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An inequality for contact CR-warped product submanifolds of nearly cosymplectic manifolds

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Abstract

Recently, Chen (Monatshefte Math. 133:177-195, 2001) established general sharp inequalities for CR-warped products in a Kaehler manifold. Afterward, Mihai obtained (Geom. Dedic. 109:165-173, 2004) the same inequalities for contact CR-warped product submanifolds of Sasakian space forms and derived some applications. In this paper, we obtain an inequality for the length of the second fundamental form of the warped product submanifold of a nearly cosymplectic manifold in terms of the warping function. The equality case is also discussed.

MSC: 53C40; 53C42; 53B25

Keywords: warped product; contact CR-submanifold; contact CR-warped product; nearly cosymplectic manifold

1 Introduction

An almost contact metric structure (ϕ, ξ, η, g) satisfying $(\bar{\nabla}_X \phi)X = 0$ is called a nearly cosymplectic structure. If we consider S^5 as a totally geodesic hypersurface of S^6, then it is known that S^5 has a non-cosymplectic nearly cosymplectic structure. Almost contact manifolds with Killing structure tensors were defined in [1] as nearly cosymplectic manifolds, and it was shown that the normal nearly cosymplectic manifolds are cosymplectic (see also [2]). Later on, Blair and Showers [3] studied nearly cosymplectic structure (ϕ, ξ, η, g) on a manifold $\bar{M}$ with η closed from the topological viewpoint.

On the other hand, Chen [4] has introduced the notion of CR-warped product submanifolds in a Kaehler manifold. He has established a sharp relationship between the squared norm of the second fundamental form and the warping function. Later on, Mihai [5] studied contact CR-warped products and obtained the same inequality for contact CR-warped product submanifolds isometrically immersed in Sasakian space forms. Motivated by the studies of these authors, many articles dealing with the existence or non-existence of warped products in different settings have appeared; one of them is [3]. In this paper, we obtain an inequality for the length of the second fundamental form in terms of the warping function for contact CR-warped product submanifolds in a more general setting, *i.e.*, nearly cosymplectic manifold.

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2 Preliminaries

A $(2m+1)$-dimensional smooth manifold $\bar{M}$ is said to have an *almost contact structure* if on $\bar{M}$ there exist a tensor field ϕ of type $(1,1)$, a vector field ξ and a 1-form η satisfying [6]

$$\phi^2 = -I + \eta \otimes \xi, \qquad \phi\xi = 0, \qquad \eta \circ \phi = 0, \qquad \eta(\xi) = 1. \tag{2.1}$$

There always exists a Riemannian metric g on $\bar{M}$ satisfying the following compatibility condition:

$$\eta(X) = g(X,\xi), \qquad g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.2}$$

where X and Y are vector fields on $\bar{M}$ [6].

An almost contact structure (ϕ,ξ,η) is said to be *normal* if the almost complex structure J on the product manifold $\bar{M} \times \mathbb{R}$ given by

$$J\left(X, f\frac{d}{dt}\right) = \left(\phi X - f\xi, \eta(X)\frac{d}{dt}\right),$$

where f is a smooth function on $\bar{M} \times \mathbb{R}$, has no torsion, *i.e.*, J is integrable, the condition for normality in terms of ϕ, ξ and η is $[\phi,\phi] + 2d\eta \otimes \xi = 0$ on $\bar{M}$, where $[\phi,\phi]$ is the Nijenhuis tensor of ϕ. Finally, the *fundamental 2-form* Φ is defined by $\Phi(X,Y) = g(X,\phi Y)$.

An almost contact metric structure (ϕ,ξ,η,g) is said to be *cosymplectic* if it is normal and both Φ and η are closed [6]. The structure is said to be *nearly cosymplectic* if ϕ is Killing, *i.e.*, if

$$(\bar{\nabla}_X \phi)Y + (\bar{\nabla}_Y \phi)X = 0 \tag{2.3}$$

for any X, Y tangent to $\bar{M}$, where $\bar{\nabla}$ is the Riemannian connection of the metric g on $\bar{M}$. Equation (2.3) is equivalent to $(\bar{\nabla}_X\phi)X = 0$ for each vector field X tangent to $\bar{M}$. The structure is said to be *closely cosymplectic* if ϕ is Killing and η is closed. It is well known that an almost contact metric manifold is *cosymplectic* if and only if $\bar{\nabla}\phi$ vanishes identically, *i.e.*, $(\bar{\nabla}_X\phi)Y = 0$ and $\bar{\nabla}_X\xi = 0$.

Proposition 2.1 [6] *On a nearly cosymplectic manifold, the vector field ξ is Killing.*

From the above proposition, we have $g(\bar{\nabla}_X\xi, X) = 0$ for any vector field X tangent to $\bar{M}$, where $\bar{M}$ is a nearly cosymplectic manifold.

Let M be a submanifold of an almost contact metric manifold $\bar{M}$ with induced metric g, and let ∇ and $\nabla^\perp$ be the induced connections on the tangent bundle TM and the normal bundle $T^\perp M$ of M, respectively. Denote by $\mathcal{F}(M)$ the algebra of smooth functions on M and by $\Gamma(TM)$ the $\mathcal{F}(M)$-module of smooth sections of TM over M. Then the Gauss and Weingarten formulas are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{2.4}$$

$$\bar{\nabla}_X N = -A_N X + \nabla_X^\perp N, \tag{2.5}$$

for each $X, Y \in \Gamma(TM)$ and $N \in \Gamma(T^\perp M)$, where h and A_N are the second fundamental form and the shape operator (corresponding to the normal vector field N), respectively, for the immersion of M into $\bar{M}$. They are related as

$$g\big(h(X,Y),N\big) = g(A_N X, Y), \tag{2.6}$$

where g denotes the Riemannian metric on $\bar{M}$ as well as induced on M.

For any $X \in \Gamma(TM)$, we write

$$\phi X = TX + FX, \tag{2.7}$$

where TX is the tangential component and FX is the normal component of ϕX.

A submanifold M tangent to the structure vector field ξ is said to be *invariant* (resp. *anti-invariant*) if $\phi(T_x M) \subset T_x M$, $\forall x \in M$ (resp. $\phi(T_x M) \subset T_x^\perp M$, $\forall x \in M$).

A submanifold M tangent to the structure vector field ξ of an almost contact metric manifold $\bar{M}$ is called a *contact CR-submanifold* (or *semi-invariant submanifold*) if there exists a pair of orthogonal differentiable distributions $\mathcal{D}$ and $\mathcal{D}^\perp$ on M such that

(i) $TM = \mathcal{D} \oplus \mathcal{D}^\perp \oplus \langle\xi\rangle$, where $\langle\xi\rangle$ is the one-dimensional distribution spanned by ξ;
(ii) $\mathcal{D}$ is invariant under ϕ, *i.e.*, $\phi(\mathcal{D}_x) \subseteq \mathcal{D}_x$, $\forall x \in M$;
(iii) $\mathcal{D}^\perp$ is anti-invariant under ϕ, *i.e.*, $\phi(\mathcal{D}_x^\perp) \subset T_x^\perp M$, $\forall x \in M$.

A contact CR-submanifold is *invariant* if $\mathcal{D}^\perp = \{0\}$ and *anti-invariant* if $\mathcal{D} = \{0\}$, respectively. It is called a *proper contact CR-submanifold* if neither $\mathcal{D} = \{0\}$ nor $\mathcal{D}^\perp = \{0\}$. Moreover, if μ is the ϕ-invariant subspace in the normal bundle $T^\perp M$, then in the case of a contact CR-submanifold, the normal bundle $T^\perp M$ can be decomposed as

$$T^\perp M = F\mathcal{D}^\perp \oplus \mu. \tag{2.8}$$

Bishop and O'Neill [7] introduced the notion of warped product manifolds. They defined these manifolds as follows. Let (N_1, g_1) and (N_2, g_2) be two Riemannian manifolds and $f > 0$ be a differentiable function on N_1. Consider the product manifold $N_1 \times N_2$ with its projections $\pi_1 : N_1 \times N_2 \to N_1$ and $\pi_2 : N_1 \times N_2 \to N_2$. Then the warped product of N_1 and N_2 denoted by $M = N_1 \times_f N_2$ is a Riemannian manifold $N_1 \times N_2$ equipped with the Riemannian structure such that

$$g(X,Y) = g_1(\pi_{1\star}X, \pi_{1\star}Y) + (f \circ \pi_1)^2 g_2(\pi_{2\star}X, \pi_{2\star}Y)$$

for each $X, Y \in \Gamma(TM)$ and $\star$ is the symbol for the tangent map. Thus, we have

$$g = g_1 + f^2 g_2. \tag{2.9}$$

The function f is called the *warping function* of the warped product [7]. A warped product $N_1 \times_f N_2$ is said to be *trivial* if the warping function f is constant.

We recall the following general result on warped product manifolds for later use.

Lemma 2.1 *Let $M = N_1 \times_f N_2$ be a warped product manifold with the warping function f, then*

(i) $\nabla_X Y \in \Gamma(TN_1)$ *is the lift of* $\nabla_X Y$ *on* N_1,
(ii) $\nabla_X Z = \nabla_Z X = (X \ln f)Z$,
(iii) $\nabla_Z W = \nabla_Z^{N_2} W - g(Z,W)\nabla \ln f$

for each $X, Y \in \Gamma(TN_1)$ *and* $Z, W \in \Gamma(TN_2)$, *where* $\nabla \ln f$ *is the gradient of the function* $\ln f$ *and* ∇ *and* ∇^{N_2} *denote the Levi-Civita connections on* M *and* N_2, *respectively.*

3 Contact CR-warped product submanifolds

In this section, we consider the warped product submanifolds $M = N_1 \times {}_f N_2$ of a nearly cosymplectic manifold $\bar{M}$, where N_1 and N_2 are Riemannian submanifolds of $\bar{M}$. In the above product, if we assume $N_1 = N_T$ and $N_2 = N_\perp$, then the warped product of N_1 and N_2 becomes a contact CR-warped product. In this section, we discuss the contact CR-warped products and obtain an inequality for the squared norm of the second fundamental form. For the general case of warped product submanifolds of a nearly cosymplectic manifold, we have the following result.

Theorem 3.1 [8] *A warped product submanifold* $M = N_1 \times {}_f N_2$ *of a nearly cosymplectic manifold* $\bar{M}$ *is a usual Riemannian product if the structure vector field* ξ *is tangent to* N_2, *where* N_1 *and* N_2 *are the Riemannian submanifolds of* $\bar{M}$.

If we consider $\xi \in \Gamma(TN_1)$, then for any $X \in \Gamma(TN_2)$, we have

$$\bar{\nabla}_X \xi = \nabla_X \xi + h(X, \xi).$$

Taking the inner product with $X \in \Gamma(TN_2)$, then by Proposition 2.1 and Lemma 2.1(ii), we obtain that $(\xi \ln f)\|X\|^2 = 0$. This means that either $\dim N_2 = 0$, which is not possible for warped products, or

$$\xi \ln f = 0. \tag{3.1}$$

Now, we consider the warped product contact CR-submanifolds of the types $M = N_\perp \times {}_f N_T$ and $M = N_T \times {}_f N_\perp$ of a nearly cosymplectic manifold $\bar{M}$. In [8], the present author has proved that the warped product contact CR-submanifolds of the first type are usual Riemannian products of $N_\perp$ and N_T, where $N_\perp$ and N_T are anti-invariant and invariant submanifolds of $\bar{M}$, respectively. In the following, we consider the contact CR-warped product submanifolds $M = N_T \times {}_f N_\perp$ and obtain a general inequality. First, we have the following preparatory result for later use.

Lemma 3.1 [8] *Let* $M = N_T \times {}_f N_\perp$ *be a contact CR-warped product submanifold of a nearly cosymplectic manifold* $\bar{M}$. *If* $X, Y \in \Gamma(TN_T)$ *and* $Z, W \in \Gamma(TN_\perp)$, *then*
(i) $g(h(X,Y), FZ) = 0$,
(ii) $g(h(\phi X, Z), FZ) = (X \ln f)\|Z\|^2$.

If we replace X by ϕX in (ii) of Lemma 3.1, then we get

$$g\big(h(X,Z), FZ\big) = -(\phi X \ln f)\|Z\|^2. \tag{3.2}$$

For a Riemannian manifold of dimension m and a smooth function f on M, we recall

(i) ∇f, the gradient of f, is defined by

$$g(\nabla f, X) = X(f), \quad \forall X \in \Gamma(TM). \tag{3.3}$$

(ii) Δf, the Laplacian of f, is defined by

$$\Delta f = \sum_{i=1}^{m} \{(\nabla_{e_i} e_i) f - e_i e_i(f)\} = -\operatorname{div} \nabla f, \tag{3.4}$$

where ∇ is the Levi-Civita connection on M and $\{e_1, \ldots, e_m\}$ is an orthonormal frame on M.

As a consequence, we have

$$\|\nabla f\|^2 = \sum_{i=1}^{m} \big(e_i(f)\big)^2. \tag{3.5}$$

Now, we prove the main result of this section using the above results.

Theorem 3.2 *Let $M = N_T \times_f N_\perp$ be a contact CR-warped product submanifold of a nearly cosymplectic manifold $\bar{M}$. Then we have*

(i) *The length of the second fundamental form of M satisfies the inequality*

$$\|h\|^2 \geq 2q \|\nabla \ln f\|^2, \tag{3.6}$$

where q is the dimension of $N_\perp$ and $\nabla \ln f$ is the gradient of $\ln f$.

(ii) *If the equality sign of (3.6) holds identically, then N_T is a totally geodesic submanifold and $N_\perp$ is a totally umbilical submanifold of $\bar{M}$. Moreover, M is a minimal submanifold of $\bar{M}$.*

Proof Let $\bar{M}$ be a $(2m+1)$-dimensional nearly cosymplectic manifold and $M = N_T \times_f N_\perp$ be an n-dimensional contact CR-warped product submanifolds of $\bar{M}$. Let us consider the $\dim N_T = 2p+1$ and $\dim N_\perp = q$, then $n = 2p+1+q$. Let $\{e_1, \ldots, e_p, \phi e_1 = e_{p+1}, \ldots, \phi e_p = e_{2p}, e_{2p+1} = \xi\}$ and $\{e_{(2p+1)+1}, \ldots, e_n\}$ be the local orthonormal frames on N_T and $N_\perp$, respectively. Then the orthonormal frames in the normal bundle $T^\perp M$ of $F\mathcal{D}^\perp$ and μ are $\{Fe_{(2p+1)+1}, \ldots, Fe_n\}$ and $\{e_{n+q+1}, \ldots, e_{2m+1}\}$, respectively. Then the length of the second fundamental form h is defined as

$$\|h\|^2 = \sum_{r=n+1}^{2m+1} \sum_{i,j=1}^{n} g\big(h(e_i, e_j), e_r\big)^2. \tag{3.7}$$

For the assumed frames, the above equation can be written as

$$\|h\|^2 = \sum_{r=n+1}^{n+q} \sum_{i,j=1}^{n} g\big(h(e_i, e_j), e_r\big)^2 + \sum_{r=n+q+1}^{2m+1} \sum_{i,j=1}^{n} g\big(h(e_i, e_j), e_r\big)^2. \tag{3.8}$$

The first term on the right-hand side of the above equality is the $F\mathcal{D}^{\perp}$-component and the second term is the μ-component. Here, we equate the $F\mathcal{D}^{\perp}$-component, then we have

$$\|h\|^2 \geq \sum_{r=n+1}^{n+q} \sum_{i,j=1}^{n} g\big(h(e_i, e_j), e_r\big)^2. \tag{3.9}$$

The above equation can be written for the given frame of $F\mathcal{D}^{\perp}$ as

$$\|h\|^2 \geq \sum_{l=(2p+1)+1}^{n} \sum_{i,j=1}^{n} g\big(h(e_i, e_j), Fe_l\big)^2.$$

Let us decompose the above equation in terms of the components of $h(\mathcal{D}, \mathcal{D})$, $h(\mathcal{D}, \mathcal{D}^{\perp})$ and $h(\mathcal{D}^{\perp}, \mathcal{D}^{\perp})$, then we have

$$\begin{aligned} \|h\|^2 \geq{}& \sum_{l=2p+2}^{n} \sum_{i,j=1}^{2p+1} g\big(h(e_i, e_j), Fe_l\big)^2 \\ &+ 2 \sum_{l=2p+2}^{n} \sum_{i=1}^{2p+1} \sum_{j=2p+2}^{n} g\big(h(e_i, e_j), Fe_l\big)^2 \\ &+ \sum_{l=2p+2}^{n} \sum_{i,j=2p+2}^{n} g\big(h(e_i, e_j), Fe_l\big)^2. \end{aligned} \tag{3.10}$$

Using Lemma 3.1(i), the first term of (3.10) is identically zero and we shall compute the next term and leave the third term

$$\|h\|^2 \geq 2 \sum_{l=2p+2}^{n} \sum_{i=1}^{2p+1} \sum_{j=2p+2}^{n} g\big(h(e_i, e_j), Fe_l\big)^2.$$

As $j, l = 2p + 2, \ldots, n$, then we can write the above equation for one summation, and using (3.2), we obtain

$$\|h\|^2 \geq 2 \sum_{i=1}^{2p+1} \sum_{l=2p+2}^{n} (-\phi e_i \ln f)^2 g(e_l, e_l)^4. \tag{3.11}$$

Using the fact that ξ is tangent to N_T and $\xi \ln f = 0$, the above equation can be written for the given frame of the distribution $\mathcal{D}$ as

$$\|h\|^2 \geq 2 \sum_{l=2p+2}^{n} \left[\sum_{i=1}^{p} (e_i \ln f)^2 g(e_l, e_l)^4 + \sum_{i=1}^{p} (\phi e_i \ln f)^2 g(e_l, e_l)^4 \right]. \tag{3.12}$$

Then from (3.5), the above expression will be

$$\|h\|^2 \geq 2 \sum_{l=2p+2}^{n} \|\nabla \ln f\|^2 g(e_l, e_l)^4 = 2q \|\nabla \ln f\|^2,$$

which proves the inequality (3.6). Let us denote by $h^\perp$, the second fundamental form of $N_\perp$ in M, then by (2.4), we have

$$g\bigl(h^\perp(Z,W),X\bigr) = g(\nabla_Z W,X) = -(X \ln f)g(Z,W)$$

for any $X \in \Gamma(TN_T)$ and $Z, W \in \Gamma(TN_\perp)$. Thus, on using (3.3), we obtain

$$g\bigl(h^\perp(Z,W),X\bigr) = g(\nabla_Z W,X) = -g(\nabla \ln f, X)g(Z,W),$$

or equivalently,

$$h^\perp(Z,W) = -g(Z,W)\nabla \ln f. \tag{3.13}$$

Suppose the equality case holds in (3.6), then from (3.8) and (3.10), we obtain

$$h(\mathcal{D},\mathcal{D}) = 0, \qquad h\bigl(\mathcal{D}^\perp,\mathcal{D}^\perp\bigr) = 0, \qquad h\bigl(\mathcal{D},\mathcal{D}^\perp\bigr) \subset F\mathcal{D}^\perp. \tag{3.14}$$

As N_T is a totally geodesic submanifold in M (by Lemma 2.1(i)), using this fact with the first part of (3.14), we get N_T is totally geodesic in $\bar{M}$. On the other hand, the second condition of (3.14) with (3.13) implies that $N_\perp$ is totally umbilical in $\bar{M}$. Moreover, from (3.14), we get M is a minimal submanifold of $\bar{M}$. This proves the theorem completely. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
SU carried out the whole research and drafted the manuscript. KAK has given the idea of this problem and checked the calculations. All authors read and approved the final manuscript.

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