Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Periodic solutions to a generalized Liénard neutral functional differential system with p-Laplacian

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Abstract

By means of the generalized Mawhin's continuation theorem, we present some sufficient conditions which guarantee the existence of at least one T-periodic solution for a generalized Liénard neutral functional differential system with p-Laplacian.

MSC: 34B15; 34L30

Keywords: periodic solutions; neutral equations; generalized Mawhin's continuation theorem

1 Introduction

This paper is devoted to investigating the following p-Laplacian Liénard neutral differential system:

$$\Bigl(\varphi_p\bigl(\bigl(x(t)-Bx(t-\tau)\bigr)'\bigr)\Bigr)'+f\bigl(x(t)\bigr)x'(t)+g\bigl(x\bigl(t-\gamma(t)\bigr)\bigr)=e(t), \tag{1.1}$$

where $x(t)=(x_1(t),x_2(t),\dots,x_n(t))^\top$;

$$\varphi_p:\mathbb{R}^n\to\mathbb{R}^n,\qquad \varphi_p(x)=|x|^{p-2}x=\left(\sqrt{\sum_{i=1}^n x_i^2}\right)^{p-2}x,\quad p>1;$$

$$f(x)=\bigl(f_1(x_1),f_2(x_2),\dots,f_n(x_n)\bigr),\quad f_i(x_i)\in C(\mathbb{R},\mathbb{R}), i=1,2,\dots,n;$$

$$g(x)=\bigl(g_1(x_1),g_2(x_2),\dots,g_n(x_n)\bigr)^\top,\quad g_i(x_i)\in C(\mathbb{R},\mathbb{R}), i=1,2,\dots,n;$$

$e\in C(\mathbb{R},\mathbb{R}^n)$ with $e(t+T)=e(t)$; $\gamma\in C(\mathbb{R},\mathbb{R})$ with $\gamma(t+T)=\gamma(t)$; τ is a given constant; $B=[b_{ij}]_{n\times n}$ is a real matrix with $|B|=(\sum_{i=1}^n\sum_{j=1}^n|b_{ij}|^2)^{1/2}$.

When the matrix B is a constant, Zhang [1] studied the properties of a difference operator A and obtained the following results: define the operator A on C_T

$$A:C_T\to C_T,\qquad [Ax](t)=x(t)-cx(t-\tau),\quad \forall t\in\mathbb{R}, \tag{1.2}$$

where $C_T=\{x:x\in C(\mathbb{R},\mathbb{R}),x(t+T)\equiv x(t)\}$, c is a constant. If $|c|\neq 1$, then A has a unique continuous bounded inverse A^{-1} satisfying

$$[A^{-1}f](t)=\begin{cases}\sum_{j\geq 0}c^jf(t-j\tau), & \text{if } |c|<1,\forall f\in C_T,\\ -\sum_{j\geq 1}c^{-j}f(t+j\tau), & \text{if } |c|>1,\forall f\in C_T.\end{cases}$$

On the basis of Zhang's work, Lu [2] further studied the properties of the difference operator A and gave the following inequality properties for A:
(1) $\|A^{-1}\| \leq \frac{1}{|1-|k||}$;
(2) $\int_0^T |[A^{-1}f](t)|\,dt \leq \frac{1}{|1-|k||}\int_0^T |f(t)|\,dt, \forall f \in C_T$;
(3) $\int_0^T |[A^{-1}f](t)|^2\,dt \leq \frac{1}{|1-|k||}\int_0^T |f(t)|^2\,dt, \forall f \in C_T$.
After that, by using the above results, many researchers studied the existence of periodic solutions for some kinds of differential equations; see [3–7]. In a recent paper [8], when the constant c of (1.2) is a variable $c(t)$, we generalized the results of [1] and obtained the following results. If $|c(t)| \neq 1$, then the operator A has continuous inverse A^{-1} on C_T, satisfying

(1)

$$[A^{-1}f](t) = \begin{cases} f(t) + \sum_{j=1}^{\infty}\prod_{i=1}^{j} c(t-(i-1)\tau)f(t-j\tau), & c_0 < 1, \forall f \in C_T, \\ -\frac{f(t+\tau)}{c(t+\tau)} - \sum_{j=1}^{\infty}\prod_{i=1}^{j+1}\frac{1}{c(t+i\tau)}f(t+j\tau+\tau), & \sigma > 1, \forall f \in C_T. \end{cases}$$

(2)

$$\int_0^T \big|[A^{-1}f](t)\big|\,dt \leq \begin{cases} \frac{1}{1-c_0}\int_0^T |f(t)|\,dt, & c_0 < 1, \forall f \in C_T, \\ \frac{1}{\sigma-1}\int_0^T |f(t)|\,dt, & \sigma > 1, \forall f \in C_T. \end{cases}$$

Using the above results, we have obtained some existence results of periodic solutions for first-order, second-order and p-Laplacian neutral equations with a variable parameter; see [9–11].

However, when B of (1.1) is a matrix, there are few existence results of periodic solutions for neutral differential systems. In [12], when B is a symmetric matrix, the authors studied a second-order p-Laplacian neutral functional differential system and obtained the existence of periodic solutions. In [13], when B is a general matrix, the authors studied a second-order neutral differential system. But for p-Laplacian functional differential system, to the best our knowledge, there are no results on the existence of periodic solutions. Hence, in this paper, we will study system (1.1) and obtain the existence of periodic solutions by using the generalization of Mawhin's continuation theorem.

2 Main lemmas

In this section, we give some notations and lemmas which will be used in this paper. Let

$$C_T = \{x | x \in C(\mathbb{R}, \mathbb{R}^n), x(t+T) \equiv x(t)\},$$
$$C_T^1 = \{x | x \in C^1(\mathbb{R}, \mathbb{R}^n), x(t+T) = x(t)\},$$

$X = C_T^1$ with the norm $\|x\| = \max\{|x|_0, |x'|_0\}$, $Z = C_T$ with the norm

$$|x|_0 = \max_{0\leq t\leq T}|x(t)|, \quad |x(t)| = \left(\sum_{i=1}^n x_i^2\right)^{1/2},$$

U is a complex such that

$$UBU^{-1} = E_\lambda = \operatorname{diag}(J_1, J_2, \ldots, J_n) \tag{2.1}$$

is a Jordan's normal matrix, where

$$J_i = \begin{pmatrix} \lambda_i & 1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_i & 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & \lambda_i & 1 \\ 0 & 0 & 0 & 0 & \cdots & \lambda_i \end{pmatrix}_{n_i \times n_i}$$

with $\sum_{i=1}^{l} n_i = n$, $\{\lambda_i : i = 1, 2, \ldots, l\}$ is the set of eigenvalues of matrix B. Let

$$A_1 : C_T \to C_T, \qquad [A_1 x](t) = x(t) - Bx(t - \tau). \tag{2.2}$$

Furthermore, we suppose that $\gamma(t) \in C^1(\mathbb{R}, \mathbb{R})$ with $\gamma'(t) < 1$, $\forall t \in \mathbb{R}$. It is obvious that the function $t - \gamma(t)$ has a unique inverse denoted by $\mu(t)$.

Lemma 2.1 ([13]) *Suppose that the matrix U and the operator A_1 are defined by* (2.1) *and* (2.2), *respectively, and for all $i = 1, 2, \ldots, l$, $|\lambda_i| \neq 1$. Then A_1 has its inverse $A_1^{-1} : C_T \to C_T$ with the following properties*:

(1) $\|A_1^{-1}\| \leq |U^{-1}||U|\sigma_0$, $\sigma_0 = \sum_{i=1}^{l} \sum_{j=1}^{n_i} \sum_{k=1}^{j} \frac{1}{|1-\lambda_i|^k}$.

(2) *For all $f \in C_T$, $\int_0^T |[A_1^{-1} f](s)|^p \, ds \leq |U^{-1}|^p |U|^p \sigma_1 \int_0^T |f(s)|^p \, ds$, $p \in [1, +\infty)$, where*

$$\sigma_1 = \begin{cases} \sum_{i=1}^{l} \sum_{j=1}^{n_i} (\sum_{k=1}^{j} \frac{1}{|1-\lambda_i|^k})^2, & p = 2, \\ n^{\frac{2-p}{2}} [\sum_{i=1}^{l} \sum_{j=1}^{n_i} (\sum_{k=1}^{j} \frac{1}{|1-\lambda_i|^k})^q]^{\frac{p}{q}}, & p \in [1, 2), \\ [\sum_{i=1}^{l} \sum_{j=1}^{n_i} (\sum_{k=1}^{j} \frac{1}{|1-\lambda_i|^k})^q]^{\frac{p}{q}}, & p \in [2, +\infty) \end{cases}$$

and $q > 0$ is a constant with $1/p + 1/q = 1$.

(3) $A_1^{-1} f \in C_T^1$, $[A_1^{-1} f]'(t) = [A_1^{-1} f'](t)$, *for all $f \in C_T^1$, $t \in \mathbb{R}$.*

Definition 2.1 ([14]) Let X and Z be two Banach spaces with norms $\|\cdot\|_X$, $\|\cdot\|_Z$, respectively. A continuous operator

$$M : X \cap \operatorname{dom} M \to Z$$

is said to be quasi-linear if

(i) $\operatorname{Im} M := M(X \cap \operatorname{dom} M)$ is a closed subset of Z;

(ii) $\operatorname{Ker} M := \{x \in X \cap \operatorname{dom} M : Mx = 0\}$ is linearly homeomorphic to $\mathbb{R}^n$, $n < \infty$.

Definition 2.2 ([14]) Let $\Omega \subset X$ be an open and bounded set with the origin $\theta \in \Omega$, $N_\lambda : \bar{\Omega} \to Z$, $\lambda \in [0, 1]$ is said to be M-compact in $\bar{\Omega}$ if there exists a subset Z_1 of Z satisfying $\dim Z_1 = \dim \operatorname{Ker} M$ and an operator $R : \bar{\Omega} \times [0, 1] \to X_2$ being continuous and compact such that for $\lambda \in [0, 1]$,

(a) $(I - Q)N_\lambda(\bar{\Omega}) \subset \operatorname{Im} M \subset (I - Q)Z$,

(b) $QN_\lambda x = 0$, $\lambda \in (0, 1) \Leftrightarrow QNx = 0$, $\forall x \in \Omega$,

(c) $R(\cdot, 0) \equiv 0$ and $R(\cdot, \lambda)|_{\sum_\lambda} = (I - P)|_{\sum_\lambda}$,

(d) $M[P + R(\cdot, \lambda)] = (I - Q)N_\lambda$, $\lambda \in [0, 1]$, where X_2 is the complement space of $\operatorname{Ker} M$ in X, *i.e.*, $X = \operatorname{Ker} M \oplus X_2$; P, Q are two projectors satisfying $\operatorname{Im} P = \operatorname{Ker} M$, $\operatorname{Im} Q = Z_1$, $N = N_1$, $\sum_\lambda = \{x \in \bar{\Omega} : Mx = N_\lambda x\}$.

Lemma 2.2 ([14]) *Let X and Z be two Banach spaces with norms $\|\cdot\|_X$, $\|\cdot\|_Z$, respectively and $\Omega \subset X$ be an open and bounded nonempty set. Suppose*

$$M : X \cap \operatorname{dom} M \to Z$$

is quasi-linear and $N_\lambda : \bar{\Omega} \to Z$, $\lambda \in [0,1]$ is M-compact in $\bar{\Omega}$. In addition, if the following conditions hold:

(A_1) $Mx \neq N_\lambda x$, $\forall (x,\lambda) \in \partial\Omega \times (0,1)$;
(A_2) $QNx \neq 0$, $\forall x \in \operatorname{Ker} M \cap \partial\Omega$;
(A_3) $\deg\{JQN, \Omega \cap \operatorname{Ker} M, 0\} \neq 0$, $J : \operatorname{Im} Q \to \operatorname{Ker} M$ *is a homeomorphism.*

Then the abstract equation $Mx = Nx$ has at least one solution in $\operatorname{dom} M \cap \bar{\Omega}$.

Lemma 2.3 ([15]) *Let $s, \sigma \in C(\mathbb{R},\mathbb{R})$ with $s(t+T) \equiv s(t)$ and $\sigma(t+T) \equiv \sigma(t)$. Suppose that the function $t - \sigma(t)$ has a unique inverse $\mu(t)$, $\forall t \in \mathbb{R}$. Then $s(\mu(t+T)) \equiv s(\mu(t))$.*

For fixed $l \in Z$ and $a \in \mathbb{R}^n$, define

$$G_l(a) = \frac{1}{T}\int_0^T \varphi_p^{-1}\big(a + l(t)\big)\,dt.$$

Lemma 2.4 ([16]) *The function G_l has the following properties*:

(1) *For any fixed $l \in Z$, there must be a unique $\tilde{a} = \tilde{a}(l)$ such that the equation*

$$G_l(a) = 0.$$

(2) *The function $\tilde{a} : Z \to \mathbb{R}^n$ defined as above is continuous and sends bounded sets into bounded sets.*

Lemma 2.5 ([17]) *Let $p \in (1,+\infty)$ be a constant, $s \in C(\mathbb{R},\mathbb{R})$ such that $s(t) \equiv s(t+T)$, $u \in X$. Then*

$$\int_0^T \big|u(t) - u\big(t - s(t)\big)\big|^p\,dt \le 2\Big(\max_{t\in[0,T]}\big|s(t)\big|\Big)^p \int_0^T \big|u'(t)\big|^p\,dt.$$

3 Main results

For convenience of applying Lemma 2.2, the operators A, M, N_λ are defined by

$$A : Z \to Z, \qquad (Ax)(t) = x(t) - Bx(t-\tau), \quad t \in \mathbb{R}, \tag{3.1}$$

$$M : \operatorname{dom} M \cap X \to Z, \qquad (Mx)(t) = \big(\varphi_p\big[(Ax)'\big]\big)'(t), \quad t \in \mathbb{R}, \tag{3.2}$$

$$N_\lambda : Z \to Z, \qquad (N_\lambda x)(t) = -\lambda f\big(x(t)\big)x'(t) - \lambda g\big(x\big(t - \gamma(t)\big)\big) + \lambda e(t), \quad t \in \mathbb{R}, \lambda \in [0,1], \tag{3.3}$$

where $\operatorname{dom} M = \{x \in X : \varphi_p[(Ax)'] \in C_T^1\}$. For convenience of the proof, let

$$F(t,x) = -f\big(x(t)\big)x'(t) - g\big(x\big(t-\gamma(t)\big)\big) + e(t), \tag{3.4}$$

then $(N_\lambda x)(t) = \lambda F$. By (3.1)-(3.3), Eq. (1.1) is equivalent to the operator equation $Nx = Mx$, where $N_1 = N$. Then we have

$$\operatorname{Ker} M = \big\{x \in \operatorname{dom} M \cap X : x(t) = a, a \in \mathbb{R}^n, t \in \mathbb{R}\big\},$$
$$\operatorname{Im} M = \left\{z \in Z : \int_0^T z(s)\,ds = \theta\right\}.$$

Since $\operatorname{Ker} M \cong \mathbb{R}^n$, $\operatorname{Im} M$ is a closed set in Z, then we have the following.

Lemma 3.1 *Let M be as defined by* (3.2), *then M is a quasi-linear operator.*

Let

$$P : X \to \operatorname{Ker} M, \qquad (Px)(t) = x(0), \quad t \in \mathbb{R},$$
$$Q : Z \to Z/\operatorname{Im} M, \qquad (Qz)(t) = \frac{1}{T}\int_0^T z(s)\,ds, \quad t \in \mathbb{R}.$$

Lemma 3.2 *If f, g, e, γ satisfy the above conditions, then N_λ is M-compact.*

Proof Let $Z_1 = \operatorname{Im} Q$. For any bounded set $\bar{\Omega} \subset X \neq \emptyset$, define $R : \bar{\Omega} \times [0,1] \to \operatorname{Ker} P$,

$$R(x,\lambda)(t) = A^{-1}\left\{\int_0^t \varphi_q\left[a_x + \int_0^s \lambda\big(F\big(r,x(r)\big) - (QF)(r)\big)\,dr\right]ds\right\}, \quad t \in [0,T],$$

where F is defined by (3.4) and a_x is a constant vector in $\mathbb{R}^n$ which depends on x. By Lemma 2.4, we know that a_x exists uniquely. Hence, $R(x,\lambda)(t)$ is well defined.

We first show that $R(\cdot,\lambda)$ is completely continuous on $\bar{\Omega} \times [0,1]$. Let

$$G_\lambda(t) = \int_0^t \varphi_q\left[a_x + \int_0^s \lambda\big(F\big(r,x(r)\big) - (QF)(r)\big)\,dr\right]ds, \quad t \in [0,T],$$

we have

$$R(x,\lambda)(t) = \big[A^{-1}G_\lambda\big](t).$$

From the properties of f, g, e, γ, obviously, $\forall x \in \bar{\Omega}$, $G_\lambda(t) \in C_T$. Then by Lemma 2.1 $R(x,\lambda)$ is uniformly bounded on $\bar{\Omega} \times [0,1]$. Now, we show $R(x,\lambda)$ is equicontinuous. $\forall t_1, t_2 \in [0,T]$, $\varepsilon > 0$ is sufficiently small, then there exists $\delta > 0$, for $|t_1 - t_2| < \delta$, by $G_\lambda, A^{-1}G_\lambda \in C_T$ we have

$$\big|\big[A^{-1}G_\lambda\big](t_1) - \big[A^{-1}G_\lambda\big](t_2)\big| < \varepsilon.$$

Hence, $R(x,\lambda)$ is equicontinuous on $\bar{\Omega} \times [0,1]$. By using the Arzelà-Ascoli theorem, we have $R(x,\lambda)$ is completely continuous on $\bar{\Omega} \times [0,1]$.

Secondly, we show that N_λ is M-compact in four steps, *i.e.*, the conditions of Definition 2.2 are all satisfied.

Step 1. By $Q^2 = Q$, we have $Q(I-Q)N_\lambda(\bar{\Omega}) = \theta$, so $(I-Q)N_\lambda(\bar{\Omega}) \subset \operatorname{Ker} Q = \operatorname{Im} M$, here θ is an n-dimension zero vector. On the other hand, $\forall z \in \operatorname{Im} M$. Clearly, $Qz = \theta$, so $z = z - Qz = (I-Q)z$, then $z \in (I-Q)Z$. So, we have

$$(I-Q)N_\lambda(\bar{\Omega}) \subset \operatorname{Im} M \subset (I-Q)Z.$$

Step 2. We show that $QN_\lambda x = \theta$, $\lambda \in (0,1) \Leftrightarrow QNx = \theta$, $\forall x \in \Omega$. Because $QN_\lambda x = \frac{1}{T}\int \lambda F\,dr = \theta$, we get $\frac{1}{T}\int F\,dr = \theta$, *i.e.*, $QNx = \theta$. The inverse is true.

Step 3. When $\lambda = 0$, from the above proof, we have $a_x = \theta$. So, we get $R(\cdot,0) = \theta$. $\forall x \in \sum_\lambda = \{x \in \bar{\Omega} : Mx = N_\lambda x\}$, we have $(\varphi_p[(Ax)'])' = \lambda F$ and $QF = \theta$. In this case, when $a_x = \varphi_p[(Ax)'(0)]$, we have

$$\begin{aligned}
G_\lambda(T) &= \int_0^T \varphi_q\left[a_x + \int_0^s \lambda\bigl(F\bigl(r,x(r)\bigr) - (QF)(r)\bigr)\,dr\right] ds \\
&= \int_0^T \varphi_q\left[\varphi_p\bigl[(Ax)'(0)\bigr] + \int_0^s \lambda F\bigl(r,x(r)\bigr)\,dr\right] ds \\
&= \int_0^T \varphi_q\left[\varphi_p\bigl[(Ax)'(0)\bigr] + \int_0^s \bigl(\varphi_p\bigl[(Ax)'(r)\bigr]\bigr)'\,dr\right] ds \\
&= \int_0^T (Ax)'(s)\,ds \\
&= (Ax)(T) - (Ax)(0) = \theta.
\end{aligned}$$

Hence,

$$\begin{aligned}
R(x,\lambda)(t) &= A^{-1}\left\{\int_0^t \varphi_q\left[\varphi_p\bigl[(Ax)'(0)\bigr] + \int_0^s \lambda\bigl(F\bigl(r,x(r)\bigr) - (QF)(r)\bigr)\,dr\right] ds\right\} \\
&= A^{-1}\left\{\int_0^t \varphi_q\left[\varphi_p\bigl[(Ax)'(0)\bigr] + \int_0^s \lambda F\bigl(r,x(r)\bigr)\,dr\right] ds\right\} \\
&= A^{-1}\left\{\int_0^t \varphi_q\left[\varphi_p\bigl[(Ax)'(0)\bigr] + \int_0^s \bigl(\varphi_p\bigl[(Ax)'(r)\bigr]\bigr)'\,dr\right] ds\right\} \\
&= A^{-1}\left\{\int_0^t (Ax)'(s)\,ds\right\} \\
&= A^{-1}\bigl[(Ax)(t) - (Ax)(0)\bigr] \\
&= \bigl[(I-P)x\bigr](t).
\end{aligned}$$

Step 4. $\forall x \in \bar{\Omega}$, we have

$$\begin{aligned}
&M\bigl[Px + R(x,\lambda)\bigr](t) \\
&\quad = \left(\varphi_p\left(\left[(Ax)(0) + AA^{-1}\left\{\int_0^t \varphi_q\left[a_x + \int_0^s \lambda\bigl(F\bigl(r,x(r)\bigr) - (QF)(r)\bigr)\,dr\right] ds\right\}\right]'\right)\right)' \\
&\quad = \left(\varphi_p\left(\left\{\int_0^t \varphi_q\left[a_x + \int_0^s \lambda\bigl(F\bigl(r,x(r)\bigr) - (QF)(r)\bigr)\,dr\right] ds\right\}'\right)\right)'
\end{aligned}$$

$$= \left(\varphi_p\left(\varphi_q\left[a_x + \int_0^t \lambda\big(F\big(r,x(r)\big) - (QF)(r)\big)\,dr\right]\right)\right)'$$
$$= \left(a_x + \int_0^t \lambda\big(F\big(r,x(r)\big) - (QF)(r)\big)\,dr\right)'$$
$$= \big[(I-Q)N_\lambda x\big](t).$$

Hence, N_λ is M-compact in $\bar{\Omega}$. □

Theorem 3.3 *Suppose that* $\int_0^T e(s)\,ds = \theta$, $\lambda_1, \lambda_2, \ldots, \lambda_l$ *are eigenvalues of the matrix* B *with* $|\lambda_i| \neq 1$, $i = 1, 2, \ldots l$, *and there exist positive constants* $D > 0$, $l > 0$ *and* $\sigma > 0$ *such that*

(H_1) $x_i g_i(x_i) > 0$, $\forall x_i \in \mathbb{R}$, $|x_i| > D$, *for each* $i = 1, 2, \ldots, n$,
(H_2) $|g_i(u_1) - g_i(u_2)| \leq l|u_1 - u_2|$, $u_1, u_2 \in \mathbb{R}$, *for each* $i = 1, 2, \ldots, n$,
(H_3) $|f_i(x_i)| \geq \sigma$, $x_i \in \mathbb{R}$, *for each* $i = 1, 2, \ldots, n$.

Then Eq. (1.1) *has at least one* T*-periodic solution if one of the following two conditions is satisfied*:

$$\sigma > \sqrt{2}l \max_{t\in[0,T]} \big|\gamma(t)\big| \quad \textit{for } 1 < q < 2 \quad \textit{or}$$
$$\sigma > \sqrt{2}l \max_{t\in[0,T]} \big|\gamma(t)\big|, \qquad \big|U^{-1}\big||U|\sigma_0\sqrt{n}Tf_{R_2} < 1 \quad \textit{for } q = 2,$$

where $f_{R_2} = \max_{|x|\leq R_2} |f(x)|$, R_2 *is defined by* (3.14).

Proof We complete the proof in three steps.

Step 1. Let $\Omega_1 = \{x \in \operatorname{dom} M : Mx = N_\lambda x, \lambda \in (0,1)\}$. We show that Ω_1 is a bounded set. If $x \in \Omega_1$, then $Mx = N_\lambda x$, *i.e.*,

$$\big(\varphi_p\big[(Ax)'\big]\big)' = -\lambda f\big(x(t)\big)x'(t) - \lambda g\big(x\big(t - \gamma(t)\big)\big) + \lambda e(t). \tag{3.5}$$

Integrating both sides of (3.5) over $[0, T]$, we have

$$\int_0^T g\big(x\big(t - \gamma(t)\big)\big)\,dt = \theta,$$

which together with assumption (H_1) leads to the fact that there exists a point $\xi_i \in \mathbb{R}$ such that

$$\big|x_i\big(\xi_i - \gamma(\xi_i)\big)\big| \leq D, \quad \text{for each } i = 1, 2, \ldots, n.$$

Let $\xi_i - \gamma(\xi_i) = kT + \eta_i$, $k \in \mathbb{Z}$, $\eta_i \in [0, T]$. Then

$$\big|x_i(\eta_i)\big| \leq D, \quad \text{for each } i = 1, 2, \ldots, n.$$

Thus,

$$|x_i| \leq D + \int_0^T \big|x_i'(s)\big|\,ds, \quad \text{for each } i = 1, 2, \ldots, n. \tag{3.6}$$

By (3.6), we have

$$\begin{aligned}|x| &= \left(x_1^2 + x_2^2 + \cdots + x_n^2\right)^{\frac{1}{2}} \\ &\leq \sqrt{n}\left(D + \int_0^T \left|x'(s)\right| ds\right)\end{aligned} \tag{3.7}$$

and

$$|x|_0 \leq \sqrt{n}\left(D + \int_0^T \left|x'(s)\right| ds\right). \tag{3.8}$$

On the other hand, multiplying the two sides of Eq. (3.5) by $[x'(t)]^\top$ from the left side and integrating them over $[0, T]$, we have

$$\begin{aligned}\int_0^T \left[x'(t)\right]^\top \left(\varphi_p\left[(Ax)'\right]\right)' dt = &-\lambda \int_0^T \left[x'(t)\right]^\top f\left(x(t)\right)x'(t)\, dt \\ &- \lambda \int_0^T \left[x'(t)\right]^\top g\left(x\left(t - \gamma(t)\right)\right) dt \\ &+ \lambda \int_0^T \left[x'(t)\right]^\top e(t)\, dt.\end{aligned} \tag{3.9}$$

Let $\omega(t) = \varphi_p[(Ax)'(t)]$, then

$$\int_0^T \left[x'(t)\right]^\top \left(\varphi_p\left[(Ax)'\right]\right)' dt = \int_0^T \left\{A^{-1}\left(\varphi_q\left(\omega(t)\right)\right)\right\}^\top d\omega(t) = 0.$$

By (3.9), we have

$$\begin{aligned}&\left|\sum_{i=1}^n \int_0^T f_i\left(x_i(t)\right)\left[x_i'(t)\right]^2 dt\right| \\ &\quad\leq \left|\int_0^T \left[x'(t)\right]^\top g\left(x\left(t - \gamma(t)\right)\right) dt\right| + \left|\int_0^T \left[x'(t)\right]^\top e(t)\, dt\right|.\end{aligned} \tag{3.10}$$

By assumption (H_3), we have

$$\begin{aligned}\sigma \sum_{i=1}^n \int_0^T \left|x_i'(t)\right|^2 dt &\leq \sum_{i=1}^n \int_0^T \left|f_i\left(x_i(t)\right)\right|\left[x_i'(t)\right]^2 dt \\ &= \left|\sum_{i=1}^n \int_0^T f_i\left(x_i(t)\right)\left[x_i'(t)\right]^2 dt\right|.\end{aligned} \tag{3.11}$$

From (3.10) and (3.11), we have

$$\sigma \sum_{i=1}^n \int_0^T \left|x_i'(t)\right|^2 dt \leq \int_0^T \left|\left[x'(t)\right]^\top g\left(x\left(t - \gamma(t)\right)\right)\right| dt + \int_0^T \left|\left[x'(t)\right]^\top e(t)\right| dt. \tag{3.12}$$

From $\int_0^T [x'(t)]^\top g(x(t))\,dt = 0$, assumption ($H_2$), Lemma 2.5 and (3.12), we have

$$\begin{aligned}
&\sigma \sum_{i=1}^n \int_0^T \left|x_i'(t)\right|^2 dt \\
&\quad\le \int_0^T \left|\left[x'(t)\right]^\top \left[g\left(x(t)\right) - g\left(x\left(t-\gamma(t)\right)\right)\right]\right| dt + \int_0^T \left|\left[x'(t)\right]^\top e(t)\right| dt \\
&\quad\le \sum_{i=1}^n l \int_0^T \left|x_i'(t)\right|\left|x_i(t) - x_i\left(t-\gamma(t)\right)\right| dt + \left(\int_0^T \left|x'(t)\right|^2 dt\right)^{1/2}\left(\int_0^T \left|e(t)\right|^2 dt\right)^{1/2} \\
&\quad\le \sum_{i=1}^n l \left(\int_0^T \left|x_i'(t)\right|^2 dt\right)^{1/2}\left(\int_0^T \left|x_i(t) - x_i\left(t-\gamma(t)\right)\right|^2 dt\right)^{1/2} \\
&\qquad + \left(\int_0^T \left|x'(t)\right|^2 dt\right)^{1/2}\left(\int_0^T \left|e(t)\right|^2 dt\right)^{1/2} \\
&\quad\le \sum_{i=1}^n \sqrt{2}l \max_{t\in[0,T]}\left|\gamma(t)\right| \int_0^T \left|x_i'(t)\right|^2 dt + \left(\int_0^T \left|x'(t)\right|^2 dt\right)^{1/2}\left(\int_0^T \left|e(t)\right|^2 dt\right)^{1/2}. \qquad (3.13)
\end{aligned}$$

Since $\sigma > \sqrt{2}l \max_{t\in[0,T]} |\gamma(t)|$, by (3.13), there exists a positive constant R_1 such that

$$\int_0^T \left|x'(s)\right| ds \le R_1.$$

Then by (3.8),

$$|x|_0 \le \sqrt{n}(D + R_1) := R_2. \qquad (3.14)$$

By (3.5), we have

$$\left|\left(\varphi_p\left[(Ax)'\right]\right)'\right| \le f_{R_2}\left|x'(t)\right| + g_{R_2} + |e|_0,$$

where $f_{R_2} = \max_{|x|\le R_2} |f(x)|$, $g_{R_2} = \max_{|x|\le R_2} |g(x)|$. Take $\varphi_p[(Ax)'(t)] = y(t)$, then

$$\left|y'\right|_0 \le f_{R_2}\left|x'(t)\right| + g_{R_2} + |e|_0 \qquad (3.15)$$

and $(Ax)'(t) = \varphi_q(y(t))$. Because there exists a $t_i \in [0, T]$ such that $y(t_i) = 0$, $i = 1, 2, \ldots, n$, so by (3.15), we get

$$\left|y(t)\right| \le \sqrt{n}T\left|y'\right|_0 \le \sqrt{n}Tf_{R_2}\left|x'(t)\right| + \sqrt{n}Tg_{R_2} + \sqrt{n}T|e|_0$$

and

$$\left|(Ax)'(t)\right| \le \left(\sqrt{n}Tf_{R_2}\left|x'(t)\right| + \sqrt{n}Tg_{R_2} + \sqrt{n}T|e|_0\right)^{q-1}. \qquad (3.16)$$

By (3.16) and Lemma 2.1, we have

$$\begin{aligned}
\left|x'(t)\right| &= \left|\left[A^{-1}Ax'\right](t)\right| \le \left|U^{-1}\right||U|\sigma_0\left|(Ax)'(t)\right| \\
&\le \left|U^{-1}\right||U|\sigma_0\left(\sqrt{n}Tf_{R_2}\left|x'(t)\right| + \sqrt{n}Tg_{R_2} + \sqrt{n}T|e|_0\right)^{q-1}. \qquad (3.17)
\end{aligned}$$

Now, we consider $(\sqrt{n}Tf_{R_2}|x'(t)| + \sqrt{n}Tg_{R_2} + \sqrt{n}T|e|_0)^{q-1}$. In the formal case, we get

$$\begin{aligned}&\left(\sqrt{n}Tf_{R_2}\left|x'(t)\right| + \sqrt{n}Tg_{R_2} + \sqrt{n}T|e|_0\right)^{q-1}\\&\quad = \left(\sqrt{n}Tf_{R_2}\left|x'(t)\right|\right)^{q-1}\left(1 + \frac{\sqrt{n}Tg_{R_2} + \sqrt{n}T|e|_0}{\sqrt{n}Tf_{R_2}|x'(t)|}\right)^{q-1}.\end{aligned} \tag{3.18}$$

By classical elementary inequalities, we see that there is a constant $h(p) > 0$, which is dependent on p only, such that

$$(1+x)^p < 1 + (1+p)x, \quad \forall x \in \big(0, h(p)\big]. \tag{3.19}$$

Case 2.1. If $\frac{\sqrt{n}Tg_{R_2}+\sqrt{n}T|e|_0}{\sqrt{n}Tf_{R_2}|x'(t)|} > h$, then

$$\left|x'(t)\right| < \frac{\sqrt{n}Tg_{R_2} + \sqrt{n}T|e|_0}{\sqrt{n}Tf_{R_2}h} := M_1. \tag{3.20}$$

Case 2.2. If $\frac{\sqrt{n}Tg_{R_2}+\sqrt{n}T|e|_0}{\sqrt{n}Tf_{R_2}|x'(t)|} \le h$, by (3.18) and (3.19), we have

$$\begin{aligned}&\left(\sqrt{n}Tf_{R_2}\left|x'(t)\right| + \sqrt{n}Tg_{R_2} + \sqrt{n}T|e|_0\right)^{q-1}\\&\quad = \left(\sqrt{n}Tf_{R_2}\left|x'(t)\right|\right)^{q-1}\left(1 + \frac{\sqrt{n}Tg_{R_2} + T|e|_0}{\sqrt{n}Tf_{R_2}\left|x'(t)\right|}\right)^{q-1}\\&\quad \le \left(\sqrt{n}Tf_{R_2}\left|x'(t)\right|\right)^{q-1}\left(1 + \frac{q(\sqrt{n}Tg_{R_2} + \sqrt{n}T|e|_0)}{\sqrt{n}Tf_{R_2}\left|x'(t)\right|}\right)\\&\quad = (\sqrt{n}Tf_{R_2})^{q-1}\left|x'(t)\right|^{q-1} + q\left(\sqrt{n}Tg_{R_2} + \sqrt{n}T|e|_0\right)(\sqrt{n}Tf_{R_2})^{q-2}\left|x'(t)\right|^{q-2}.\end{aligned} \tag{3.21}$$

From (3.17) and (3.21), we have

$$\begin{aligned}\left|x'(t)\right| &\le \left|U^{-1}\right||U|\sigma_0(\sqrt{n}Tf_{R_2})^{q-1}\left|x'(t)\right|^{q-1}\\&\quad + \left|U^{-1}\right||U|\sigma_0 q\left(\sqrt{n}Tg_{R_2} + T|e|_0\right)(\sqrt{n}Tf_{R_2})^{q-2}\left|x'(t)\right|^{q-2}.\end{aligned} \tag{3.22}$$

When $q = 2$, from $|U^{-1}||U|\sigma_0\sqrt{n}Tf_{R_2} < 1$, we know that there exists a constant $M_2 > 0$ such that

$$\left|x'(t)\right| \le M_2. \tag{3.23}$$

When $1 < q < 2$, there must be a constant $M_3 > 0$ such that

$$\left|x'(t)\right| \le M_3. \tag{3.24}$$

Hence, from (3.14), (3.20), (3.23) and (3.24), we have

$$\|x\| < \max\{R_2, M_1, M_2, M_3\} + 1 := L.$$

Step 2. Let $\Omega_2 = \{x \in \operatorname{Ker} M : QNx = \theta\}$, we shall prove that Ω_2 is a bounded set. $\forall x \in \Omega_2$, then $x = a$, $a \in \mathbb{R}^n$, we have $g_i(a_i) = 0$ for each $i = 1, 2, \ldots, n$. By assumption (H_1) we have $|a_i| \le D$ and $|a| \le \sqrt{n}D$. So, Ω_2 is a bounded set.

Step 3. Let $\Omega = \{x \in X : \|x\| < L\}$, then $\Omega_1 \cup \Omega_2 \subset \Omega$, $\forall (x, \lambda) \in \partial\Omega \times (0, 1)$. From the above proof, $Mx \neq N_\lambda x$ is satisfied. Obviously, condition (A_2) of Lemma 2.2 is also satisfied. Now, we prove that condition (A_3) of Lemma 2.2 is satisfied. Take the homotopy

$$H(x, \mu) = \mu x - (1 - \mu) JQNx, \quad x \in \bar{\Omega} \cap \operatorname{Ker} M, \mu \in [0, 1],$$

where $J : \operatorname{Im} Q \to \operatorname{Ker} M$ is a homeomorphism with $Ja = a$, $a \in \mathbb{R}^n$. $\forall x \in \partial\Omega \cap \operatorname{Ker} M$, we have $x = a_1 \in \mathbb{R}^n$, $|a_1| = L > D$, then

$$\begin{aligned} H(x, \mu) &= a_1\mu - (1 - \mu)\frac{1}{T}\int_0^T \bigl(-g(a_1) + e(t)\bigr)\,dt \\ &= a_1\mu + (1 - \mu)g(a_1), \end{aligned}$$

then we have

$$a_1^\top H(x, \mu) = a_1^\top a_1\mu + (1 - \mu)a_1^\top g(a_1).$$

By using assumption (H_1), we have $H(x, \mu) \neq 0$. And then, by the degree theory,

$$\begin{aligned} \deg\{JQN, \Omega \cap \operatorname{Ker} M, 0\} &= \deg\bigl\{H(\cdot, 0), \Omega \cap \operatorname{Ker} M, 0\bigr\} \\ &= \deg\bigl\{H(\cdot, 1), \Omega \cap \operatorname{Ker} M, 0\bigr\} \\ &= \deg\{I, \Omega \cap \operatorname{Ker} M, 0\} \neq 0. \end{aligned}$$

Applying Lemma 2.2, we complete the proof. □

Remark Assumption (H_1) guarantees that condition (A_2) of Lemma 2.2 is satisfied. Furthermore, using assumptions (H_1)-(H_3), we can easily estimate prior boud of the solution to Eq. (1.1).

As an application, we consider the following example:

$$\bigl(\varphi_p\bigl[\bigl(x(t) - Bx(t - \pi)\bigr)'\bigr]\bigr)' + f\bigl(x(t)\bigr)x'(t) + g\bigl(x(t - \pi)\bigr) = e(t), \tag{3.25}$$

where

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \in \mathbb{R}^3, \qquad g(x) = \begin{pmatrix} \frac{1}{100}x_1 \\ \frac{1}{100}x_2 \end{pmatrix}, \qquad B = \begin{pmatrix} -1 & -3 \\ -4 & 0 \end{pmatrix},$$

$e(t) = (\sin t, \cos t)^\top$, $\tau = \gamma = \pi$, $p = 1.5$, $T = 2\pi$, $f(x) = (5 + \sin x_1, 10 + \cos x_2)$.

Obviously, $\lambda_1 = 3 \neq \pm 1$, $\lambda_2 = -4 \neq \pm 1$,

$$\int_0^{2\pi} e(t)\,dt = \begin{pmatrix} \int_0^{2\pi} \cos t\,dt \\ \int_0^{2\pi} \sin t\,dt \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Since

$$x_1 g_1(x_1) = \frac{1}{100}x_1^2 > 0 \quad \text{for} |x_1| > D > 0, \qquad x_2 g_2(x_2) = \frac{1}{100}x_2^2 > 0 \quad \text{for } |x_2| > D > 0,$$

so assumption (H_1) is satisfied. Take $l = \frac{1}{100}$, then

$$\left|g_i(u_1) - g_i(u_2)\right| \le \frac{1}{100}|u_1 - u_2|, \quad u_1, u_2 \in \mathbb{R}, \text{ for each } i = 1, 2,$$

and assumption (H_2) is satisfied. Take $\sigma = 4$, then

$$\left|f_1(x_1)\right| = |5 + \sin x_1| \ge 4, \qquad \left|f_2(x_2)\right| = |10 + \cos x_2| \ge 4,$$

and assumption (H_3) is satisfied. Hence, assumptions (H_1)-(H_3) are all satisfied. Take

$$U = \begin{pmatrix} -1 & 1 \\ 4 & 3 \end{pmatrix}, \qquad U^{-1} = \begin{pmatrix} -\frac{3}{7} & \frac{1}{7} \\ \frac{4}{7} & \frac{1}{7} \end{pmatrix}$$

such that

$$UBU^{-1} = \begin{pmatrix} 3 & 0 \\ 0 & -4 \end{pmatrix}.$$

Take $\gamma(t) = \pi$, then

$$\sigma > \sqrt{2}l \max_{t\in[0,T]} \left|\gamma(t)\right|.$$

By using Theorem 3.3, we know that Eq. (3.25) has at least one 2π-periodic solution.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The first author QY gave an example for verifying the paper's results. The corresponding author BD gave the proof for all the theorems. QY and BD read and approved the final manuscript.

Acknowledgements
This work was supported by NSF of Jiangsu education office (11KJB110002), Postdoctoral Fundation of Jiangsu (1102096C), Postdoctoral Fundation of China (2012M511296) and Jiangsu province fund (BK2011407).

Received: 24 April 2012 Accepted: 8 November 2012 Published: 27 November 2012

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doi:10.1186/1029-242X-2012-270
Cite this article as: Yang and Du: **Periodic solutions to a generalized Liénard neutral functional differential system with *p*-Laplacian.** *Journal of Inequalities and Applications* 2012 **2012**:270.