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Every n-dimensional normed space is the space $\mathbb{R}^n$ endowed with a normal norm

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Abstract

Recently, Alonso showed that every two-dimensional normed space is isometrically isomorphic to a generalized Day-James space introduced by Nilsrakoo and Saejung. In this paper, we consider the result of Alonso for n-dimensional normed spaces.
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A norm $\|\cdot\|$ on $\mathbb{R}^2$ is said to be absolute if $\|(x,y)\| = \|(|x|,|y|)\|$ for all $(x,y) \in \mathbb{R}^2$, and normalized if $\|(1,0)\| = \|(0,1)\| = 1$. The set of all absolute normalized norms on $\mathbb{R}^2$ is denoted by AN_2. Bonsall and Duncan [1] showed the following characterization of absolute normalized norms on $\mathbb{R}^2$. Namely, the set AN_2 of all absolute normalized norms on $\mathbb{R}^2$ is in a one-to-one correspondence with the set Ψ_2 of all convex functions ψ on $[0,1]$ satisfying $\max\{1-t,t\} \le \psi(t) \le 1$ for all $t \in [0,1]$ (*cf.* [2]). The correspondence is given by the equation $\psi(t) = \|(1-t,t)\|$ for all $t \in [0,1]$. Note that the norm $\|\cdot\|_\psi$ associated with the function $\psi \in \Psi_2$ is given by

$$\|(x,y)\|_\psi = \begin{cases} (|x|+|y|)\psi\left(\frac{|y|}{|x|+|y|}\right), & \text{if } (x,y) \neq (0,0), \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

The Day-James space ℓ_p-ℓ_q is defined for $1 \le p, q \le \infty$ as the space $\mathbb{R}^2$ endowed with the norm

$$\|(x,y)\|_{p,q} = \begin{cases} \|(x,y)\|_p, & \text{if } xy \ge 0, \\ \|(x,y)\|_q, & \text{if } xy \le 0. \end{cases}$$

James [3] considered the space ℓ_p-ℓ_q with $p^{-1}+q^{-1}=1$ as an example of a two-dimensional normed space where Birkhoff orthogonality is symmetric. Recall that if x,y are elements of a real normed space X, then x is said to be Birkhoff-orthogonal to y, denoted by $x \perp_B y$, if $\|x+\lambda y\| \ge \|x\|$ for all $\lambda \in \mathbb{R}$. Birkhoff orthogonality is *homogeneous*, that is, $x \perp_B y$ implies $\alpha x \perp_B \beta y$ for any real numbers α and β. However, Birkhoff orthogonality is not *symmetric* in general, that is, $x \perp_B y$ does not imply $y \perp_B x$. More details about Birkhoff orthogonality can be found in Birkhoff [4], Day [5, 6] and James [3, 7, 8].

In 2006, Nilsrakoo and Saejung [9] introduced and studied generalized Day-James spaces ℓ_φ-ℓ_ψ, where ℓ_φ-ℓ_ψ is defined for $\varphi, \psi \in \Psi_2$ as the space $\mathbb{R}^2$ endowed with the

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norm

$$\big\|(x,y)\big\|_{\varphi,\psi} = \begin{cases} \|(x,y)\|_\varphi, & \text{if } xy \geq 0, \\ \|(x,y)\|_\psi, & \text{if } xy \leq 0. \end{cases}$$

Recently, Alonso [10] showed that every two-dimensional normed space is isometrically isomorphic to a generalized Day-James space. In this paper, we consider the result of Alonso for n-dimensional spaces.

First, we give a characterization of generalized Day-James spaces.

Proposition 1 *Let $\|\cdot\|$ be a norm on $\mathbb{R}^2$. Then the space $(\mathbb{R}^2, \|\cdot\|)$ is a generalized Day-James space if and only if $\|\cdot\|_\infty \leq \|\cdot\| \leq \|\cdot\|_1$.*

Proof If $(\mathbb{R}^2, \|\cdot\|)$ is a generalized Day-James space, then one can easily have $\|\cdot\|_\infty \leq \|\cdot\| \leq \|\cdot\|_1$. So, we assume that $\|\cdot\|_\infty \leq \|\cdot\| \leq \|\cdot\|_1$. Let

$$\varphi(t) = \big\|(1-t,t)\big\| \quad \text{and} \quad \psi(t) = \big\|(1-t,-t)\big\|$$

for all $t \in [0,1]$, respectively. Then, clearly, we have $\varphi, \psi \in \Psi_2$ and $\|\cdot\| = \|\cdot\|_{\varphi,\psi}$. Hence, the space $(\mathbb{R}^2, \|\cdot\|)$ is a generalized Day-James space. □

Motivated by this fact, we consider the following

Definition 2 A norm $\|\cdot\|$ on $\mathbb{R}^n$ is said to be normal if it satisfies $\|\cdot\|_\infty \leq \|\cdot\| \leq \|\cdot\|_1$.

We recall some notions about multilinear forms. Let X be a real vector space. Then a real-valued function F on X^n is said to be an n-linear form if it is linear separately in each variable, that is,

$$F\big(x_1,\ldots,x_{i-1}, \alpha x_i + x_i', x_{i+1},\ldots,x_n\big)$$
$$= \alpha F\big(x_1,\ldots,x_{i-1},x_i,x_{i+1},\ldots,x_n\big) + F\big(x_1,\ldots,x_{i-1},x_i',x_{i+1},\ldots,x_n\big)$$

for each $i \in \{1,2,\ldots,n\}$. If $F : X^n \to \mathbb{R}$ is an n-linear form, then F is said to be alternating if

$$F(x_1,\ldots,x_i,x_{i+1},\ldots,x_n) = -F(x_1,\ldots,x_{i+1},x_i,\ldots,x_n)$$

for each $i \in \{1,2,\ldots,n\}$ or, equivalently, $F(x_1,x_2,\ldots,x_n) = 0$ if $x_i = x_j$ for some i,j with $i \neq j$. Furthermore, F is said to be bounded if

$$\|F\| := \sup\big\{\big|F(x_1,x_2,\ldots,x_n)\big| : (x_1,x_2,\ldots,x_n) \in (S_X)^n\big\} < \infty,$$

where S_X denotes the unit sphere of X. If F is bounded, then we have

$$\big|F(x_1,x_2,\ldots,x_n)\big| \leq \|F\|\|x_1\|\|x_2\|\cdots\|x_n\|$$

for all $(x_1,x_2,\ldots,x_n) \in X^n$.

For our purpose, we give another simple proof of the following result of Day [5]. For each subset A of a normed space, let $[A]$ denote the closed linear span of A. If M, N are subspaces of a real normed space X, then M is said to be Birkhoff orthogonal to N, denoted by $M \perp_B N$, if $\|x + y\| \geq \|x\|$ for all $x \in M$ and all $y \in N$. In particular, $x \perp_B M$ denotes $[\{x\}] \perp_B M$.

Lemma 3 *Let X be an n-dimensional real normed space. Then there exists a basis $\{e_1, e_2, \ldots, e_n\}$ for X such that $\|e_i\| = 1$ and $e_i \perp_B [\{e_k\}_{k \neq i}]$ for all $i = 1, 2, \ldots, n$.*

Proof Let $\{u_1, u_2, \ldots, u_n\}$ be a basis for X. Then each vector $x \in X$ is uniquely expressed in the form $x = \sum_{k=1}^{n} \alpha_k(x) u_k$. Define the function F on X^n by

$$F(x_1, x_2, \ldots, x_n) = \begin{vmatrix} \alpha_1(x_1) & \alpha_2(x_1) & \ldots & \alpha_n(x_1) \\ \alpha_1(x_2) & \alpha_2(x_2) & \ldots & \alpha_n(x_2) \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_1(x_n) & \alpha_2(x_n) & \ldots & \alpha_n(x_n) \end{vmatrix}$$

for all $(x_1, x_2, \ldots, x_n) \in X^n$. Then it is easy to check that F is an alternating bounded n-linear form. Since F is jointly continuous on the compact subset $(S_X)^n$ of X^n, there exists $(e_1, e_2, \ldots, e_n) \in (S_X)^n$ such that

$$F(e_1, e_2, \ldots, e_n) = \|F\| > 0.$$

For all $i = 1, 2, \ldots, n$ and all $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{R}^n$, we have

$$\|F\| \left\| \sum_{k=1}^{n} \alpha_k e_k \right\| \geq \left| F\left(e_1, \ldots, e_{i-1}, \sum_{k=1}^{n} \alpha_k e_k, e_{i+1}, \ldots, e_n \right) \right|$$
$$= \left| \sum_{k=1}^{n} \alpha_k F(e_1, \ldots, e_{i-1}, e_k, e_{i+1}, \ldots, e_n) \right|$$
$$= \left| \alpha_i F(e_1, e_2, \ldots, e_n) \right| = \|F\| |\alpha_i|.$$

Thus, we obtain

$$\left\| \sum_{k=1}^{n} \alpha_k e_k \right\| \geq |\alpha_i| = \|\alpha_i e_i\|,$$

for all $i = 1, 2, \ldots, n$ and all $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{R}^n$. This means that $e_i \perp_B [\{e_k\}_{k \neq i}]$ for all $i = 1, 2, \ldots, n$. □

Now, we state the main theorem.

Theorem 4 *Every n-dimensional normed space is isometrically isomorphic to the space $\mathbb{R}^n$ endowed with a normal norm.*

Proof By Lemma 3, there exists an n-tuple $(e_1, e_2, \ldots, e_n)$ of elements of S_X such that $e_i \perp_B [\{e_k\}_{k \neq i}]$ for all $i = 1, 2, \ldots, n$. Since $e_i \perp_B [\{e_k\}_{k \neq i}]$, we have

$$\left\| \sum_{k=1}^n \alpha_k e_k \right\| \geq \|\alpha_i e_i\| = |\alpha_i|,$$

for all $i = 1, 2, \ldots, n$ and all $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{R}^n$. Hence, we obtain

$$\left\| \sum_{k=1}^n \alpha_k e_k \right\| \geq \max\{|\alpha_1|, |\alpha_2|, \ldots, |\alpha_n|\}$$

for all $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{R}^n$. From this fact, we note that $\{e_1, e_2, \ldots, e_n\}$ is linearly independent, that is, a basis for X.

Define the norm $\|\cdot\|_0$ on $\mathbb{R}^n$ by the formula

$$\|(\alpha_1, \alpha_2, \ldots, \alpha_n)\|_0 = \left\| \sum_{k=1}^n \alpha_k e_k \right\|$$

for all $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{R}^n$. Then, clearly, $\|\cdot\|_0$ is normal and X is isometrically isomorphic to the space $(\mathbb{R}^n, \|\cdot\|_0)$. This completes the proof. □

Since the space $\mathbb{R}^2$ endowed with a normal norm is a generalized Day-James space by Proposition 1, we have the result of Alonso as a corollary.

Corollary 5 ([10]) *Every two-dimensional real normed space is isometrically isomorphic to a generalized Day-James space.*

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