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A complete convergence theorem for weighted sums of arrays of rowwise negatively dependent random variables

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Abstract

In this article, applying moment inequality of negatively dependent (ND) random variables which obtained by Asadian et al., the complete convergence theorem for weighted sums of arrays of rowwise ND random variables is discussed. As a result, the complete convergence theorem for ND arrays of random variables is extended. Our results generalize and improve those on complete convergence theorem previously obtained by Hu et al., Ahmed et al, Volodin, and Sung from the independent and identically distributed case to ND sequences.

Mathematical Subject Classification: 62F12.

Keywords: negatively dependent arrays of random variables, complete convergence theorem, moment inequality

1 Introduction

Random variables X and Y are said to be negatively dependent (ND) if

$$P(X \le x, Y \le y) \le P(X \le x)P(Y \le y) \tag{1.1}$$

for all $x, y \in \mathbb{R}$. A collection of random variables is said to be pairwise negatively dependent (PND) if every pair of random variables in the collection satisfies (1.1).

It is important to note that (1.1) implies

$$P(X > x, Y > y) \le P(X > x)P(Y > y) \tag{1.2}$$

for all $x, y \in \mathbb{R}$. Moreover, it follows that (1.2) implies (1.1), and hence, (1.1) and (1.2) are equivalent. However, (1.1) and (1.2) are not equivalent for a collection of three or more random variables. Consequently, the following definition is needed to define sequences of ND random variables.

Definition 1. Random variables $X_1, \ldots, X_n$ are said to be ND if for all real $x_1, \ldots, x_n$,

$$P\left(\bigcap_{j=1}^{n}(X_j \le x_j)\right) \le \prod_{j=1}^{n} P(X_j \le x_j),$$
$$P\left(\bigcap_{j=1}^{n}(X_j > x_j)\right) \le \prod_{j=1}^{n} P(X_j > x_j).$$

An infinite sequence of random variables $\{X_n;\ n \geq 1\}$ is said to be ND if every finite subset $X_1, ..., X_n$ is ND.

Definition 2. Random variables $X_1, X_2, ..., X_n$, $n \geq 2$ are said to be negatively associated (NA) if for every pair of disjoint subsets A_1 and A_2 of $\{1, 2, ..., n\}$,

$$\operatorname{cov}\left(f_1(X_i; i \in A_1),\ f_2(X_j; j \in A_2)\right) \leq 0,$$

where f_1 and f_2 are increasing for every variable (or decreasing for every variable), such that this covariance exists. An infinite sequence of random variables $\{X_n;\ n \geq 1\}$ is said to be NA if every finite subfamily is NA.

The definition of PND was given by Lehmann [1], the concept of ND and NA was introduced by Joag-Dev and Proschan [2]. These concepts of dependence random variables are very useful to reliability theory and applications.

Obviously, NA implies ND from the definition of NA and ND. But ND does not imply NA, so ND is much weaker than NA. Because of the wide applications of ND random variables, the notions of ND random variables have received more and more attention recently. A series of useful results have been established [3-12]. Hence, the extending the limit properties of independent variables to the case of ND variables is highly desirable and of considerably significance in the theory and application.

The concept of complete convergence of a sequence of random variables was introduced by Hsu and Robbins [13] as follows. A sequence $\{X_n;\ n \geq 1\}$ of random variables converges completely to the constant a if $\sum_{n=1}^{\infty} P(|X_n - a| > \varepsilon) < \infty$ for all $\epsilon > 0$. In view of the Borel-Cantelli lemma, this implies that $X_n \to a$ almost surely. The converse is true if $\{X_n;\ n \geq 1\}$ are independent random variables. Thus, complete convergence is one of the most important problems in probability theory. Hsu and Robbins [13] proved that the sequence of arithmetic means of independent and identically distributed (i.i.d.) random variables converges completely to the expected value if the variance of the summands is finite. Baum and Katz [14] proved that if $\{X, X_n;\ n \geq 1\}$ is a sequence of i.i.d. random variables with mean zero, then $E|X|^{p(t+2)} < \infty$ $(1 \leq p < 2,\ t \geq 1)$ is equivalent to the condition that $\sum_{n=1}^{\infty} n^t P\left(\left|\sum_{i=1}^{n} X_i\right| / n^{1/p} > \varepsilon\right) < \infty$ for all $\epsilon > 0$. Some recent results can be found in [12,15-17].

In this article we study the complete convergence for ND random variables. Our results generalize and improve those on complete convergence theorem previously obtained by Hu et al. [16], Ahmed et al. [15], Volodin [17] and Sung [18] from the i.i.d. case to ND sequences.

2 Main results

Theorem 1. Let $\{X_{nk};\ k, n \geq 1\}$ be an array of rowwise ND random variables, there exist a r.v. X and a positive constant c satisfying

$$P(|X_{nk}| \geq x) \leq cP(|X| \geq x) \text{ for all } n, \quad k \geq 1, \quad x > 0. \tag{2.1}$$

Suppose that $\beta > -1$, and that $\{a_{nk};\ k, n \geq 1\}$ is an array of constants such that

$$\sup_{k \geq 1} |a_{nk}| = O(n^{-\gamma}) \quad \text{for some } \gamma > 0, \tag{2.2}$$

and

$$\sum_{k=1}^{\infty} |a_{nk}|^{\theta} = O(n^{\alpha}), \quad \text{for some } \alpha < 2\gamma \text{ and some } 0 < \theta < \min(2, 2 - \alpha/\gamma). \tag{2.3}$$

(i) If $1 + \alpha + \beta > 0$ and

$$E|X|^{\nu} < \infty, \quad \nu = \theta + \frac{1 + \alpha + \beta}{\gamma}. \tag{2.4}$$

When $\nu \geq 1$, further assume that $EX_{nk} = 0$ for any $n, k \geq 1$. Then

$$\sum_{n=1}^{\infty} n^{\beta} P\left(\left|\sum_{k=1}^{\infty} a_{nk} X_{nk}\right| > \varepsilon\right) < \infty, \quad \forall \varepsilon > 0. \tag{2.5}$$

(ii) If $1 + \alpha + \beta = 0$ and

$$E(|X|^{\theta} \ln(1 + |X|)) < \infty. \tag{2.6}$$

When $\nu = \theta \geq 1$, further assume that $EX_{nk} = 0$ for any $n, k \geq 1$. Then (2.5) holds.

Remark 2. Theorem 1 generalize and improve those on complete convergence theorem previously obtained by Hu et al. [16], Ahmed et al. [15], Volodin [17], and Sung [18] from the i.i.d. case to ND arrays.

By using Theorrem 1, we can extend the well-known Baum and Katz [14] complete convergence theorem from the i.i.d. case to ND random variables.

Corollary 3. Let $\{X_n, n \in N\}$ be a sequence of ND random variables, there exist a r.v. X and a constant c satisfying $P(|X_n| \geq x) \leq cP(|X| \geq x)$ for all $n \geq 1$, $x > 0$. Suppose $\gamma > 1/2$ and $\gamma p > 1$; and if $p \geq 1$ then assume also that $EX_n = 0$ for any $n \geq 1$. If $E|X|^p < \infty$, then

$$\sum_{n=1}^{\infty} n^{\gamma p - 2} P(|S_n| > \varepsilon n^{\gamma}) < \infty, \quad \forall \varepsilon > 0,$$

where $S_n = \sum_{k=1}^{n} X_k$.

3 Proofs

In the following, let $a_n \ll b_n$ denote that there exists a constant $c > 0$ such that $a_n \leq cb_n$ for sufficiently large n. The symbol c stands for a generic positive constant which may differ from one place to another.

Lemma 1. [3] Let $X_1, ..., X_n$ be ND random variables and let $\{f_n; n \geq 1\}$ be a sequence of Borel functions all of which are monotone increasing (or all are monotone decreasing). Then $\{f_n(X_n); n \geq 1\}$ is still a sequence of ND r.v.'s.

Lemma 2. [9] Let $\{X_n; n \geq 1\}$ be an ND sequence with $EX_n = 0$ and $E|X_n|^p < \infty$, $p \geq 2$. Then

$$E|S_n|^p \leq c_p \left\{\sum_{i=1}^{n} E|X_i|^p + \left(\sum_{i=1}^{n} EX_i^2\right)^{p/2}\right\},$$

where $c_p > 0$ depends only on p.

Lemma 3. [19] Let $\{X_n; n \geq 1\}$ be an arbitrary sequence of random variables. If there exist a r.v. X and a positive constant c such that $P(|X_n| \geq x) \leq cP(|X| \geq x)$ for and $n \geq 1$ and $x > 0$. Then for any $u > 0$, $t > 0$, and $n \geq 1$,

$$E|X_n|^u I_{(|X_n| \leq t)} \leq c\left(E|X|^u I_{(|X| \leq t)} + t^u P(|X| > t)\right),$$

and

$$E|X_n|^u I_{(|X_n|>t)} \leq cE|X|^u I_{(|X|>t)}.$$

Proof of Theorem 1. Let $a_{nk}^{+} = \max(a_{nk}, 0) \geq 0$ and $a_{nk}^{-} = \max(-a_{nk}, 0) \geq 0$. From (2.2), (2.5) and

$$\sum_{n=1}^{\infty} n^{\beta} P\left(\left|\sum_{k=1}^{\infty} a_{nk}X_{nk}\right| > \varepsilon\right) \leq \sum_{n=1}^{\infty} n^{\beta} P\left(\left|\sum_{k=1}^{\infty} a_{nk}^{+}X_{nk}\right| > \varepsilon/2\right) + \sum_{n=1}^{\infty} n^{\beta} P\left(\left|\sum_{k=1}^{\infty} a_{nk}^{-}X_{nk}\right| > \varepsilon/2\right),$$

without loss of generality, for all i, $n \geq 1$, we can assume that $a_{ni} > 0$ and

$$\sup_{k\geq 1} a_{nk} = n^{-\gamma}. \tag{3.1}$$

For any k, $n \geq 1$, let

$$Y_{nk} = -a_{nk}^{-1} I_{(a_{nk}X_{nk}<-1)} + X_{nk} I_{(a_{nk}|X_{nk}|\leq 1)} + a_{nk}^{-1} I_{(a_{nk}X_{nk}>1)}.$$

Then for any $n \geq 1$,

$$\left\{\left|\sum_{k=1}^{\infty} a_{nk}X_{nk}\right| > \varepsilon\right\} = \left\{\forall k \geq 1, |a_{nk}X_{nk}| \leq 1, \left|\sum_{k=1}^{\infty} a_{nk}Y_{nk}\right| > \varepsilon\right\} \cup \{\exists k \geq 1, |a_{nk}X_{nk}| > 1\}.$$

Hence

$$\begin{aligned}\sum_{n=1}^{\infty} n^{\beta} P\left(\sum_{k=1}^{\infty} |a_{nk}X_{nk}| > \varepsilon\right) &\leq \sum_{n=1}^{\infty} n^{\beta} \sum_{k=1}^{\infty} P(|a_{nk}X_{nk}| > 1) + \sum_{n=1}^{\infty} n^{\beta} P\left(\left|\sum_{k=1}^{\infty} a_{nk}Y_{nk}\right| > \varepsilon\right) \\ &\hat{=} J_1 + J_2.\end{aligned} \tag{3.2}$$

Therefore, in order to prove (2.5), it suffices to prove that $J_1 < \infty$ and $J_2 < \infty$.

(i) If $1 + \alpha + \beta > 0$, by Lemma 3, (2.1), (2.3), (2.4), (3.1), and the Markov inequality, we have

$$\begin{aligned} J_1 &\ll \sum_{n=1}^{\infty} n^{\beta} \sum_{k=1}^{\infty} P(|a_{nk}X| > 1) \\ &\leq \sum_{n=1}^{\infty} n^{\beta} \sum_{k=1}^{\infty} E|a_{nk}X|^{\theta} I_{(|X|>a_{nk}^{-1})} \\ &\leq \sum_{n=1}^{\infty} n^{\beta} E|X|^{\theta} I_{(|X|>n^{\gamma})} \sum_{k=1}^{\infty} a_{nk}^{\theta} \\ &\ll \sum_{n=1}^{\infty} n^{\beta+\alpha} E|X|^{\theta} I_{(|X|>n^{\gamma})} \\ &= \sum_{n=1}^{\infty} n^{\beta+\alpha} \sum_{j=n}^{\infty} E|X|^{\theta} I_{(j^{\gamma}<|X|\leq(j+1)^{\gamma})} \\ &= \sum_{j=1}^{\infty} E|X|^{\theta} I_{(j^{\gamma}<|X|\leq(j+1)^{\gamma})} \sum_{n=1}^{j} n^{\beta+\alpha} \\ &\leq \sum_{j=1}^{\infty} j^{1+\alpha+\beta} E|X|^{\theta} I_{(j^{\gamma}<|X|\leq(j+1)^{\gamma})} \\ &\ll \sum_{j=1}^{\infty} E|X|^{\theta+(1+\alpha+\beta)/\gamma} I_{(j^{\gamma}<|X|\leq(j+1)^{\gamma})} \\ &< \infty. \end{aligned} \tag{3.3}$$

Next we prove that $J_2 < \infty$ for $\nu < 1$ and $\nu \geq 1$, respectively. Put $N(nk) = \sharp\{i; (nk)^{\gamma} \leq a_{ni}^{-1} < (n(k+1))^{\gamma}\},\ k,\ n \geq 1$.

(a) If $\nu < 1$. Choose t such that $\nu < t < 1$. By the Markov inequality, the c_γ inequality, Lemma 3 and the process of proof of (3.3), we have

$$\begin{aligned}
J_2 &\ll \sum_{n=1}^{\infty} n^{\beta} E\left|\sum_{k=1}^{\infty} a_{nk} Y_{nk}\right|^{t} \leq \sum_{n=1}^{\infty} n^{\beta} \sum_{k=1}^{\infty} E|a_{nk} Y_{nk}|^{t} \\
&\ll \sum_{n=1}^{\infty} n^{\beta} \sum_{k=1}^{\infty} \left\{E|a_{nk} X_{nk}|^{t} I_{(a_{nk}|X_{nk}| \leq 1)} + P\left(|a_{nk} X_{nk}| > 1\right)\right\} \\
&\ll \sum_{n=1}^{\infty} n^{\beta} \sum_{k=1}^{\infty} \left\{E|a_{nk} X|^{t} I_{(a_{nk}|X| \leq 1)} + P\left(|a_{nk} X| > 1\right)\right\} \\
&= \sum_{n=1}^{\infty} n^{\beta} \sum_{j=1}^{\infty} \sum_{(nj)^{\gamma} \leq a_{nk}^{-1} < (n(j+1))^{\gamma}} \left\{a_{nk}^{t} E|X|^{t} I_{\left(|X| \leq a_{nk}^{-1}\right)} + P\left(|a_{nk} X| > 1\right)\right\} \\
&\ll \sum_{n=1}^{\infty} n^{\beta} \sum_{j=1}^{\infty} N(nj)(nj)^{-\gamma t} E|X|^{t} I_{\left(|X| < (n(j+1))^{\gamma}\right)} \\
&= \sum_{n=1}^{\infty} n^{\beta} \sum_{j=1}^{\infty} N(nj)(nj)^{-\gamma t} \sum_{i=1}^{n(j+1)} E|X|^{t} I_{\left((i-1)^{\gamma} \leq |X| < i^{\gamma}\right)} \\
&\leq \sum_{n=1}^{\infty} n^{\beta} \sum_{j=1}^{\infty} N(nj)(nj)^{-\gamma t} \sum_{i=1}^{2n} E|X|^{t} I_{\left((i-1)^{\gamma} \leq |X| < i^{\gamma}\right)} \\
&\quad + \sum_{n=1}^{\infty} n^{\beta} \sum_{j=1}^{\infty} N(nj)(nj)^{-\gamma t} \sum_{i=2n+1}^{n(j+1)} E|X|^{t} I_{\left((i-1)^{\gamma} \leq |X| < i^{\gamma}\right)} \\
&\hat{=} J_{21} + J_{22}.
\end{aligned} \tag{3.4}$$

Since $t > \nu$ and $\gamma > 0$, so $t > \theta$, $(1 + 1/k)^{-\gamma\theta} \geq 2^{-\gamma\theta}$ and $(nk)^{\gamma(t-\theta)} \geq (nj)^{\gamma(t-\theta)}$, $\forall k \geq j$. Therefore, by (2.3) we have

$$\begin{aligned}
n^{\alpha} &\gg \sum_{i=1}^{\infty} a_{ni}^{\theta} = \sum_{k=1}^{\infty} \sum_{(nk)^{\gamma} \leq a_{ni}^{-1} < (n(k+1))^{\gamma}} a_{ni}^{\theta} \\
&\geq \sum_{k=1}^{\infty} N(nk)\left(n(k+1)\right)^{-\gamma\theta} \\
&\geq \sum_{k=1}^{\infty} N(nk) 2^{-\gamma\theta} (nk)^{-\gamma\theta} \\
&\gg \sum_{k=j}^{\infty} N(nk)(nk)^{-\gamma t}(nk)^{\gamma(t-\theta)} \\
&\geq \sum_{k=j}^{\infty} N(nk)(nk)^{-\gamma t}(nj)^{\gamma(t-\theta)}.
\end{aligned}$$

Hence,

$$\sum_{k=j}^{\infty} N(nk)(nk)^{-\gamma t} \ll n^{\alpha - \gamma(t-\theta)} j^{-\gamma(t-\theta)}, \quad \forall j \in N. \tag{3.5}$$

Combining with (2.4) and $t > \nu = \theta + \dfrac{1+\alpha+\beta}{\gamma}$, i.e. $\alpha + \beta - \gamma(t - \theta) < -1$, we can get that

$$\begin{aligned} J_{21} &\ll \sum_{n=1}^{\infty} n^{\beta} n^{\alpha-\gamma(t-\theta)} \sum_{i=1}^{2n} E|X|^t I_{((i-1)^{\gamma} \le |X| < i^{\gamma})} \\ &\ll \sum_{i=2}^{\infty} E|X|^t I_{((i-1)^{\gamma} \le |X| < i^{\gamma})} \sum_{n=[i/2]}^{\infty} n^{\beta+\alpha-\gamma(t-\theta)} \\ &\ll \sum_{i=2}^{\infty} i^{\beta+\alpha-\gamma(t-\theta)+1} E|X|^t I_{((i-1)^{\gamma} \le |X| < i\gamma)} \\ &\ll \sum_{i=2}^{\infty} E|X|^{\theta+(1+\alpha+\beta)/\gamma} I_{((i-1)^{\gamma} \le |X| < i\gamma)} \\ &< \infty. \end{aligned} \tag{3.6}$$

By (3.5),

$$\begin{aligned} J_{22} &= \sum_{n=1}^{\infty} n^{\beta} \sum_{i=2n+1}^{\infty} E|X|^t I_{((i-1)^{\gamma} \le |X| < i^{\gamma})} \sum_{j=\left[\frac{i}{n}-1\right]}^{\infty} N(nj)(nj)^{-\gamma t} \\ &\ll \sum_{n=1}^{\infty} n^{\beta} \sum_{i=2n+1}^{\infty} n^{\alpha-\gamma(t-\theta)} \left(\frac{i}{n}\right)^{-\gamma(t-\theta)} E|X|^t I_{((i-1)^{\gamma} \le |X| < i^{\gamma})} \\ &\ll \sum_{i=2}^{\infty} i^{-\gamma(t-\theta)} E|X|^t I_{((i-1)^{\gamma} \le |X| < i^{\gamma})} \sum_{n=1}^{[i/2]} n^{\alpha+\beta} \\ &\ll \sum_{i=2}^{\infty} i^{1+\alpha+\beta-\gamma(t-\theta)} E|X|^t I_{((i-1)^{\gamma} \le |X| < i^{\gamma})} \\ &\ll \sum_{i=2}^{\infty} E|X|^{\theta+(1+\alpha+\beta)/\gamma} I_{((i-1)^{\gamma} \le |X| < i^{\gamma})} \\ &< \infty. \end{aligned} \tag{3.7}$$

By (3.2), (3.3), (3.4), (3.6), and (3.7), (2.5) holds.

(b) If $\nu \ge 1$. Since $EX_{nk} = 0$, $E|X|^{\nu} < \infty$, and $\nu \ge 1$, $\nu > \theta$, $\beta + 1 > 0$, $\gamma > 0$, by (2.3), (2.4), (3.1), and Lemma 3, we have

$$\begin{aligned} \left|\sum_{k=1}^{\infty} E_{a_n k} Y_{nk}\right| &\le \left|\sum_{k=1}^{\infty} E a_{nk} X_{nk} I_{(|a_{nk}X_{nk}| \le 1)}\right| + \sum_{k=1}^{\infty} P\left(|a_{nk}X_{nk}| > 1\right) \\ &= \left|\sum_{k=1}^{\infty} E a_{nk} X_{nk} I_{(|a_{nk}X_{nk}| > 1)}\right| + \sum_{k=1}^{\infty} EI\left(|a_{nk}X_{nk}| > 1\right) \\ &\ll \sum_{k=1}^{\infty} E|a_{nk}X_{nk}|^{\nu} I_{(|a_{nk}X_{nk}| > 1)} \\ &\le \sup_{k \ge 1} a_{nk}^{\nu-\theta} \sum_{k=1}^{\infty} a_{nk}^{\theta} E|X|^{\nu} I_{(|X| > a_{nk}^{-1})} \\ &\ll n^{-\gamma(\nu-\theta)+\alpha} E|X|^{\nu} I_{(|X| > n^{\gamma})} = n^{-(1+\beta)} E|X|^{\nu} I_{(|X| > n^{\gamma})} \\ &\to 0,\ n \to \infty. \end{aligned} \tag{3.8}$$

Thus, in order to prove $J_2 < \infty$, we only need to prove that for all $\epsilon > 0$,

$$J_2^* = \sum_{n=1}^{\infty} n^{\beta} P\left(\left|\sum_{k=1}^{\infty}(a_{nk}Y_{nk} - Ea_{nk}Y_{nk})\right| > \varepsilon\right) < \infty. \tag{3.9}$$

Obviously, Y_{nk} is monotonic on X_{nk}. By Lemma 1, $\{a_{nk}\ Y_{nk} - Ea_{nk}\ Y_{nk};\ k,\ n \geq 1\}$ is also an array of rowwise ND random variables with $E(a_{nk}\ Y_{nk} - Ea_{nk}\ Y_{nk}) = 0$. And note that $\gamma(2 - \theta) - \alpha = \gamma(2 - \alpha/\gamma - \theta) > 0$ from $\theta < 2 - \alpha/\gamma$ and $1 + \beta > 0$ from $\beta > -1$, let $t > \max\left(2, \dfrac{2(1+\beta)}{\gamma(2-\theta)-\alpha}\right)$ in Lemma 2, by the Markov inequality, the c_r inequality, we have

$$\begin{aligned} J_2^* &\ll \sum_{n=1}^{\infty} n^{\beta} E\left|\sum_{k=1}^{\infty}(a_{nk}Y_{nk} - Ea_{nk}Y_{nk})\right|^t \\ &\ll \sum_{n=1}^{\infty} n^{\beta}\left\{\sum_{k=1}^{\infty} E|a_{nk}Y_{nk}|^t + \left(\sum_{k=1}^{\infty} E|a_{nk}Y_{nk}|^2\right)^{t/2}\right\} \\ &\triangleq J_{21}^* + J_{22}^*. \end{aligned} \tag{3.10}$$

From the process of the proof of (3.4)-(3.7), we know that

$$J_{21}^* < \infty. \tag{3.11}$$

Since $\theta < \min(2, \nu)$ and $E|X|^{\nu} < \infty$, by Lemma 3, (2.1), (2.3), and (3.1), we have

$$\begin{aligned} \sum_{k=1}^{\infty} E|a_{nk}Y_{nk}|^2 &\ll \sum_{k=1}^{\infty} E\left(|a_{nk}X|^2 I_{(|a_{nk}X|\le 1)} + \sum_{k=1}^{\infty} P\left(|a_{nk}X| > 1\right)\right. \\ &\ll \begin{cases} \sum_{k=1}^{\infty} E|a_{nk}X|^2 \ll \sum_{k=1}^{\infty} |a_{nk}|^{\theta} \sup_{k\geq 1}|a_{nk}|^{2-\theta} \ll n^{\alpha-\gamma(2-\theta)}, \ \nu \geq 2, \\ \sum_{k=1}^{\infty} E|a_{nk}X|^{\nu} \ll \sum_{k=1}^{\infty} |a_{nk}|^{\theta} \sup_{k\geq 1}|a_{nk}|^{\nu-\theta} \ll n^{\alpha-\gamma(\nu-\theta)}, \ \nu = 2. \end{cases} \end{aligned}$$

By the definition of t, $t(\gamma(2 - \theta) - \alpha)/2 - \beta > 1$ and $t(1 + \beta)/2 - \beta > 1$, hence

$$J_{22}^* \ll \begin{cases} \sum_{n=1}^{\infty} \frac{1}{n^{t(\gamma(2-\theta)-\alpha)/2-\beta}}, \ \nu \geq 2, \\ \sum_{n=1}^{\infty} \frac{1}{n^{t(1+\beta)/2-\beta}}, \quad \nu < 2 \end{cases} < \infty. \tag{3.12}$$

By (3.10)-(3.12), we have (3.9), therefore, (2.5) holds.

(ii) If $1 + \alpha + \beta = 0$, then $\sum_{n=1}^{j} n^{\alpha+\beta} \ll \ln j$, similar to proof of (3.3), we have

$$J_1 = \sum_{n=1}^{\infty} n^{\beta} \sum_{k=1}^{\infty} P\left(|a_{nk}X_{nk}| > 1\right) \ll E\left(|X|^{\theta} \ln(1 + |X|)\right) < \infty$$

from (2.6).

(a) When $\nu = \theta < 1$, similar to the corresponding part of the proof of (3.6) and (3.7), we get that

$$J_{21} \ll E\left(|X|^{\theta}\right) < \infty,$$

and

$$J_{22} \ll E\left(|X|^{\theta} \ln(1 + |X|)\right) < \infty,$$

from (2.6). Therefore, (2.5) holds.

(b) When $\nu = \theta \geq 1$. Since $EX_{nk} = 0$, $E|X|^{\nu} < \infty$, $1 + \alpha + \beta = 0$, $\theta < 2$, and $\beta > -1$, $\nu = \theta$, (3.8) remains true. Therefore, we only need to prove (3.9). By Lemmas 2 and 3, $J_1 < \infty$, noting that $\nu = \theta$ and $\alpha + \beta = -1$, we have

$$\begin{aligned}
J_2^* &\ll \sum_{n=1}^{\infty} n^{\beta} E\left(\sum_{k=1}^{\infty}(a_{nk}Y_{nk} - Ea_{nk}Y_{nk})\right)^2 \ll \sum_{n=1}^{\infty} n^{\beta} \sum_{k=1}^{\infty} E|a_{nk}Y_{nk}|^2 \\
&\ll \sum_{n=1}^{\infty} n^{\beta} \sum_{k=1}^{\infty} P(|a_{nk}X_{nk}| > 1) + \sum_{n=1}^{\infty} n^{\beta} \sum_{k=1}^{\infty} E|a_{nk}X|^2 I_{(|a_{nk}X| \leq 1)} \\
&= J_1 + \sum_{n=1}^{\infty} n^{\beta} \sum_{k=1}^{\infty} a_{nk}^2 EX^2 I_{(|X| \leq a_{nk}^{-1})} \\
&\ll \sum_{n=1}^{\infty} n^{\beta} \sum_{j=1}^{\infty} \sum_{(nj)^{\gamma} \leq a_{nk}^{-1} < (n(j+1))^{\gamma}} a_{nk}^2 EX^2 I_{(|X| \leq a_{nk}^{-1})} \\
&\ll \sum_{n=1}^{\infty} n^{\beta} \sum_{j=1}^{\infty} N(nj)(nj)^{-2\gamma} EX^2 I_{(|X| < (n(j+1))^{\gamma})} \\
&= \sum_{n=1}^{\infty} n^{\beta} \sum_{j=1}^{\infty} N(nj)(nj)^{-2\gamma} \sum_{i=1}^{n(j+1)} EX^2 I_{((i-1)^{\gamma} \leq |X| < i^{\gamma})} \\
&\leq \sum_{n=1}^{\infty} n^{\beta} \sum_{j=1}^{\infty} N(nj)(nj)^{-2\gamma} \sum_{i=1}^{2n} EX^2 I_{((i-1)^{\gamma} \leq |X| < i^{\gamma})} \\
&\quad + \sum_{n=1}^{\infty} n^{\beta} \sum_{j=1}^{\infty} N(nj)(nj)^{-2\gamma} \sum_{i=2n+1}^{n(j+1)} EX^2 I_{((i-1)^{\gamma} \leq |X| < i^{\gamma})} \\
&\stackrel{\wedge}{=} J_{21}^* + J_{22}^*.
\end{aligned} \tag{3.13}$$

Since $\nu = \theta < 2$, hence (3.5) also holds for $t = 2$. Combining with (2.6) and $\alpha+\beta = -1$, we can get that

$$\begin{aligned}
J_{21}^* &\ll \sum_{n=1}^{\infty} n^{\beta} n^{\alpha - \gamma(2-\theta)} \sum_{i=1}^{2n} EX^2 I_{((i-1)^{\gamma} \leq |X| < i^{\gamma})} \\
&\ll \sum_{n=1}^{\infty} EX^2 I_{((i-1)^{\gamma} \leq |X| < i^{\gamma})} \sum_{n=[i/2]}^{\infty} n^{-1-\gamma(2-\theta)} \\
&\ll \sum_{i=2}^{\infty} i^{-\gamma(2-\theta)} EX^2 I_{((i-1)^{\gamma} \leq |X| < i^{\gamma})} \\
&\ll \sum_{i=2}^{\infty} E|X|^{\theta} I_{((i-1)^{\gamma} \leq |X| < i^{\gamma})} \\
&< \infty.
\end{aligned} \tag{3.14}$$

By (3.5),

$$
\begin{aligned}
J_{22}^{*} &= \sum_{n=1}^{\infty} n^{\beta} \sum_{i=2n+1}^{\infty} EX^2 I_{((i-1)^{\gamma} \le |X| < i^{\gamma})} \sum_{j=\left[\frac{i}{n}-1\right]}^{\infty} N(nj)(nj)^{-2\gamma} \\
&\ll \sum_{n=1}^{\infty} n^{\beta} \sum_{i=2n+1}^{\infty} n^{\alpha-\gamma(2-\theta)} \left(\frac{i}{n}\right)^{-\gamma(2-\theta)} E|X|^2 I_{((i-1)^{\gamma} \le |X| < i^{\gamma})} \\
&\ll \sum_{i=2}^{\infty} i^{-\gamma(2-\theta)} EX^2 I_{((i-1)^{\gamma} \le |X| < i^{\gamma})} \sum_{n=1}^{[i/2]} n^{-1} \\
&\ll \sum_{i=2}^{\infty} i^{-\gamma(2-\theta)} \ln i EX^2 I_{((i-1)^{\gamma} \le |X| < i^{\gamma})} \\
&\ll \sum_{i=2}^{\infty} E|X|^{\theta} \ln(1+|X|)\, I_{((i-1)^{\gamma} \le |X| < i^{\gamma})} \\
&< \infty.
\end{aligned}
\tag{3.15}
$$

By (3.13)-(3.15), (3.9) holds.

Proof of Corollary 3. Let $a_{nk} = \begin{cases} n^{-\gamma}, & k \le n, \\ 0, & k > n, \end{cases}$ $\quad 0 \le \alpha < 2\gamma, \quad \theta = \frac{1-\alpha}{\gamma}, \quad \beta = \gamma p - 2$. Then $\beta > -1$, $0 < \theta < 2 - \alpha/\gamma$, $1 + \alpha + \beta > 0$, $\theta + (1 + \alpha + \beta)/\gamma = p$, and $\sum_{k=1}^{\infty} |a_{nk}|^{\theta} = n^{\alpha}$. Thus, the conditions of Theorem 1 hold, by Theorem 1,

$$
\sum_{n=1}^{\infty} n^{\gamma p-2} P\left(|S_n| > \varepsilon n^{\gamma}\right) < \infty, \quad \forall \varepsilon > 0.
$$

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The author declares that she has no competing interests.

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