) Using simple techniques of integration and the hypothesis of p_1, we have the identities

$$\int_c^d f\big(g(s)\big)p_1(s)\,ds = \int_c^{\frac{c+d}{2}} \big[f\big(g(s)\big)+f\big(g(c+d-s)\big)\big]p_1(s)\,ds \tag{3.3}$$

and

$$\int_c^{\frac{c+d}{2}} p_1(s)\,ds = \frac{1}{2}\int_c^d p_1(s)\,ds. \tag{3.4}$$

Proceeding as in the proof of Theorem 1, we also obtain the inequality (2.4). Multiplying the inequality (2.4) by $p_1(s)$, integrating it over s on $[c, \frac{c+d}{2}]$ and using the above identities, we obtain the inequality (3.1).

(2) Proceeding as in the proof of Theorem 1, we also obtain the inequality (2.5). Multiplying the inequality (2.5) by $p_1(s)$, integrating it over s on $[c, \frac{c+d}{2}]$ and using the above identities, we obtain the inequality (3.2). This completes the proof. □

Remark 6

(1) Let $c = a$, $d = b$ and let the functions $g(s) = s$ and $p_1(s) = p(s)$ on $[a,b]$. Then Theorem 5 reduces to Fejér inequality (1.5).

(2) Let the function $p_1(s) \equiv \frac{1}{d-c}$ on $[c,d]$. Then Theorem 5 reduces to Theorem 1.

Theorem 6 *Let the functions f, g, p_1, WH_g be defined as in the first page. Then*:

(1) *The function WH_g is convex on* $[0,1]$.

(2) *The function WH_g is increasing on* $[0,1]$ *and, for all $t \in [0,1]$, we have*

$$\begin{aligned} f\left(g\left(\frac{c+d}{2}\right)\right)\int_c^d p_1(s)\,ds &= WH_g(0) \\ &\le WH_g(t) \\ &\le WH_g(1) = \int_c^d f\big(g(s)\big)p_1(s)\,ds. \end{aligned} \tag{3.5}$$

Proof

(1) It is easily observed from the convexity of f and the hypothesis of p_1 that the function WH_g is convex on $[0,1]$.

(2) Using simple techniques of integration and the hypothesis of p_1, we have the following identity:

$$WH_g(t) = \int_c^{\frac{c+d}{2}} \left[f\left(tg(s) + (1-t)g\left(\frac{c+d}{2}\right)\right) + f\left(tg(c+d-s) + (1-t)g\left(\frac{c+d}{2}\right)\right)\right] p_1(s)\,ds$$

for all $t \in [0,1]$.

Let $t_1 < t_2$ in $[0,1]$. Proceeding as in the proof of Theorem 2, we also obtain the inequality (2.7). Multiplying the inequality (2.7) by $p_1(s)$, integrating it over s on $[c, \frac{c+d}{2}]$ and using the above identity, we obtain

$$WH_g(t_1) \le WH_g(t_2).$$

Thus, the function WH_g is increasing on $[0,1]$ and from which the inequality (3.5) holds. This completes the proof. □

Remark 7

(1) In Theorem 6, the inequality (3.5) refines the inequality (3.1).

(2) Let the function $p_1(s) \equiv \frac{1}{d-c}$ on $[c,d]$. Then Theorem 6 reduces to Theorem 2.

Theorem 7 *Let the functions f, g, p_1, WP_g be defined as in the first and second pages. Then*:

(1) *The function WP_g is convex on* $[0,1]$.

(2) *The function WP_g is increasing on* $[0,1]$ *and, for all $t \in [0,1]$, we have*

$$\begin{aligned}\int_c^d f\big(g(s)\big)p_1(s)\,ds &= WP_g(0) \\ &\le WP_g(t) \\ &\le WP_g(1) = \frac{f(g(c)) + f(g(d))}{2}\int_c^d p_1(s)\,ds \end{aligned} \tag{3.6}$$

as the function g is monotonic on $[c,d]$.

Proof

(1) It is easily observed from the convexity of f and the hypothesis of p_1 that the function WP_g is convex on $[0,1]$.

(2) Using simple techniques of integration and the hypothesis of p_1, we have the following identity:

$$WP_g(t) = \int_c^{\frac{c+d}{2}} \left[f\big(tg(c) + (1-t)g(s)\big) + f\big(tg(d) + (1-t)g(c+d-s)\big)\right] p_1(s)\,ds$$

for all $t \in [0,1]$.

Let $t_1 < t_2$ in $[0,1]$. Proceeding as in the proof of Theorem 3, we also obtain the inequality (2.9). Multiplying the inequality (2.9) by $p_1(s)$, integrating it over s on $[c, \frac{c+d}{2}]$ and using the above identity, we obtain

$$WP_g(t_1) \le WP_g(t_2).$$

Thus, the function WP_g is increasing on $[0,1]$ and from which the inequality (3.6) holds. This completes the proof. □

Remark 8

(1) In Theorem 7, the inequality (3.6) refines the inequality (3.2).

(2) Let the function $p_1(s) \equiv \frac{1}{d-c}$ on $[c,d]$. Then Theorem 7 reduces to Theorem 3.

Remark 9 Let $c = a$, $d = b$ and let the functions $g(s) = s$ and $p_1(s) = p(s)$ on $[a,b]$. Then Theorems 6 and 7 reduce to Theorem E.

Theorem 8 *Let the functions f, g, p_1, WH_g, WF_g be defined as in the first page. Then we have the following results*:

(1) *The function WF_g is convex on* $[0,1]$ *and symmetric about* $\frac{1}{2}$.

(2) *The function WF_g is decreasing on* $[0,\frac{1}{2}]$ *and increasing on* $[\frac{1}{2},1]$,

$$\sup_{t\in[0,1]} WF_g(t) = WF_g(0) = WF_g(1) = \int_c^d f\big(g(s)\big)p_1(s)\,ds$$

and

$$\inf_{t\in[0,1]} WF_g(t) = WF_g\left(\frac{1}{2}\right) = \int_c^d \int_c^d f\left(\frac{g(s)+g(u)}{2}\right)p_1(s)p_1(u)\,ds\,du.$$

(3) *We have*

$$WH_g(t)\int_c^d p_1(s)\,ds \le WF_g(t) \quad \big(t\in[0,1]\big) \tag{3.7}$$

and

$$f\left(g\left(\frac{c+d}{2}\right)\right)\left(\int_c^d p_1(s)\,ds\right)^2 \le WF_g\left(\frac{1}{2}\right). \tag{3.8}$$

Proof

(1)-(2) Proceeding as in the proof of Theorem 4, the parts (1) and (2) hold.

(3) Using the substitution rules for integration and the hypothesis of p_1, we have the identity

$$\begin{aligned} WF_g(t) = \int_c^d \int_c^{\frac{c+d}{2}} &\big[f\big(tg(s)+(1-t)g(u)\big) \\ &+ f\big(tg(s)+(1-t)(c+d-u)\big)\big]p_1(u)p_1(s)\,du\,ds \end{aligned} \tag{3.9}$$

for all $t \in [0,1]$. Proceeding as in the proof of Theorem 4, we also obtain the inequality (2.17). Multiplying the inequality (2.17) by $p_1(u)p_1(s)$, integrating it over s on $[c,d]$, over u on $[c, \frac{c+d}{2}]$ and using the identities (3.4) and (3.9), we obtain the inequality (3.7).

From the inequalities (3.5), (3.7) and the monotonicity of WH_g, we derive the inequality (3.8).

This completes the proof. □

Remark 10

(1) Theorem 8 refines the inequality (3.1).
(2) Let the function $p_1(s) \equiv \frac{1}{d-c}$ on $[c,d]$. Then Theorem 8 reduces to Theorem 2.
(3) Let $c = a$, $d = b$ and the functions $g(s) = s$ and $p_1(s) = p(s)$ on $[a,b]$. Then Theorem 8 reduces to Theorem F.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors did not provide this information.

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References

1. Hadamard, J: Étude sur les propriétés des fonctions entières en particulier d'une fonction considérée par Riemann. J. Math. Pures Appl. **58**, 171-215 (1893)
2. Dragomir, SS: Two mappings in connection to Hadamard's inequalities. J. Math. Anal. Appl. **167**, 49-56 (1992)
3. Dragomir, SS: A refinement of Hadamard's inequality for isotonic linear functionals. Tamkang. J. Math. **24**, 101-106 (1993)
4. Dragomir, SS: On the Hadamard's inequality for convex on the co-ordinates in a rectangle from the plane. Taiwan. J. Math. **5**(4), 775-788 (2001)
5. Dragomir, SS: Further properties of some mapping associated with Hermite-Hadamard inequalities. Tamkang. J. Math. **34**(1), 45-57 (2003)
6. Dragomir, SS, Cho, Y-J, Kim, S-S: Inequalities of Hadamard's type for Lipschitzian mappings and their applications. J. Math. Anal. Appl. **245**, 489-501 (2000)
7. Dragomir, SS, Milošević, DS, Sándor, J: On some refinements of Hadamard's inequalities and applications. Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. **4**, 3-10 (1993)
8. Fejér, L: Über die Fourierreihen, II. Math. Naturwiss Anz Ungar. Akad. Wiss. **24**, 369-390 (1906) (In Hungarian)
9. Hwang, D-Y, Tseng, K-L, Yang, G-S: Some Hadamard's inequalities for co-ordinated convex functions in a rectangle from the plane. Taiwan. J. Math. **11**(1), 63-73 (2007)
10. Tseng, K-L, Hwang, S-R, Dragomir, SS: On some new inequalities of Hermite-Hadamard-Fejér type involving convex functions. Demonstr. Math. **XL**(1), 51-64 (2007)
11. Tseng, K-L, Yang, G-S, Hsu, K-C: On some inequalities of Hadamard's type and applications. Taiwan. J. Math. **13**(6B), 1929-1948 (2009)
12. Yang, G-S, Hong, M-C: A note on Hadamard's inequality. Tamkang. J. Math. **28**(1), 33-37 (1997)
13. Yang, G-S, Tseng, K-L: On certain integral inequalities related to Hermite-Hadamard inequalities. J. Math. Anal. Appl. **239**, 180-187 (1999)
14. Yang, G-S, Tseng, K-L: Inequalities of Hadamard's type for Lipschitzian mappings. J. Math. Anal. Appl. **260**, 230-238 (2001)
15. Yang, G-S, Tseng, K-L: On certain multiple integral inequalities related to Hermite-Hadamard inequalities. Util. Math. **62**, 131-142 (2002)
16. Yang, G-S, Tseng, K-L: Inequalities of Hermite-Hadamard-Fejér type for convex functions and Lipschitzian functions. Taiwan. J. Math. **7**(3), 433-440 (2003)

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