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On a Hilbert-type inequality with a homogeneous kernel in $\mathbb{R}^2$ and its equivalent form

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Abstract

By using the way of weight functions and the technique of real analysis, a new integral inequality with a homogeneous kernel and the best constant factor in $\mathbb{R}^2$ is given. The equivalent form and the reverses are considered.

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1. Introduction

One hundred years ago, Hilbert proved the following classic inequality [1]

$$\sum_n \sum_m \frac{a_m b_n}{m+n} \le \pi \left(\sum_n \alpha_n^2 \right)^{1/2} \left(\sum_n b_n^2 \right)^{1/2}. \tag{1.1}$$

The inequality (1.1) may be classified into several types (discrete and integral etc.), which is of great importance in analysis and its applications [1,2]. Ever since the advent of inequality (1.1), all kinds of improvements and extensions can be seen in [3-12]. Note that the kernel of (1.1) is homogeneous of degree -1. In 2009, [13] reviews the negative degree homogeneous kernel of the parameterized Hilbert-type inequalities.

In recent years, many authors have started on Hilbert-type inequality of 0-degree homo-geneous kernel and non-homogeneous kernel. They even established inequalities in $\mathbb{R}^2$. In 2008, Yang [14] obtained the improved inequality as follows: If $p, r > 1$, $(1/p) + (1/q) = 1$, $(1/r) + (1/s) = 1$, $0 < \lambda < 1$ and the right-hand side integrals are convergent, then

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{f(x)g(y)}{|x+y|^{\lambda}} \mathrm{d}x\mathrm{d}y < k_\lambda(r) \left(\int_{-\infty}^{\infty} |x|^{p(1-\frac{\lambda}{r})} f^p(x) \mathrm{d}x \right)^{1/p} \left(\int_{-\infty}^{\infty} |x|^{q(1-\frac{\lambda}{s})} g^q(x) \mathrm{d}x \right)^{1/q}, \tag{1.2}$$

where the constant factor $k_\lambda(r) = B(\frac{\lambda}{r}, \frac{\lambda}{s}) + B(1-\lambda, \frac{\lambda}{r}) + B(1-\lambda, \frac{\lambda}{s})$ is the best possible.

Motivated by (1.2) and the technique of real analysis, we establish a new inequality in $\mathbb{R}^2$ with a homogeneous kernel of 0-degree. Furthermore, the equivalent form and the corresponding reverse inequalities are also considered.

In what follows, α_1, α_2 will be real numbers such that $0 < \alpha_1 < \alpha_2 < \pi$.

2. Lemmas

LEMMA 2.1. *If* $k := 2\ln(4\cos\frac{\alpha_1}{2}\sin\frac{\alpha_2}{2}) - \alpha_1\cot\alpha_1 + (\pi-\alpha_2)\cot\alpha_2$, *the weight function*

$$\varpi(x) := \int_{-\infty}^{\infty} \min_{i\in\{1,2\}} \frac{\min\{x^2, y^2\}}{x^2+2xy\cos\alpha_i+y^2}\cdot\frac{1}{|y|}\mathrm{d}y, x\in(-\infty,\infty), \tag{2.1}$$

then for all $x \in (-\infty, 0) \cup (0, \infty)$

$$\varpi(x) = k. \tag{2.2}$$

Proof. If $x \in (-\infty, 0)$, then

$$\varpi(x) = \int_{-\infty}^{0} \min_{i\in\{1,2\}} \frac{\min\{x^2, y^2\}}{x^2+2xy\cos\alpha_i+y^2}\cdot\frac{1}{-y}\mathrm{d}y + \int_{0}^{\infty} \min_{i\in\{1,2\}} \frac{\min\{x^2, y^2\}}{x^2+2xy\cos\alpha_i+y^2}\cdot\frac{1}{y}\mathrm{d}y.$$

Letting $u = y/x$ for the first integrals and $u = -y/x$ for the second integrals gives

$$\begin{aligned}
\varpi(x) &= \int_0^\infty \min_{i\in\{1,2\}} \frac{\min\{1,u^2\}}{u^2+2u\cos\alpha_i+1}\cdot\frac{1}{u}\mathrm{d}u + \int_0^\infty \min_{i\in\{1,2\}} \frac{\min\{1,u^2\}}{u^2-2u\cos\alpha_i+1}\cdot\frac{1}{u}\mathrm{d}u \\
&= \int_0^1 \frac{u}{u^2+2u\cos\alpha_1+1}\mathrm{d}u + \int_1^\infty \frac{u^{-1}}{u^2+2u\cos\alpha_1+1}\mathrm{d}u \\
&\quad + \int_0^1 \frac{u}{u^2-2u\cos\alpha_2+1}\mathrm{d}u + \int_1^\infty \frac{u^{-1}}{u^2-2u\cos\alpha_2+1}\mathrm{d}u. \\
&= \int_0^1 \frac{u}{u^2+2u\cos\alpha_1+1}\mathrm{d}u + \int_0^1 \frac{u}{u^2+2u\cos\alpha_1+1}\mathrm{d}u \\
&\quad + \int_0^1 \frac{u}{u^2-2u\cos\alpha_2+1}\mathrm{d}u + \int_0^1 \frac{u}{u^2-2u\cos\alpha_2+1}\mathrm{d}u \\
&= 2\left(\int_0^1 \frac{u}{u^2+2u\cos\alpha_1+1}\mathrm{d}u + \int_0^1 \frac{u}{u^2-2u\cos\alpha_2+1}\mathrm{d}u\right) \\
&= 2\left[\left(\ln 2\cos\frac{\alpha_1}{2} - \frac{\alpha_1}{2}\cot\alpha_1\right) + \left(\ln 2\sin\frac{\alpha_2}{2} - \frac{\alpha_2}{2}\cot\alpha_2 + \frac{\pi}{2}\cot\alpha_2\right)\right] \\
&= 2\ln\left(4\cos\frac{\alpha_1}{2}\sin\frac{\alpha_2}{2}\right) - \alpha_1\cot\alpha_1 + (\pi-\alpha_2)\cot\alpha_2 = k.
\end{aligned} \tag{2.3}$$

Similarly, $\varpi(x) = k$ for $x \in (0, \infty)$. Hence (2.2) is valid for $x \in (-\infty, 0) \cup (0, \infty)$. □

Note. (i) It is obvious that $\varpi(0) = 0$. (ii) If $\alpha_1 = \alpha_2 = \alpha$, then

$$\min_{i\in\{1,2\}} \frac{\min\{x^2, y^2\}}{x^2 + 2xy\cos\alpha_i + y^2} = \frac{\min\{x^2, y^2\}}{x^2 + 2xy\cos\alpha + y^2},$$

and $\varpi(x) = 2\ln(2\sin\alpha) + (\pi - 2\alpha)\cot\alpha$.

LEMMA 2.2. *If* $p > 1, \frac{1}{p} + \frac{1}{q} = 1$ *and* $f(x)$ *is a nonnegative measurable function in* $(-\infty,\infty)$, *then for all* $x \in (-\infty, 0) \cup (0, \infty)$

$$\begin{aligned} J := \int_{-\infty}^{\infty} |y|^{-1} \left(\int_{-\infty}^{\infty} \min_{i\in\{1,2\}} \frac{\min\{x^2, y^2\}}{x^2 + 2xy\cos\alpha_i + y^2} f(x)\mathrm{d}x \right)^p \mathrm{d}y \\ \leq k^p \int_{-\infty}^{\infty} |x|^{p-1} f^p(x)\mathrm{d}x. \end{aligned} \tag{2.4}$$

Proof. By Hölder's inequality with weight [15] and Lemma 2.1, we obtain

$$\begin{aligned} &\left(\int_{-\infty}^{\infty} \min_{i\in\{1,2\}} \frac{\min\{x^2, y^2\}}{x^2 + 2xy\cos\alpha_i + y^2} f(x)\mathrm{d}x \right)^p \\ &= \left[\int_{-\infty}^{\infty} \min_{i\in\{1,2\}} \frac{\min\{x^2, y^2\}}{x^2 + 2xy\cos\alpha_i + y^2} \left(\frac{|x|^{1/q}}{|y|^{1/p}} f(x) \right) \left(\frac{|y|^{1/p}}{|x|^{1/q}} \right) \mathrm{d}x \right]^p \\ &\leq \int_{-\infty}^{\infty} \min_{i\in\{1,2\}} \frac{\min\{x^2, y^2\}}{x^2 + 2xy\cos\alpha_i + y^2} \frac{|x|^{p-1}}{|y|} f^p(x)\mathrm{d}x \\ &\times \left(\int_{-\infty}^{\infty} \min_{i\in\{1,2\}} \frac{\min\{x^2, y^2\}}{x^2 + 2xy\cos\alpha_i + y^2} \frac{|y|^{q-1}}{|x|} \mathrm{d}x \right)^{p-1} \\ &= k^{p-1} |y| \int_{-\infty}^{\infty} \min_{i\in\{1,2\}} \frac{\min\{x^2, y^2\}}{x^2 + 2xy\cos\alpha_i + y^2} \frac{|x|^{p-1}}{|y|} f^p(x)\mathrm{d}x. \end{aligned} \tag{2.5}$$

By Fubini theorem, we find

$$\begin{aligned} J &\leq k^{p-1} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \min_{i\in\{1,2\}} \frac{\min\{x^2, y^2\}}{x^2 + 2xy\cos\alpha_i + y^2} \frac{|x|^{p-1}}{|y|} f^p(x)\mathrm{d}x \right) \mathrm{d}y \\ &= k^{p-1} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \min_{i\in\{1,2\}} \frac{\min\{x^2, y^2\}}{x^2 + 2xy\cos\alpha_i + y^2} \frac{|x|^{p-1}}{|y|} \mathrm{d}y \right) f^p(x)\mathrm{d}x \\ &= k^p \int_{-\infty}^{\infty} |x|^{p-1} f^p(x)\mathrm{d}x. \end{aligned}$$

□

LEMMA 2.3. *If* $0 < p < 1, \frac{1}{p} + \frac{1}{q} = 1$ *and* $g(x)$ *is a nonnegative measurable function in* $(-\infty,\infty)$, *then for all* $x \in (-\infty, 0) \cup (0, \infty)$

$$J \geq k^p \int_{-\infty}^{\infty} |x|^{p-1} f^p(x)\mathrm{d}x, \tag{2.6}$$

$$\begin{aligned} L := \int_{-\infty}^{\infty} |x|^{-1} \left(\int_{-\infty}^{\infty} \min_{i \in \{1,2\}} \frac{\min\{x^2, y^2\}}{x^2 + 2xy\cos\alpha_i + y^2} g(y)\mathrm{d}y \right)^q \mathrm{d}x \\ \leq k^q \int_{-\infty}^{\infty} |y|^{q-1} g^q(y)\mathrm{d}y, \end{aligned} \tag{2.7}$$

where $k = 2\ln(4\cos\frac{\alpha_1}{2}\sin\frac{\alpha_2}{2}) - \alpha_1\cot\alpha_1 + (\pi - \alpha_2)\cot\alpha_2$.

Proof. It can be completed similarly by following the proof of Lemma 2.2 as long as applying the reverse Hölder's inequality [15], hence we omit the details. Since $q < 0$, thus (2.7) takes the positive inequality. □

3. Main results and applications

THEOREM 3.1. *If* $p > 1, \frac{1}{p} + \frac{1}{q} = 1, f, g \geq 0$ *such that* $0 < \int_{-\infty}^{\infty} |x|^{p-1} f^p(x)\mathrm{d}x < \infty$ *and* $0 < \int_{-\infty}^{\infty} |x|^{q-1} g^q(x)\mathrm{d}x < \infty$, *then we obtain the following equivalent inequalities*

$$\begin{aligned} I := \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \min_{i \in \{1,2\}} \frac{\min\{x^2, y^2\}}{x^2 + 2xy\cos\alpha_i + y^2} f(x)g(y)\mathrm{d}x\mathrm{d}y \\ < k\left(\int_{-\infty}^{\infty} |x|^{p-1} f^p(x)\mathrm{d}x \right)^{1/p} \left(\int_{-\infty}^{\infty} |x|^{q-1} g^q(x)\mathrm{d}x \right)^{1/q}, \end{aligned} \tag{3.1}$$

$$\begin{aligned} J = \int_{-\infty}^{\infty} |y|^{-1} \left(\int_{-\infty}^{\infty} \min_{i \in \{1,2\}} \frac{\min\{x^2, y^2\}}{x^2 + 2xy\cos\alpha_i + y^2} f(x)\mathrm{d}x \right)^p \mathrm{d}y \\ < k^p \int_{-\infty}^{\infty} |x|^{p-1} f^p(x)\mathrm{d}x, \end{aligned} \tag{3.2}$$

where the constant factors $k = 2\ln(4\cos\frac{\alpha_1}{2}\sin\frac{\alpha_2}{2}) - \alpha_1\cot\alpha_1 + (\pi - \alpha_2)\cot\alpha_2$ *and* k^p *are both the best possible.*

Proof. If (2.5) takes the form of equality for some $y \in (-\infty,0) \cup (0,\infty)$, then there exist constants A and B such that they are not all zero and

$$A\frac{|x|^{p-1}}{|y|} \cdot f^p(x) = B\frac{|y|^{q-1}}{|x|} \text{a.e. in}(-\infty,\infty) \times (-\infty,\infty).$$

i.e., $A|x|^p f^p(x) = B|y|^q$ a.e. in $(-\infty,\infty) \times (-\infty, \infty)$. We conform that $A \neq 0$ (otherwise $B = A = 0$). Then $|x|^{p-1} f^p(x) = \frac{B|y|^q}{A|x|}$ a.e. in$(-\infty,\infty)$, which contradicts the fact that $0 < \int_{-\infty}^{\infty} |x|^{p-1} f^p(x)\mathrm{d}x < \infty$. Hence (2.5) takes a strict inequality and the same as (2.4), thus (3.2) is valid.

By Hölder's inequality with weight [15], we find

$$I = \int_{-\infty}^{\infty} \left(|y|^{-1+(1/q)} \int_{-\infty}^{\infty} \min_{i\in\{1,2\}} \frac{\min\{x^2, y^2\}}{x^2 + 2xy\cos\alpha_i + y^2} f(x)\mathrm{d}x \right) |y|^{1-(1/q)} g(y)\mathrm{d}y \le J^{1/p} \left(\int_{-\infty}^{\infty} |y|^{q-1} g^q(y)\mathrm{d}y \right)^{1/q}. \tag{3.3}$$

By (3.2), we obtain (3.1). On the other hand, suppose that (3.1) is valid. Let

$$g(y) := |y|^{-1} \left(\int_{-\infty}^{\infty} \min_{i\in\{1,2\}} \frac{\min\{x^2, y^2\}}{x^2 + 2xy\cos\alpha_i + y^2} f(x)\mathrm{d}x \right)^{p-1},$$

then $J = \int_{-\infty}^{\infty} |y|^{q-1} g^q(y)\mathrm{d}y$. In view of (2.4), $J < \infty$. If $J = 0$, then (3.2) is naturally valid; if $J > 0$, by (3.1), then

$$0 < \int_{-\infty}^{\infty} |y|^{q-1} g^q(y)\mathrm{d}y = J = I < k \left(\int_{-\infty}^{\infty} |x|^{p-1} f^p(x)\mathrm{d}x \right)^{1/p} \left(\int_{-\infty}^{\infty} |x|^{q-1} g^q(x)\mathrm{d}x \right)^{1/q} \tag{3.4}$$

$$J^{1/p} = \left(\int_{-\infty}^{\infty} |y|^{q-1} g^q(y)\mathrm{d}y \right)^{1/p} < k \left(\int_{-\infty}^{\infty} |x|^{p-1} f^p(x)\mathrm{d}x \right)^{1/p}. \tag{3.5}$$

Hence we obtain (3.2). Thus (3.2) and (3.1) are equivalent.

For any $\varepsilon > 0$, suppose that

$$\tilde{f}(x) = \begin{cases} x^{-1-\frac{2\varepsilon}{p}}, & x \in [1, \infty), \\ 0, & x \in (-1, 1), \\ (-x)^{-1-\frac{2\varepsilon}{p}}, & x \in (-\infty, -1], \end{cases} \quad \tilde{g}(x) = \begin{cases} x^{-1-\frac{2\varepsilon}{q}}, & x \in [1, \infty), \\ 0, & x \in (-1, 1), \\ (-x)^{-1-\frac{2\varepsilon}{q}}, & x \in (-\infty, -1]. \end{cases}$$

Then we get the following inequality

$$H(\varepsilon) := \left(\int_{-\infty}^{\infty} |x|^{p-1} \tilde{f}^p(x)\mathrm{d}x \right)^{\frac{1}{p}} \left(\int_{-\infty}^{\infty} |x|^{q-1} \tilde{g}^q(x)\mathrm{d}x \right)^{\frac{1}{q}} = \frac{1}{\varepsilon}, \tag{3.6}$$

$$I(\varepsilon) := \int_{-\infty}^{\infty} \min_{i\in\{1,2\}} \frac{\min\{x^2, y^2\}}{x^2 + 2xy\cos\alpha_i + y^2} \tilde{f}(x)\tilde{g}(y)\mathrm{d}x\mathrm{d}y = I_1 + I_2 + I_3 + I_4, \tag{3.7}$$

where

$$I_1 := \int_{-\infty}^{-1} (-y)^{-1-\frac{2\varepsilon}{q}} \left(\int_{-\infty}^{-1} \min_{i\in\{1,2\}} \frac{\min\{x^2, y^2\}}{x^2 + 2xy\cos\alpha_i + y^2} (-x)^{-1-\frac{2\varepsilon}{p}} \mathrm{d}x \right) \mathrm{d}y,$$

$$I_2 := \int_{-\infty}^{-1} (-y)^{-1-\frac{2\varepsilon}{q}} \left(\int_{1}^{\infty} \min_{i\in\{1,2\}} \frac{\min\{x^2, y^2\}}{x^2 + 2xy\cos\alpha_i + y^2} x^{-1-\frac{2\varepsilon}{p}} \mathrm{d}x \right) \mathrm{d}y,$$

$$I_3 := \int_{1}^{\infty} y^{-1-\frac{2\varepsilon}{q}} \left(\int_{-\infty}^{-1} \min_{i\in\{1,2\}} \frac{\min\{x^2, y^2\}}{x^2 + 2xy\cos\alpha_i + y^2} (-x)^{-1-\frac{2\varepsilon}{p}} \mathrm{d}x \right) \mathrm{d}y,$$

$$I_4 := \int_{1}^{\infty} y^{-1-\frac{2\varepsilon}{q}} \left(\int_{1}^{\infty} \min_{i\in\{1,2\}} \frac{\min\{x^2, y^2\}}{x^2 + 2xy\cos\alpha_i + y^2} x^{-1-\frac{2\varepsilon}{p}} \mathrm{d}x \right) \mathrm{d}y.$$

By Fubini theorem [16], it follows

$$I_1 = I_4 = \int_{1}^{\infty} y^{-1-\frac{2\varepsilon}{q}} \left(\int_{1}^{\infty} \min_{i\in\{1,2\}} \frac{\min\{x^2, y^2\}}{x^2 + 2xy\cos\alpha_i + y^2} x^{-1-\frac{2\varepsilon}{p}} \mathrm{d}x \right) \mathrm{d}y$$

$$= \int_{1}^{\infty} y^{-1-2\varepsilon} \left(\int_{\frac{1}{y}}^{\infty} \min_{i\in\{1,2\}} \frac{\min\{u^2, 1\}}{u^2 + 2u\cos\alpha_i + 1} u^{-1-\frac{2\varepsilon}{p}} \mathrm{d}u \right) \mathrm{d}y \quad (u = x/y)$$

$$= \int_{1}^{\infty} y^{-1-2\varepsilon} \left(\int_{\frac{1}{y}}^{1} \frac{u^2}{u^2 + 2u\cos\alpha_1 + 1} u^{-1-\frac{2\varepsilon}{p}} \mathrm{d}u \right) \mathrm{d}y$$

$$+ \int_{1}^{\infty} y^{-1-2\varepsilon} \left(\int_{1}^{\infty} \frac{1}{u^2 + 2u\cos\alpha_1 + 1} u^{-1-\frac{2\varepsilon}{p}} \mathrm{d}u \right) \mathrm{d}y$$

$$= \int_{0}^{1} \left(\int_{\frac{1}{u}}^{\infty} y^{-1-2\varepsilon} \mathrm{d}y \right) \frac{u^{1-\frac{2\varepsilon}{p}}}{u^2 + 2u\cos\alpha_1 + 1} \mathrm{d}u + \frac{1}{2\varepsilon} \int_{1}^{\infty} \frac{u^{-1-\frac{2\varepsilon}{p}}}{u^2 + 2u\cos\alpha_1 + 1} \mathrm{d}u$$

$$= \frac{1}{2\varepsilon} \left(\int_{0}^{1} \frac{u^{1+\frac{2\varepsilon}{q}}}{u^2 + 2u\cos\alpha_1 + 1} \mathrm{d}u + \int_{1}^{\infty} \frac{1}{u^2 + 2u\cos\alpha_1 + 1} u^{-1-\frac{2\varepsilon}{p}} \mathrm{d}u \right).$$

$$I_2 = I_3 = \int_{1}^{\infty} y^{-1-\frac{2\varepsilon}{q}} \left(\int_{1}^{\infty} \min_{i\in\{1,2\}} \frac{\min\{x^2, y^2\}}{x^2 - 2xy\cos\alpha_i + y^2} x^{-1-\frac{2\varepsilon}{p}} \mathrm{d}x \right) \mathrm{d}y$$

$$= \frac{1}{2\varepsilon} \left(\int_{0}^{1} \frac{u^{1+\frac{2\varepsilon}{q}}}{u^2 - 2u\cos\alpha_2 + 1} \mathrm{d}u + \int_{1}^{\infty} \frac{1}{u^2 - 2u\cos\alpha_2 + 1} u^{-1-\frac{2\varepsilon}{p}} \mathrm{d}u \right).$$

If the constant factor k in (3.1) is not the best possible, then there exists a constant $0 < M \le k$, such that

$$I := \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \min_{i\in\{1,2\}} \frac{\min\{x^2, y^2\}}{x^2 + 2xy\cos\alpha_i + y^2} f(x)g(y)\mathrm{d}x\mathrm{d}y < M\left(\int_{-\infty}^{\infty} |x|^{p-1} f^p(x)\mathrm{d}x\right)^{1/p}\left(\int_{-\infty}^{\infty} |x|^{q-1} g^q(x)\mathrm{d}x\right)^{1/q},$$

In view of (3.6) and (3.7), we obtain

$$\int_0^1 \frac{u^{1+\frac{2\varepsilon}{q}}}{u^2 + 2u\cos\alpha_1 + 1}\mathrm{d}u + \int_1^\infty \frac{1}{u^2 + 2u\cos\alpha_1 + 1} u^{-1-\frac{2\varepsilon}{p}}\mathrm{d}u + \int_0^1 \frac{u^{1+\frac{2\varepsilon}{q}}}{u^2 - 2u\cos\alpha_2 + 1}\mathrm{d}u + \int_1^\infty \frac{1}{u^2 - 2u\cos\alpha_2 + 1} u^{-1-\frac{2\varepsilon}{p}}\mathrm{d}u = \varepsilon I(\varepsilon) < \varepsilon k H(\varepsilon) = k \tag{3.8}$$

By (3.8), (2.3) and Fatou lemma [16], we find

$$\begin{aligned} k &= \int_0^1 \frac{u}{u^2 + 2u\cos\alpha_1 + 1}\mathrm{d}u + \int_1^\infty \frac{u^{-1}}{u^2 + 2u\cos\alpha_1 + 1}\mathrm{d}u \\ &\quad + \int_0^1 \frac{u}{u^2 - 2u\cos\alpha_2 + 1}\mathrm{d}u + \int_1^\infty \frac{u^{-1}}{u^2 - 2u\cos\alpha_2 + 1}\mathrm{d}u \\ &= \int_0^1 \lim_{\varepsilon\to 0^+} \frac{u^{1+\frac{2\varepsilon}{q}}}{u^2 + 2u\cos\alpha_i + 1}\mathrm{d}u + \int_1^\infty \lim_{\varepsilon\to 0^+} \frac{1}{u^2 + 2u\cos\alpha_i + 1} u^{-1-\frac{2\varepsilon}{p}}\mathrm{d}u \\ &\quad + \int_0^1 \lim_{\varepsilon\to 0^+} \frac{u^{1+\frac{2\varepsilon}{q}}}{u^2 - 2u\cos\alpha_i + 1}\mathrm{d}u + \int_1^\infty \lim_{\varepsilon\to 0^+} \frac{1}{u^2 - 2u\cos\alpha_i + 1} u^{-1-\frac{2\varepsilon}{p}}\mathrm{d}u \\ &\le \frac{\lim}{\varepsilon \to 0^+}\left(\int_0^1 \frac{u^{1+\frac{2\varepsilon}{q}}}{u^2 + 2u\cos\alpha_i + 1}\mathrm{d}u + \int_1^\infty \frac{1}{u^2 + 2u\cos\alpha_i + 1} u^{-1-\frac{2\varepsilon}{p}}\mathrm{d}u \right. \\ &\quad \left. + \int_0^1 \frac{u^{1+\frac{2\varepsilon}{q}}}{u^2 - 2u\cos\alpha_i + 1}\mathrm{d}u + \int_1^\infty \frac{1}{u^2 - 2u\cos\alpha_i + 1} u^{-1-\frac{2\varepsilon}{p}}\mathrm{d}u\right) \le M. \end{aligned}$$

Hence k is the best value of (3.1). We conform that k^p is also the best value of (3.2). Otherwise, we can get a contradiction by (3.3) that (3.1) is not the best possible. □

THEOREM 3.2. *If* $0 < p < 1, \frac{1}{p} + \frac{1}{q} = 1, f, g \ge 0$ *such that* $0 < \int_{-\infty}^{\infty} |x|^{p-1} f^p(x)\mathrm{d}x < \infty$ *and* $0 < \int_{-\infty}^{\infty} |x|^{q-1} g^q(x)\mathrm{d}x < \infty$ *, then we have the following equivalent inequalities*

$$
\begin{aligned}
I &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \min_{i\in\{1,2\}} \frac{\min\{x^2,y^2\}}{x^2+2xy\cos\alpha_i+y^2} f(x)g(y)\mathrm{d}x\mathrm{d}y \\
&< k\left(\int_{-\infty}^{\infty}|x|^{p-1}f^p(x)\mathrm{d}x\right)^{1/p}\left(\int_{-\infty}^{\infty}|x|^{q-1}g^q(x)\mathrm{d}x\right)^{1/q},
\end{aligned}
\tag{3.9}
$$

$$
\begin{aligned}
J &= \int_{-\infty}^{\infty}|y|^{-1}\left(\int_{-\infty}^{\infty}\min_{i\in\{1,2\}} \frac{\min\{x^2,y^2\}}{x^2+2xy\cos\alpha_i+y^2} f(x)\mathrm{d}x\right)^p \mathrm{d}y \\
&> k^p\int_{-\infty}^{\infty}|x|^{p-1}f^p(x)\mathrm{d}x,
\end{aligned}
\tag{3.10}
$$

$$
\begin{aligned}
L &:= \int_{-\infty}^{\infty}|x|^{-1}\left(\int_{-\infty}^{\infty}\min_{i\in\{1,2\}} \frac{\min\{x^2,y^2\}}{x^2+2xy\cos\alpha_i+y^2} g(y)\mathrm{d}y\right)^q \mathrm{d}x \\
&< k^q\int_{-\infty}^{\infty}|y|^{q-1}g^q(y)\mathrm{d}y,
\end{aligned}
\tag{3.11}
$$

where the constant factors $k = 2\ln(4\cos\frac{\alpha_1}{2}\sin\frac{\alpha_2}{2}) - \alpha_1\cot\alpha_1 + (\pi-\alpha_2)\cot\alpha_2$, *both* k^p *and* k^q *are the best possible.*

Proof. By Lemma 2.3, similar to the proof of (3.2), we obtain that (3.10) and (3.11) are valid. In view of the reverse equality of (3.3), (3.9) is valid too. On the other hand, suppose that (3.9) is valid, let $g(y)$ defined as Theorem 3.1, it is obvious $J > 0$. If $J = \infty$, then (3.10) is valid naturally; if $0 < J < \infty$, then by (3.9), we find

$$
\begin{aligned}
&\int_{-\infty}^{\infty}|y|^{q-1}g^q(y)\mathrm{d}y = J = I \\
&> k\left(\int_{-\infty}^{\infty}|x|^{p-1}f^p(x)\mathrm{d}x\right)^{1/p}\left(\int_{-\infty}^{\infty}|x|^{q-1}g^q(x)\mathrm{d}x\right)^{1/q}, \\
J^{1/p} &= \left(\int_{-\infty}^{\infty}|y|^{q-1}g^q(y)\mathrm{d}y\right)^{1/p} > k\left(\int_{-\infty}^{\infty}|x|^{p-1}f^p(x)\mathrm{d}x\right)^{1/p}.
\end{aligned}
\tag{3.12}
$$

Hence we obtain (3.10). Thus (3.10) and (3.9) are equivalent.

(3.11) and (3.9) are equivalent. In fact, we have proved (3.11) is valid above. On the other hand, suppose that (3.11) is valid, by the reverse Hölder's inequality with weight [15], we obtain

$$
\begin{aligned}
I &= \int_{-\infty}^{\infty}\left(|x|^{1-(1/p)}f(x)\mathrm{d}x\right)\left(|x|^{-1+(1/p)}\int_{-\infty}^{\infty}\min_{i\in\{1,2\}} \frac{\min\{x^2,y^2\}}{x^2+2xy\cos\alpha_i+y^2} g(y)\mathrm{d}y\right) \\
&\geq L^{1/q}\left(\int_{-\infty}^{\infty}|x|^{p-1}f^p(x)\mathrm{d}y\right)^{1/p}.
\end{aligned}
\tag{3.13}
$$

By (3.11), we obtain (3.9), and it is equivalent between (3.11) and (3.9). Thus (3.9), (3.10), and (3.11) are equivalent.

k is the best value of (3.9). In fact, If there exists a constant $M \geq k$, such that (3.9) is still valid as we replace k by M. By the reverse inequality of (3.8), we obtain

$$\begin{aligned}&\int_0^1\left(\frac{1}{u^2+2u\cos\alpha_1+1}+\frac{1}{u^2-2u\cos\alpha_2+1}\right)u^{1+\frac{2\varepsilon}{q}}\mathrm{d}u\\&+\int_1^\infty\left(\frac{1}{u^2+2u\cos\alpha_1+1}+\frac{1}{u^2-2u\cos\alpha_2+1}\right)u^{-1-\frac{2\varepsilon}{p}}\mathrm{d}u>k\end{aligned} \tag{3.14}$$

Suppose that $0<\varepsilon_0<\frac{|q|}{2}$ such that $\frac{2\varepsilon_0}{q}+1>0$. Letting $0<\varepsilon\leq\varepsilon_0$ gives $u^{\frac{2\varepsilon}{q}}\leq u^{\frac{2\varepsilon_0}{q}}\,(u\in(0,1])$ and

$$\int_0^1\left(\frac{1}{u^2+2u\cos\alpha_1+1}+\frac{1}{u^2-2u\cos\alpha_2+1}\right)u^{1+\frac{2\varepsilon_0}{q}}\mathrm{d}u\leq\frac{k}{2}\int_0^1 u^{\frac{2\varepsilon_0}{q}}=\frac{k}{2}\cdot\frac{1}{1+(2\varepsilon_0)/q}.$$

By Lebesgue control convergent theorem [16], it follows

$$\begin{aligned}&\int_0^1\left(\frac{1}{u^2+2u\cos\alpha_1+1}+\frac{1}{u^2-2u\cos\alpha_2+1}\right)u^{1+\frac{2\varepsilon}{q}}\mathrm{d}u\\&=\int_0^1\left(\frac{1}{u^2+2u\cos\alpha_1+1}+\frac{1}{u^2-2u\cos\alpha_2+1}\right)u\mathrm{d}u+o(1)(\varepsilon\to0^+).\end{aligned}$$

Then by Levi theorem [16], we obtain

$$\begin{aligned}&\int_1^\infty\left(\frac{1}{u^2+2u\cos\alpha_1+1}+\frac{1}{u^2-2u\cos\alpha_2+1}\right)u^{-1-\frac{2\varepsilon}{p}}\mathrm{d}u\\&=\int_1^\infty\left(\frac{1}{u^2+2u\cos\alpha_1+1}+\frac{1}{u^2-2u\cos\alpha_2+1}\right)\frac{1}{u}\mathrm{d}u+\tilde{o}(1)(\varepsilon\to0^+).\end{aligned}$$

By (3.14), if follows that $k \geq M$ for $\varepsilon \to 0^+$. Hence k is the best value of (3.9). Furthermore, the constant factors in (3.10) and (3.11) are both the best value too. Otherwise, by (3.3) or (3.13), we may get a contradiction that the constant factor in (3.9) is not the best possible. □

By Note (ii), Theorems 3.1 and 3.2, it follows that

COROLLARY 3.3. *If* $p>1,\frac{1}{p}+\frac{1}{q}=1, f,g\geq0$ *such that* $0<\int_{-\infty}^{\infty}|x|^{p-1}f^p(x)\mathrm{d}x<\infty$ *and* $0<\int_{-\infty}^{\infty}|x|^{q-1}g^q(x)\mathrm{d}x<\infty$ *, then we obtain the following equivalent inequalities*

$$\begin{aligned}&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\min\{x^2,y^2\}}{x^2+2xy\cos\alpha+y^2}f(x)g(y)\mathrm{d}x\mathrm{d}y\\&<k_1\left(\int_{-\infty}^{\infty}|x|^{p-1}f^p(x)\mathrm{d}x\right)^{1/p}\left(\int_{-\infty}^{\infty}|x|^{q-1}g^q(x)\mathrm{d}x\right)^{1/q},\end{aligned} \tag{3.15}$$

$$\int_{-\infty}^{\infty} |y|^{-1} \left(\int_{-\infty}^{\infty} \frac{\min\{x^2, y^2\}}{x^2 + 2xy\cos\alpha + y^2} f(x)\mathrm{d}x \right)^p \mathrm{d}y < k_1^p \int_{-\infty}^{\infty} |x|^{p-1} f^p(x)\mathrm{d}x, \quad (3.16)$$

where the constant factors $k_1 = 2\ln(2\sin\alpha) + (\pi - 2\alpha)\cot\alpha$ *and* k_1^p *are both the best possible. In particular, for* $\alpha = \pi/3$ *or* $2\pi/3$, *it reduces to*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\min\{x^2, y^2\}}{x^2 \pm xy + y^2} f(x)g(y)\mathrm{d}x\mathrm{d}y < \left(\ln 3 + \frac{\sqrt{3}\pi}{9}\right) \left(\int_{-\infty}^{\infty} |x|^{p-1} f^p(x)\mathrm{d}x \right)^{1/p} \left(\int_{-\infty}^{\infty} |x|^{q-1} g^q(x)\mathrm{d}x \right)^{1/q}, \quad (3.17)$$

$$\int_{-\infty}^{\infty} |y|^{-1} \left(\int_{-\infty}^{\infty} \frac{\min\{x^2, y^2\}}{x^2 \pm xy + y^2} f(x)\mathrm{d}x \right)^p \mathrm{d}y < \left(\ln 3 + \frac{\sqrt{3}\pi}{9}\right)^p \int_{-\infty}^{\infty} |x|^{p-1} f^p(x)\mathrm{d}x. \quad (3.18)$$

COROLLARY 3.4. *If* $0 < p < 1, \frac{1}{p} + \frac{1}{q} = 1, f, g \geq 0$ *such that* $0 < \int_{-\infty}^{\infty} |x|^{p-1} f^p(x)\mathrm{d}x < \infty$ *and* $0 < \int_{-\infty}^{\infty} |x|^{q-1} g^q(x)\mathrm{d}x < \infty$, *then we have the following equivalent inequalities*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\min\{x^2, y^2\}}{x^2 + 2xy\cos\alpha + y^2} f(x)g(y)\mathrm{d}x\mathrm{d}y > k_1 \left(\int_{-\infty}^{\infty} |x|^{p-1} f^p(x)\mathrm{d}x \right)^{1/p} \left(\int_{-\infty}^{\infty} |x|^{q-1} g^q(x)\mathrm{d}x \right)^{1/q}, \quad (3.19)$$

$$\int_{-\infty}^{\infty} |y|^{-1} \left(\int_{-\infty}^{\infty} \frac{\min\{x^2, y^2\}}{x^2 + 2xy\cos\alpha + y^2} f(x)\mathrm{d}x \right)^p \mathrm{d}y > k_1^p \int_{-\infty}^{\infty} |x|^{p-1} f^p(x)\mathrm{d}x, \quad (3.20)$$

$$\int_{-\infty}^{\infty} |x|^{-1} \left(\int_{-\infty}^{\infty} \frac{\min\{x^2, y^2\}}{x^2 + 2xy\cos\alpha + y^2} g(y)\mathrm{d}y \right)^q \mathrm{d}x < k_1^q \int_{-\infty}^{\infty} |y|^{q-1} g^q(y)\mathrm{d}y, \quad (3.21)$$

where the constant factors $k_1 = 2\ln(2\sin\alpha) + (\pi - 2\alpha)\cot\alpha$, k_1^p *and* k_1^q *are both the best possible. In particular, for* $\alpha = \pi/3$ *or* $2\pi/3$, *it reduces to*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\min\{x^2, y^2\}}{x^2 \pm xy + y^2} f(x)g(y)\mathrm{d}x\mathrm{d}y > \left(\ln 3 + \frac{\sqrt{3}\pi}{9}\right) \left(\int_{-\infty}^{\infty} |x|^{p-1} f^p(x)\mathrm{d}x \right)^{1/p} \left(\int_{-\infty}^{\infty} |x|^{q-1} g^q(x)\mathrm{d}x \right)^{1/q}, \quad (3.22)$$

$$\int_{-\infty}^{\infty} |y|^{-1} \left(\int_{-\infty}^{\infty} \frac{\min\{x^2, y^2\}}{x^2 \pm xy + y^2} f(x) \mathrm{d}x \right)^p \mathrm{d}y > \left(\ln 3 + \frac{\sqrt{3}\pi}{9} \right)^p \int_{-\infty}^{\infty} |x|^{p-1} f^p(x) \mathrm{d}x, \quad (3.23)$$

$$\int_{-\infty}^{\infty} |x|^{-1} \left(\int_{-\infty}^{\infty} \frac{\min\{x^2, y^2\}}{x^2 \pm xy + y^2} g(y) \mathrm{d}y \right)^q \mathrm{d}x < \left(\ln 3 + \frac{\sqrt{3}\pi}{9} \right)^q \int_{-\infty}^{\infty} |y|^{q-1} g^q(y) \mathrm{d}y. \quad (3.24)$$

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Competing interests
The authors declare that they have no competing interests.

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