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L_p Blaschke-Minkowski homomorphisms

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Abstract
In this paper, we introduce the concept of L_p Blaschke-Minkowski homomorphisms and show that those maps are represented by a spherical convolution operator. And then we consider the Busemann-Petty type problem for L_p Blaschke-Minkowski homomorphisms.
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1 Introduction

The theory of real valued valuations is at the center of convex geometry. Blaschke started a systematic investigation in the 1930s, and then Hadwiger [1] focused on classifying valuations on compact convex sets in $\mathbb{R}^n$ and obtained the famous Hadwiger's characterization theorem. Schneider [2] obtained first results on convex body valued valuations with Minkowski addition in 1970s. The survey [3] and the book [4] are an excellent source for the classical theory of valuations. Some more recent results can see [1, 5–20].

An operator $Z : \mathcal{K}^n \to \mathcal{K}^n$ is called a Minkowski valuation if

$$Z(K \cup L) + Z(K \cap L) = ZK + ZL, \tag{1.1}$$

whenever $K, L, K \cup L \in \mathcal{K}^n$, and here + is the Minkowski addition.

A Minkowski valuation Z is called SO(n) equivariant, if for all $\vartheta \in \mathrm{SO}(n)$ and all $K \in \mathcal{K}^n$,

$$Z(\vartheta K) = \vartheta ZK. \tag{1.2}$$

A Minkowski valuation Z is called homogeneity of degree p, if for all $K \in \mathcal{K}^n$ and all $\lambda \geq 0$,

$$Z(\lambda K) = \lambda^p ZK. \tag{1.3}$$

A map $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ is called a Blaschke-Minkowski homomorphism if it is continuous, SO(n) equivariant and satisfies $\Phi(K \# L) = \Phi K + \Phi L$, where # denotes the Blaschke addition, *i.e.*, $S(K \# L, \cdot) = S(K, \cdot) + S(L, \cdot)$.

Obviously, a Blaschke-Minkowski homomorphism is a continuous Minkowski valuation which is SO(n) equivariant and $(n-1)$-homogeneous. Schuster introduced Blaschke-Minkowski homomorphisms and studied the Busemann-Petty type problem for them.

Theorem A [15] *If $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ be a Blaschke-Minkowski homomorphism, then there is a weakly positive $g \in \mathcal{C}(S^{n-1}, \widehat{e})$, unique up to a linear function, such that*

$$h(\Phi K, \cdot) = S(K, \cdot) * g.$$

Theorem B [16] *Let $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ be a Blaschke-Minkowski homomorphism. If $K \in \Phi\mathcal{K}^n$ and $L \in \mathcal{K}^n$, then*

$$\Phi K \subseteq \Phi L \quad \Rightarrow \quad V(K) \le V(L),$$

and $V(K) = V(L)$ if and only if $K = L$.

Recently, the investigations of convex body and star body valued valuations have received great attention from a series of articles by Ludwig [10–13]; see also [8]. She started systematic studies and established complete classifications of convex and star body valued valuations with respect to L_p Minkowski addition and L_p radial which are compatible with the action of the group $GL(n)$. Based on these results, in this article we study L_p Blaschke-Minkowski homomorphisms which are continuous, $(\frac{n}{p} - 1)$-homogeneous and $\mathrm{SO}(n)$ equivariant.

Theorem 1.1 *Let $p > 1$ and $p \neq n$. If $\Phi_p : \mathcal{K}_e^n \to \mathcal{K}_e^n$ be an L_p Blaschke-Minkowski homomorphism, then there is a nonnegative function $g \in \mathcal{C}(S^{n-1}, \widehat{e})$, such that*

$$h^p(\Phi_p K, \cdot) = S_p(K, \cdot) * g. \tag{1.4}$$

Theorem 1.2 *Let $1 < p < n$ and p is not an even integer, and let $\Phi_p : \mathcal{K}_e^n \to \mathcal{K}_e^n$ be an L_p Blaschke-Minkowski homomorphism. If $K \in \mathcal{K}_e^n$ and $L \in \Phi_p\mathcal{K}_e^n$, then*

$$\Phi_p K \subseteq \Phi_p L \quad \Rightarrow \quad V(K) \le V(L). \tag{1.5}$$

If $p > n$ and p is not an even integer, then

$$\Phi_p K \subseteq \Phi_p L \quad \Rightarrow \quad V(K) \ge V(L), \tag{1.6}$$

and $V(K) = V(L)$, if and only if $K = L$.

2 Notation and background material

Let $\mathcal{K}_0^n$ denote the set of convex bodies containing the origin in their interiors, and let $\mathcal{K}_e^n$ denote origin-symmetric convex bodies. In this paper, we restrict the dimension of $\mathbb{R}^n$ to $n \ge 3$. A convex body $K \in \mathcal{K}^n$ is uniquely determined by its support function, $h(K, \cdot)$. From the definition of $h(K, \cdot)$, it follows immediately that for $\lambda > 0$ and $\vartheta \in \mathrm{SO}(n)$,

$$h(\lambda K, u) = \lambda h(K, u) \quad \text{and} \quad h(\vartheta K, u) = h\bigl(K, \vartheta^{-1} u\bigr), \tag{2.1}$$

where ϑ^{-1} is the inverse of ϑ.

For $K, L \in \mathcal{K}_0^n$, $p \geq 1$, and $\varepsilon > 0$, the L_p Minkowski addition $K +_p \varepsilon \cdot L \in \mathcal{K}_0^n$ is defined by (see [21])

$$h(K +_p \varepsilon \cdot L, \cdot)^p = h(K, \cdot)^p + \varepsilon h(L, \cdot)^p, \tag{2.2}$$

where '$\cdot$' in $\varepsilon \cdot L$ denotes the Firey scalar multiplication, *i.e.*, $\varepsilon \cdot L = \varepsilon^{\frac{1}{p}} L$.

If $K, L \in \mathcal{K}_0^n$, then for $p \geq 1$, the L_p mixed volume, $V_p(K, L)$, of K and L is defined by (see [21])

$$V_p(K, L) = \lim_{\varepsilon \to 0^+} \frac{V(K +_p \varepsilon \cdot L) - V(K)}{\varepsilon}.$$

Corresponding to each $K \in \mathcal{K}_0^n$, there is a positive Borel measure, $S_p(K, \cdot)$, on S^{n-1} such that (see [21])

$$V_p(K, L) = \frac{1}{n} \int_{S^{n-1}} h(L, u)^p \, dS_p(K, u), \tag{2.3}$$

for each $L \in \mathcal{K}_0^n$. The measure $S_p(K, \cdot)$ is just the L_p surface area measure of K, which is absolutely continuous with respect to classical surface area measure $S(K, \cdot)$, and has a Radon-Nikodym derivative

$$\frac{dS_p(K, \cdot)}{dS(K, \cdot)} = h(K, \cdot)^{1-p}. \tag{2.4}$$

A convex body $K \in \mathcal{K}_0^n$ is said to have a p-curvature function (see [21]) $f_p(K, \cdot) : S^{n-1} \to \mathbb{R}$, if its L_p surface area measure $S_p(K, \cdot)$ is absolutely continuous with respect to spherical Lebesgue measure S and the Radon-Nikodym derivative

$$\frac{dS_p(K, \cdot)}{dS} = f_p(K, \cdot). \tag{2.5}$$

From the formula (2.3), it follows immediately that for each $K \in K_0^n$,

$$V_p(K, K) = V(K).$$

The Minkowski inequality for the L_p mixed volume states that (see [21]): For $K, L \in \mathcal{K}_0^n$, if $p \geq 1$, then

$$V_p(K, L) \geq V(K)^{\frac{n-p}{n}} V(L)^{\frac{p}{n}}, \tag{2.6}$$

if $p > 1$, equality holds if and only if K and L are dilates; if $p = 1$, equality holds if and only if K and L are homothetic.

The L_p Minkowski problem asks for necessary and sufficient conditions for a Borel measure μ on S^{n-1} to be the L_p surface area measure of a convex body. Lutwak [22] gave a weak solution to the L_p Minkowski problem as follows.

Theorem C *If μ is an even position Borel measure on S^{n-1}, which is not concentrated on any great subsphere, then for any $p > 1$ and $p \neq n$, there exists a unique origin-symmetric* *convex bodies* $K \in \mathcal{K}_e^n$, *such that*

$$S_p(K, \cdot) = \mu.$$

From (2.4), for $\lambda > 0$, we have

$$S_p(\lambda K, \cdot) = \lambda^{n-p} S_p(K, \cdot). \tag{2.7}$$

Noting the fact $S(\vartheta K, \cdot) = \vartheta S(K, \cdot)$ for $\vartheta \in \mathrm{SO}(n)$ and (2.1), one can obtain

$$S_p(\vartheta K, \cdot) = \vartheta S_p(K, \cdot), \tag{2.8}$$

where $\vartheta S_p(K, \cdot)$ is the image measure of $S_p(K, \cdot)$ under the rotation ϑ. Obviously, $S_1(K, \cdot)$ is just $S(K, \cdot)$.

The L_p Blaschke addition $K \#_p L$ of $K, L \in \mathcal{K}_0^n$ is the convex body with

$$S_p(K \#_p L, \cdot) = S_p(K, \cdot) + S_p(L, \cdot). \tag{2.9}$$

Some basic notions on spherical harmonics will be required. The article by Grinberg and Zhang [23] and the article by Schuster [16] are excellent general references on spherical harmonics. As usual, $\mathrm{SO}(n)$ and S^{n-1} will be equipped with the invariant probability measures. Let $\mathcal{C}(\mathrm{SO}(n))$, $\mathcal{C}(S^{n-1})$ be the spaces of continuous functions on $\mathrm{SO}(n)$ and S^{n-1} with uniform topology and $\mathcal{M}(\mathrm{SO}(n))$, $\mathcal{M}(S^{n-1})$ their dual spaces of signed finite Borel measures with weak* topology. The group $\mathrm{SO}(n)$ acts on these spaces by left translation, *i.e.*, for $f \in \mathcal{C}(S^{n-1})$ and $\mu \in \mathcal{M}(S^{n-1})$, we have $\vartheta f(u) = f(\vartheta^{-1}u)$, $\vartheta \in \mathrm{SO}(n)$, and $\vartheta\mu$ is the image measure of μ under the rotation ϑ.

The sphere S^{n-1} is identified with the homogeneous space $\mathrm{SO}(n)/\mathrm{SO}(n-1)$, where $\mathrm{SO}(n-1)$ denotes the subgroup of rotations leaving the pole $\widehat{e}$ of S^{n-1} fixed. The projection from $\mathrm{SO}(n)$ onto S^{n-1} is $\vartheta \mapsto \widehat{\vartheta} := \vartheta\widehat{e}$. Functions on S^{n-1} can be identified with right $\mathrm{SO}(n-1)$-invariant functions on $\mathrm{SO}(n)$, by $\check{f}(\vartheta) = f(\widehat{\vartheta})$, for $f \in \mathcal{C}(S^{n-1})$. In fact, $\mathcal{C}(S^{n-1})$ is isomorphic to the subspace of right $\mathrm{SO}(n-1)$-invariant functions in $\mathcal{C}(\mathrm{SO}(n))$.

The convolution $\mu * f \in \mathcal{C}(S^{n-1})$ of a measure $\mu \in \mathcal{M}(\mathrm{SO}(n))$ and a function $f \in \mathcal{C}(S^{n-1})$ is defined by

$$(\mu * f)(u) = \int_{\mathrm{SO}(n)} \vartheta f(u)\, d\mu(\vartheta). \tag{2.10}$$

The canonical pairing of $f \in \mathcal{C}(S^{n-1})$ and $\mu \in \mathcal{M}(S^{n-1})$ is defined by

$$\langle \mu, f \rangle = \langle f, \mu \rangle = \int_{S^{n-1}} f(u)\, d\mu(u). \tag{2.11}$$

A function $f \in \mathcal{C}(S^{n-1})$ is called zonal, if $\vartheta f = f$ for every $\vartheta \in \mathrm{SO}(n-1)$. Zonal functions depend only on the value $u \cdot \widehat{e}$. The set of continuous zonal functions on S^{n-1} will be denoted by $\mathcal{C}(S^{n-1}, \widehat{e})$ and the definition of $\mathcal{M}(S^{n-1}, \widehat{e})$ is analogous. A map $\Lambda : \mathcal{C}[-1, 1] \to \mathcal{C}(S^{n-1}, \widehat{e})$ is defined by

$$\Lambda f(u) = f(u \cdot \widehat{e}), \quad u \in S^{n-1}. \tag{2.12}$$

The map Λ is also an isomorphism between functions on $[-1,1]$ and zonal functions on S^{n-1}. If $f \in \mathcal{C}(S^{n-1})$, $\mu \in \mathcal{M}(S^{n-1},\widehat{e})$ and $\eta \in \mathrm{SO}(n)$, then

$$(f * \mu)(\widehat{\eta}) = \int_{S^{n-1}} f(\eta u)\, d\mu(u). \tag{2.13}$$

If $\mu \in \mathcal{M}(S^{n-1},\widehat{e})$, for each $f \in \mathcal{C}(S^{n-1})$ and every $\vartheta \in \mathrm{SO}(n)$, then

$$(\vartheta f) * \mu = \vartheta(f * \mu). \tag{2.14}$$

We denote $\mathcal{H}_k^n$ by the finite dimensional vector space of spherical harmonics of dimension n and order k, and let $N(n,k)$ be the dimension of $\mathcal{H}_k^n$. The space of all finite sums of spherical harmonics of dimension n is denoted by $\mathcal{H}^n$. The spaces $\mathcal{H}_k^n$ are pairwise orthogonal with respect to the usual inner product on $\mathcal{C}(S^{n-1})$. Clearly, $\mathcal{H}_k^n$ is invariant with respect to rotations.

Let $P_k^n \in \mathcal{C}[-1,1]$ denote the Legendre polynomial of dimension n and order k. The zonal function ΛP_k^n is up to a multiplicative constant the unique zonal spherical harmonic in $\mathcal{H}_k^n$. In each space $\mathcal{H}_k^n$ we choose an orthonormal basis $H_{k1},\dots,H_{kN(n,k)}$. The collection $\{H_{k1},\dots,H_{kN(n,k)} : k \in \mathbb{N}\}$ forms a complete orthogonal system in $\mathcal{L}^2(S^{n-1})$. In particular, for every $f \in \mathcal{L}^2(S^{n-1})$, the series

$$f \sim \sum_{k=0}^{\infty} \pi_k f$$

converges to f in the $\mathcal{L}^2(S^{n-1})$-norm, where $\pi_k f \in \mathcal{H}_k^n$ is the orthogonal projection of f on the space $\mathcal{H}_k^n$. Using well-known properties of the Legendre polynomials, it is not hard to show that

$$\pi_k f = N(n,k)\big(f * \Lambda P_k^n\big). \tag{2.15}$$

This leads to the spherical expansion of a measure $\mu \in \mathcal{M}(S^{n-1})$,

$$\mu \sim \sum_{k=0}^{\infty} \pi_k \mu, \tag{2.16}$$

where $\pi_k \mu \in \mathcal{H}_k^n$ is defined by

$$\pi_k \mu = N(n,k)\big(\mu * \Lambda P_k^n\big). \tag{2.17}$$

From $P_0^n(t) = 1$, $N(n,0) = 1$ and $P_1^n(t) = t$, $N(n,1) = n$, we obtain, for $\mu \in \mathcal{M}(S^{n-1})$, the following special cases of (2.18):

$$\pi_0 \mu = \mu\big(S^{n-1}\big) \quad \text{and} \quad (\pi_1 \mu)(u) = n \int_{S^{n-1}} u \cdot v \, d\mu(v). \tag{2.18}$$

Let κ_n denote the volume of the Euclidean unit ball B. By (2.3) and (2.19), for every convex body $K \in \mathcal{K}_0^n$, it follows that

$$\kappa_n \pi_0 h(K,\cdot)^p = V_p(B,K) \quad \text{and} \quad \pi_0 S_p(K,\cdot) = n V_p(K,B). \tag{2.19}$$

A measure $\mu \in \mathcal{M}(S^{n-1})$ is uniquely determined by its series expansion (2.19). Using the fact that ΛP_k^n is (essentially) the unique zonal function in $\mathcal{H}_k^n$, a simple calculation shows that for $\mu \in \mathcal{M}(S^{n-1},\widehat{e})$, formula (2.18) becomes

$$\pi_k\mu = N(n,k)\langle \mu, \Lambda P_k^n\rangle \Lambda P_k^n. \tag{2.20}$$

A zonal measure $\mu \in \mathcal{M}(S^{n-1},\widehat{e})$ is defined by its so-called Legendre coefficients $\mu_k := \langle \mu, \Lambda P_k^n\rangle$. Using $\pi_k H = H$ for every $H \in \mathcal{H}_k^n$ and the fact that spherical convolution of zonal measures is commutative, we have the Funk-Hecke theorem: If $\mu \in \mathcal{M}(S^{n-1},\widehat{e})$ and $H \in \mathcal{H}_k^n$, then $H * \mu = \mu_k H$.

A map $\Phi : \mathcal{D} \subseteq \mathcal{M}(S^{n-1}) \to \mathcal{M}(S^{n-1})$ is called a multiplier transformation [16] if there exist real numbers c_k, the multipliers of Φ, such that, for every $k \in \mathbb{N}$,

$$\pi_k \Phi\mu = c_k \pi_k \mu, \quad \forall \mu \in \mathcal{D}. \tag{2.21}$$

From the Funk-Hecke theorem and the fact that the spherical convolution of zonal measures is commutative, it follows that, for $\mu \in \mathcal{M}(S^{n-1},\widehat{e})$, the map $\Phi_\mu : \mathcal{M}(S^{n-1}) \to \mathcal{M}(S^{n-1})$, defined by $\Phi_\mu = \nu * \mu$, is a multiplier transformation. The multipliers of this convolution operator are just the Legendre coefficients of the measure μ.

3 L_p Blaschke-Minkowski homomorphisms and convolutions

The L_p Minkowski valuation was introduced by Ludwig [11]. A function $\Psi : \mathcal{K}_0^n \to \mathcal{K}_0^n$ is called an L_p Minkowski valuation if

$$\Psi(K \cup L) +_p \Psi(K \cap L) = \Psi K +_p \Psi L, \tag{3.1}$$

whenever $K, L, K \cup L \in \mathcal{K}_0^n$, and here '$+_p$' is L_p Minkowski addition.

Definition 3.1 A map $\Phi_p : \mathcal{K}_e^n \to \mathcal{K}_e^n$ satisfying the following properties (a), (b) and (c) is called an L_p Blaschke-Minkowski homomorphism.

(a) Φ_p is continuous with respect to Hausdorff metric.
(b) $\Phi_p(K \#_p L) = \Phi_p K +_p \Phi_p L$ for all $K, L \in \mathcal{K}_e^n$.
(c) Φ_p is SO(n) equivariant, *i.e.*, $\Phi_p(\vartheta K) = \vartheta \Phi_p K$ for all $\vartheta \in \mathrm{SO}(n)$ and all $K \in \mathcal{K}_e^n$.

It is easy to verify that an L_p Blaschke-Minkowski homomorphism is an L_p Minkowski valuation.

In order to prove our results, we need to quote some lemmas. We call a map $\Phi : \mathcal{M}(S^{n-1}) \to \mathcal{C}(S^{n-1})$ monotone, if non-negative measures are mapped to non-negative functions.

Lemma 3.1 *A map $\Phi : \mathcal{M}(S^{n-1}) \to \mathcal{C}(S^{n-1})$ is a monotone, linear map that is intertwines rotations if and only if there is a function $f \in \mathcal{C}(S^{n-1},\widehat{e})$, such that*

$$\Phi\mu = f * \mu. \tag{3.2}$$

Proof From the definition of spherical convolution and (2.15), it follows that mapping of form (3.2) has the desired properties. This proves the sufficiency.

Next, we prove the necessity.

Let Φ be monotone, linear and intertwines rotations. Consider the map $\phi : \mathcal{M}(S^{n-1}) \to \mathbb{R}$, $\mu \to \Phi\mu(\widehat{e})$. By the properties of Φ, the functional ϕ is positive and linear on $\mathcal{M}(S^{n-1})$, thus, by the Riesz representation theorem, there is a function $f \in \mathcal{M}_+(S^{n-1})$ such that

$$\phi(\mu) = \int_{S^{n-1}} f(u)\,d\mu(u).$$

Since ϕ is $\mathrm{SO}(n-1)$ invariant, the function f is zonal. Thus, we have for $\eta \in \mathrm{SO}(n)$

$$\Phi\mu(\eta\widehat{e}) = \Phi(\eta^{-1}\mu)(\widehat{e}) = \phi(\eta^{-1}\mu) = \int_{S^{n-1}} f(\eta u)\,d\mu(u).$$

Lemma 3.1 follows now from (2.14). □

Proof of Theorem 1.1 Suppose that a map $\Phi_p : \mathcal{K}_0^n \to \mathcal{K}_0^n$ satisfies $h(\Phi_p K, \cdot)^p = S_p(K, \cdot) * g$, where $g \in \mathcal{C}(S^{n-1}, \widehat{e})$ is a nonnegative measure. The continuity of Φ_p follows from the fact that the support function $h(K, \cdot)$ is continuous with respect to Hausdorff metric. From (2.9) and (2.1), for $\vartheta \in \mathrm{SO}(n)$, we obtain

$$h(\Phi_p \vartheta K, \cdot)^p = S_p(\vartheta K, \cdot) * g = S_p(K, \vartheta^{-1}\cdot) * g = h(\Phi_p K, \vartheta^{-1}\cdot)^p = h(\vartheta\,\Phi_p K, \cdot)^p.$$

Taking $K = L$ in (1.4), we have

$$h(\Phi_p L, \cdot)^p = S_p(L, \cdot) * g. \tag{3.3}$$

Combining with (2.2), (1.4) and (3.3), we obtain

$$\begin{aligned} h(\Phi_p K +_p \Phi_p L, \cdot)^p &= h(\Phi_p K, \cdot)^p + h(\Phi_p L, \cdot)^p \\ &= S_p(K, \cdot) * g + S_p(L, \cdot) * g \\ &= (S_p(K, \cdot) + S_p(L, \cdot)) * g \\ &= S_p(K \#_p L, \cdot) * g \\ &= h(\Phi_p(K \#_p L), \cdot)^p. \end{aligned} \tag{3.4}$$

Thus maps of the form of (1.4) are L_p Blaschke-Minkowski homomorphisms (satisfy the properties (a), (b) and (c) from Definition 3.1). Thus, we have to show that for every such operator Φ_p, there is a function $g \in \mathcal{C}(S^{n-1}, \widehat{e})$ such that (1.4) holds.

Since every positive continuous even measure on S^{n-1} can be the L_p surface area measure of some convex body, the set $\{S_p(K, \cdot) - S_p(L, \cdot), K, L \in \mathcal{K}_e^n\}$ coincides with $\mathcal{M}_e(S^{n-1})$. The operator $\bar{\Phi} : \mathcal{M}(S^{n-1}) \to \mathcal{C}(S^{n-1})$ is defined by

$$\bar{\Phi}\mu_1 = h(\Phi_p K_1, \cdot)^p - h(\Phi_p K_2, \cdot)^p, \tag{3.5}$$

where $\mu_1 = S_p(K_1, \cdot) - S_p(K_2, \cdot)$.

The operator $\bar{\Phi}$ for $\mu_2 = S_p(L_1, \cdot) - S_p(L_2, \cdot)$ immediately yields:

$$\bar{\Phi}\mu_2 = h(\Phi_p L_1, \cdot)^p - h(\Phi_p L_2, \cdot)^p. \tag{3.6}$$

Combining with (3.5), (3.6), (2.2) and (3.4), we obtain

$$\begin{aligned}\bar{\Phi}\mu_1 + \bar{\Phi}\mu_2 &= h(\Phi_p K_1,\cdot)^p - h(\Phi_p K_2,\cdot)^p + h(\Phi_p L_1,\cdot)^p - h(\Phi_p L_2,\cdot)^p \\ &= h(\Phi_p K_1 +_p \Phi_p L_1,\cdot)^p - h(\Phi_p K_2 +_p \Phi_p L_2,\cdot)^p \\ &= h\big(\Phi_p(K_1 \#_p L_1),\cdot\big)^p - h(\Phi_p(K_2 \#_p L_2),\cdot)^p \\ &= \bar{\Phi}\big(S_p(K_1\#_p L_1,\cdot) - S_p(K_2\#_p L_2,\cdot)\big) \\ &= \bar{\Phi}\big(S_p(K_1,\cdot) + S_p(L_1,\cdot) - S_p(K_2,\cdot) - S_p(L_2,\cdot)\big) \\ &= \bar{\Phi}(\mu_1 + \mu_2).\end{aligned}$$

So, the operator $\bar{\Phi}$ is linear.

Noting that Φ_p is an L_p Minkowski homomorphism and $S_p(\vartheta K,\cdot) = \vartheta S_p(K,\cdot)$, we obtain that the operator $\bar{\Phi}$ is SO(n) equivariant.

Since the cone of the L_p surface area measures of origin symmetric convex bodies is invariant under $\bar{\Phi}$, it is also monotone. Hence, by Lemma 3.1, there is a non-negative function $g \in \mathcal{C}(S^{n-1},\widehat{e})$ such that $\bar{\Phi}\mu = \mu * g$. The statement now follows from

$$\bar{\Phi}S_p(K,\cdot) = S_p(K,\cdot) * g = h(\Phi_p K,\cdot)^p.$$

Hence, it is to complete the proof. □

Lutwak, Yang and Zhang first introduced the notion of L_p-projection body (see [24]). Let $\Pi_p K$, $p \geq 1$ denote the compact convex symmetric set whose support function is given by

$$h(\Pi_p K,\theta)^p = \frac{1}{n\omega_n c_{n-2,p}} S_p(K,\cdot) * \big|\langle\theta,\cdot\rangle\big|^p, \tag{3.7}$$

where

$$c_{n,p} = \frac{\omega_{n+p}}{\omega_2\omega_n\omega_{p-1}}.$$

Obviously, $\Pi_p : \mathcal{K}_e^n \to \mathcal{K}_e^n$ is an L_p Blaschke-Minkowski homomorphism.

Lemma 3.2 [23] *If $\mu,\nu \in \mathcal{M}(S^{n-1})$ and $f \in \mathcal{C}(S^{n-1})$, then*

$$\langle\mu * \nu, f\rangle = \langle\mu, f * \nu\rangle.$$

Theorem 3.3 *If $\Phi_p : \mathcal{K}_e^n \to \mathcal{K}_e^n$ is an L_p Blaschke-Minkowski homomorphism, then for $K, L \in \mathcal{K}_e^n$,*

$$V_p(K,\Phi_p L) = V_p(L,\Phi_p K). \tag{3.8}$$

Proof Let $g \in \mathcal{C}(S^{n-1},\widehat{e})$ be the generating function of Φ_p. Using (2.3), Theorem 1.1 and

Lemma 3.2, it follows that

$$\begin{aligned} nV_p(K,\Phi_pL) &= \langle h(\Phi_pL,\cdot)^p, S_p(K,\cdot)\rangle \\ &= \langle S_p(L,\cdot)*g, S_p(K,\cdot)\rangle \\ &= \langle S_p(L,\cdot), S_p(K,\cdot)*g\rangle \\ &= \langle S_p(L,\cdot), h(\Phi_pK,\cdot)^p\rangle \\ &= nV_p(L,\Phi_pK). \end{aligned} \tag{3.9}$$

□

Using Theorem 1.1 and the fact that spherical convolution operators are multiplier transformations, one obtains the following lemma.

Lemma 3.4 *If Φ_p is an L_p Blaschke-Minkowski homomorphism, which is generated by the zonal function g, then for every origin symmetric convex body $K \in \mathcal{K}_e^n$,*

$$\pi_k h(\Phi_pK,\cdot)^p = g_k\pi_k S_p(K,\cdot), \quad k \in \mathbb{N}, \tag{3.10}$$

where the numbers g_k are the Legendre coefficients of g, i.e., $g_k = \langle g, \Lambda P_k^n\rangle$.

Proof By (2.18) and Theorem 1.1, we have

$$\pi_k h(\Phi_pK,\cdot)^p = N(n,k)\big(S_p(K,\cdot)*g*\Lambda P_k^n\big).$$

Since spherical convolution is associative and g is zonal, we obtain from (2.18):

$$\pi_k h(\Phi_pK,\cdot)^p = g_kN(n,k)\big(S_p(K,\cdot)*\Lambda P_k^n\big) = g_k\pi_k S_p(K,\cdot).$$ □

Definition 3.2 If Φ_p is an L_p Blaschke-Minkowski homomorphism, generated by the zonal function g, then we call the subset $\mathcal{K}_e^n(\Phi_p)$ of $\mathcal{K}_e^n$, defined by

$$\mathcal{K}_e^n(\Phi_p) = \big\{K \in \mathcal{K}_e^n : \pi_k S_p(K,\cdot) = 0 \text{ if } g_k = 0\big\},$$

the injectivity set of Φ_p.

It is easy to verify that for every L_p Blaschke-Minkowski homomorphism, the set is a nonempty rotation and dilatation invariant subset of which is closed under L_p Blaschke addition.

Definition 3.3 An origin-symmetric convex body $K \in \mathcal{K}_e^n$ p-polynomial if $h(K,\cdot)^p \in \mathcal{H}^n$.

Clearly, the set of p-polynomial convex bodies is dense in $\mathcal{K}_e^n$.

Let $p > 1$ and $p \neq n$ where p is not an even integer. The size of range, $\Phi_p(\mathcal{K}_e^n)$, of the L_p Blaschke-Minkowski homomorphism Φ_p will be critical. The set of origin-symmetric convex bodies whose support functions are elements of the vector space

$$\operatorname{span}\big\{\big(h(\Phi_pK,\cdot)^p - h(\Phi_pL,\cdot)^p\big)^{\frac{1}{p}} : K, L \in \mathcal{K}_e^n\big\} \tag{3.11}$$

is a large subset of $\mathcal{K}_e^n$, provided the injectivity set $\mathcal{K}_e^n(\Phi_p)$ is not too small.

Theorem 3.5 *Let $p > 1$ and $p \neq n$ where p is not an even integer. If $\Phi_p : \mathcal{K}_e^n \to \mathcal{K}_e^n$ is an L_p Blaschke-Minkowski homomorphism such that $\mathcal{K}_e^n \subseteq \mathcal{K}_e^n(\Phi_p)$, then for every p-polynomial convex body $K \in \mathcal{K}_e^n$, there exist origin-symmetry convex bodies $K_1, K_2 \in \mathcal{K}_e^n$ such that*

$$K +_p \Phi_p K_1 = \Phi_p K_2. \tag{3.12}$$

Proof Let $K \in \mathcal{K}_e^n$ be a p-polynomial convex body. From Definition 3.3, we have

$$h(K,\cdot)^p = \sum_{k=0}^{m} \pi_k h(K,\cdot)^p. \tag{3.13}$$

For $K \in \mathcal{K}_e^n$ and the properties of the orthogonal projection of f on the space $\mathcal{H}_k^n$, we have $\pi_k h(K,\cdot)^p = 0$ for all odd $k \in \mathbb{N}$. Let $g \in \mathcal{C}(S^{n-1}, \widehat{e})$ denote the generating function of Φ and let g_k denote the Legendre coefficients of g. From $\mathcal{K}_e^n \subseteq \mathcal{K}_e^n(\Phi)$ and Definition 3.2, it follows that $g_k \neq 0$ for every even $k \in \mathbb{N}$. We define

$$f := \sum_{k=0}^{m} c_k \pi_k h(K,\cdot)^p, \tag{3.14}$$

where $c_k = 0$ for odd and $c_k = g_k^{-1}$ if k is even. Since f is an even continuous function on S^{n-1} and spherical convolution operators are multiplier transformations, we have

$$f * g = \sum_{k=0}^{m} c_k g_k \pi_k h(K,\cdot)^p = \sum_{k=0}^{m} \pi_k h(K,\cdot)^p = h(K,\cdot)^p. \tag{3.15}$$

Denote by f^+ and f^- the positive and negative parts of f and let K_1 and K_2 be the convex bodies such that $S_p(K_1,\cdot) = f^-$ and $S_p(K_2,\cdot) = f^+$. By Theorem 1.1 and (2.2), it follows that

$$K +_p \Phi_p K_1 = \Phi_p K_2. \qquad \square$$

4 The Shephard-type problem

Let $\Phi_p : \mathcal{K}_e^n \to \mathcal{K}_e^n$ denote a nontrivial L_p Blaschke-Minkowski homomorphism, *i.e.*, Φ_p is continuous and SO(n) equivariant map satisfying $\Phi_p(K \#_p L) = \Phi_p K +_p \Phi_p L$ and Φ_p does not map every origin-symmetric convex body to the origin. In this section, we study the Shephard-type problem for L_p Blaschke-Minkowski homomorphisms.

Problem 4.1 Let $p > 1$, $p \neq n$ and $\Phi_p : \mathcal{K}_0^n \to \mathcal{K}_e^n$ be an L_p Blaschke-Minkowski homomorphism. Is there the implication:

If $0 < p < n$, then

$$\Phi_p K \subseteq \Phi_p L \quad \Rightarrow \quad V(K) \leq V(L)? \tag{4.1}$$

If $p > n$, then

$$\Phi_p K \subseteq \Phi_p L \quad \Rightarrow \quad V(K) \geq V(L)? \tag{4.2}$$

Proof of Theorem 1.2 For $L \in \Phi_p\mathcal{K}_e^n$ and p is not an even integer, there exists an origin-symmetric convex body L_0 such that $L = \Phi_p L_0$. Using Theorem 3.3 and the fact that the L_p mixed volume V_p is monotone with respect to set inclusion, it follows that

$$V_p(K,L) = V_p(K,\Phi_p L_0) = V_p(L_0,\Phi_p K) \le V_p(L_0,\Phi_p L) = V_p(L,\Phi_p L_0) = V(L).$$

Applying the L_p Minkowski inequality (2.6), we thus obtain that, if $1 < p < n$, then

$$V(K) \le V(L),$$

and if $p > n$, then

$$V(K) \ge V(L),$$

with equality if and only if K and L are dilates. □

An immediate consequence of Theorem 1.2 is the following.

Theorem 4.1 *Let $p > 1$, $p \ne n$, where p is not an even integer and $\Phi_p : \mathcal{K}_e^n \to \mathcal{K}_e^n$ is an L_p Blaschke-Minkowski homomorphism. If $K, L \in \Phi_p\mathcal{K}_e^n$, then*

$$\Phi_p K = \Phi_p L \quad \Leftrightarrow \quad K = L. \tag{4.3}$$

Since the L_p projection body operator Π_p is just an L_p Blaschke-Minkowski homomorphism, the L_p Aleksandrov's projection theorem is a direct corollary of Theorem 4.1.

Corollary 4.2 [25] *Let $p > 1$, $p \ne n$, where p is not an even integer, and K and L are both L_p projection bodies in $\mathbb{R}^n$. Then*

$$\Pi_p K = \Pi_p L \quad \Leftrightarrow \quad K = L.$$

Our next result shows that if the injectivity set $\mathcal{K}_e^n(\Phi_p)$ does not exhaust all of $\mathcal{K}_e^n$, in general the answer to Problem 4.1 is negative.

Theorem 4.3 *Let $1 < p < n$ where p is not an even integer. If $\mathcal{K}_e^n(\Phi_p)$ does not coincide with $\mathcal{K}_e^n$, then there exist origin-symmetric convex bodies $K, L \in \mathcal{K}_e^n$, such that*

$$\Phi_p K \subseteq \Phi_p L,$$

but

$$V(K) > V(L).$$

Proof Let $g \in \mathcal{C}(S^{n-1}, \widehat{e})$ be the generating function of Φ_p and let g_k denote its Legendre coefficients. Since $\mathcal{K}_e^n(\Phi_p) \ne \mathcal{K}_e^n$ and Φ_p is nontrivial, there exists, by Definition 3.2, an integer $k \in \mathbb{N}$, such that $g_k = 0$ and $k \ge 1$. We can choose $\alpha > 0$ such that the function

$f(u) = 1 + \alpha P_k^n(u \cdot \widehat{e})$, $u \in S^{n-1}$, is positive. According to Theorem C, there exists an origin-symmetric convex body $L \in \mathcal{K}_e^n$ with $S_p(L,\cdot) = f$.

Since $\pi_k S_p(L,\cdot) = \pi_k(1 + \alpha P_k^n(u \cdot \widehat{e})) \neq 0$, from Definition 3.2 we have that $L \notin \mathcal{K}_e^n(\Phi_p)$.

From (2.20) and the properties of the orthogonal projection on the space $\mathcal{H}_k^n$, we have that

$$nV_p(L,B) = \pi_0 S_p(L,\cdot) = 1. \tag{4.4}$$

Using the fact that: For $1 < p < n$ where p is not an even integer, an origin-symmetric convex body $L \in \mathcal{K}_e^n(\Phi_p)$ is uniquely determined by its image $\Phi_p L$, we obtain that $\Phi_p L = \Phi_p K$, where K denotes the Euclidean ball centered at the origin with L_p surface area $S_p(K) = 1$. Noting that L is just a perturb body of K, we use (4.4) and (2.6) to conclude

$$V(K)^{n-p} = \frac{1}{n^n V(B)^p} > V(L)^{n-p}.$$ □

Theorem 4.4 *Suppose* $1 < p < n$ *where* p *is not an even integer and* $\mathcal{K}_e^n \subseteq \mathcal{K}_e^n(\Phi_p)$. *If* $K \in \mathcal{K}_e^n$ *is a p-polynomial convex body which has p-positive curvature function, then if* $K \notin \Phi_p\mathcal{K}_e^n$, *there exists an origin-symmetric convex body* $L \in \mathcal{K}_e^n$, *such that*

$$\Phi_p K \subseteq \Phi_p L,$$

but

$$V(K) > V(L).$$

Proof Let $g \in \mathcal{C}(S^{n-1}, \widehat{e})$ be the generating function of Φ_p. Since $K \in \mathcal{K}_e^n$ is p-polynomial, it follows from the proof of Theorem 3.5 that there exists an even function $f \in \mathcal{H}^n$ such that

$$h(K,\cdot)^p = f * g. \tag{4.5}$$

The function must assume negative values, otherwise, by Theorem 1.1 we have $K = \Phi_p K_0$, where K_0 is the convex body with $S_p(K_0,\cdot) = f$. Let $F \in \mathcal{C}(S^{n-1})$ be a non-constant even function, such that: $F(u) \geq 0$ if $f(u) < 0$, and $F(u) = 0$ if $f(u) \geq 0$. By suitable approximation of the function F with spherical harmonics, we can find a nonnegative even function $G \in \mathcal{H}^n$ and an even function $H \in \mathcal{H}^n$ such that

$$\langle f, G\rangle < 0, \quad \text{and} \quad G = H * g. \tag{4.6}$$

Since K is a p-polynomial and has p-positive curvature, the L_p surface area measure of K has a positive density $S_p(K,\cdot)$. Thus, we can choose $\alpha > 0$ such that

$$S_p(K,\cdot) + \alpha H > 0.$$

By Theorem C, there exists an origin-symmetric convex body L such that

$$S_p(L,\cdot) = S_p(K,\cdot) + \alpha H. \tag{4.7}$$

From (4.6) and Theorem 1.1, we see that $h(\Phi_p L,\cdot)^p = h(\Phi_p K,\cdot)^p + \alpha G$.
Since $G \geq 0$, it follows that

$$\Phi_p K \subseteq \Phi_p L. \tag{4.8}$$

Applying with (2.3), (4.5), (4.7), (2.10) and (4.6), we obtain

$$\begin{aligned} n\bigl(V_p(K,L) - V(K)\bigr) &= \bigl\langle h(K,\cdot)^p, S_p(L,\cdot) - S_p(K,\cdot)\bigr\rangle \\ &= \bigl\langle h(K,\cdot)^p, \alpha H\bigr\rangle \\ &= \alpha\langle f * g, H\rangle \\ &= \alpha\langle f, H * g\rangle \\ &= \alpha\langle f, G\rangle < 0. \end{aligned} \tag{4.9}$$

To complete the proof, we can use (2.6) to conclude

$$V(K) > V(L). \qquad \square$$

In particular, we replace Φ_p by Π_p to Theorem 1.2, we have the following corollary, which was proved by Ryabogin and Zvavitch.

Corollary 4.5 [25] *Let K and L be origin-symmetric convex bodies and $1 \leq p < n$ where p is not an even integer. If L belongs to the class of L_p projection bodies, then*

$$\Pi_p K \subseteq \Pi_p L \quad \Rightarrow \quad V(K) \leq V(L).$$

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