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Some Hermite-Hadamard type inequalities for n-time differentiable (α, m)-convex functions

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Abstract
In the paper, the famous Hermite-Hadamard integral inequality for convex functions is generalized to and refined as inequalities for n-time differentiable functions which are (α, m)-convex.
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1 Introduction

Throughout this paper, we adopt the following notations:

$$\mathbb{R} = (-\infty, \infty), \qquad \mathbb{R}_0 = [0, \infty), \quad \text{and} \quad \mathbb{R}_+ = (0, \infty). \tag{1.1}$$

We recall some definitions of several convex functions.

Definition 1.1 A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex if

$$f\big(\lambda x + (1-\lambda)y\big) \le \lambda f(x) + (1-\lambda)f(y) \tag{1.2}$$

holds for all $x, y \in I$ and $\lambda \in [0,1]$.

Definition 1.2 ([1]) For $f : [0, b] \to \mathbb{R}$ and $m \in (0,1]$, if

$$f\big(\lambda x + m(1-\lambda)y\big) \le \lambda f(x) + m(1-\lambda)f(y) \tag{1.3}$$

is valid for all $x, y \in [0, b]$ and $\lambda \in [0,1]$, then we say that $f(x)$ is an m-convex function on $[0, b]$.

Definition 1.3 ([2]) For $f : [0, b] \to \mathbb{R}$ and $\alpha, m \in (0,1]$, if

$$f\big(\lambda x + m(1-\lambda)y\big) \le \lambda^\alpha f(x) + m\big(1-\lambda^\alpha\big)f(y) \tag{1.4}$$

is valid for all $x, y \in [0, b]$ and $\lambda \in [0,1]$, then we say that $f(x)$ is an (α, m)-convex function on $[0, b]$.

In recent decades, plenty of inequalities of Hermite-Hadamard type for various kinds of convex functions have been established. Some of them may be reformulated as follows.

Theorem 1.1 ([3, Theorem 2.2]) *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping and $a, b \in I^\circ$ with $a < b$. If $|f'(x)|$ is convex on $[a, b]$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \le \frac{(b-a)(|f'(a)|+|f'(b)|)}{8}. \tag{1.5}$$

Theorem 1.2 ([4, Theorem 2]) *Let $f : \mathbb{R}_0 \to \mathbb{R}$ be m-convex and $m \in (0, 1]$. If $f \in L[a, b]$ for $0 \le a < b < \infty$, then*

$$\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \le \min\left\{\frac{f(a)+mf(b/m)}{2}, \frac{mf(a/m)+f(b)}{2}\right\}. \tag{1.6}$$

Theorem 1.3 ([2, Theorem 2.2]) *Let $I \supseteq \mathbb{R}_0$ be an open interval and let $f : I \to \mathbb{R}$ be a differentiable function such that $f' \in L[a, b]$ for $0 \le a < b < \infty$. If $|f'(x)|^q$ is m-convex on $[a, b]$ for some $m \in (0, 1]$ and $q \ge 1$, then*

$$\begin{aligned}&\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \\ &\quad\le \frac{b-a}{4}\min\left\{\left[\frac{|f'(a)|^q + m|f'(b/m)|^q}{2}\right]^{1/q}, \left[\frac{m|f'(a/m)|^q + |f'(b)|^q}{2}\right]^{1/q}\right\}.\end{aligned} \tag{1.7}$$

Theorem 1.4 ([2, Theorem 3.1]) *Let $I \supseteq \mathbb{R}_0$ be an open interval and let $f : I \to \mathbb{R}$ be a differentiable function such that $f' \in L[a, b]$ for $0 \le a < b < \infty$. If $[f'(x)]^q$ is (α, m)-convex on $[a, b]$ for some $\alpha, m \in (0, 1]$ and $q \ge 1$, then*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \\ &\quad\le \frac{b-a}{2}\left(\frac{1}{2}\right)^{1-1/q}\min\left\{\left[v_1\left[f'(a)\right]^q + v_2 m\left[f'\left(\frac{b}{m}\right)\right]^q\right]^{1/q}, \right. \\ &\qquad\left.\left[v_2 m\left[f'\left(\frac{a}{m}\right)\right]^q + v_1\left[f'(b)\right]^q\right]^{1/q}\right\},\end{aligned}$$

where

$$v_1 = \frac{1}{(\alpha+1)(\alpha+2)}\left(\alpha + \frac{1}{2^\alpha}\right) \tag{1.8}$$

and

$$v_2 = \frac{1}{(\alpha+1)(\alpha+2)}\left(\frac{\alpha^2+\alpha+2}{2} - \frac{1}{2^\alpha}\right). \tag{1.9}$$

For more and detailed information on this topic, please refer to the monograph [5] and newly published papers [6–16].

In this paper, we establish some Hermite-Hadamard type integral inequalities for n-time differentiable functions which are (α, m)-convex.

2 A lemma

In order to find inequalities of Hermite-Hadamard type for (α, m)-convex functions, we need the following lemma.

Lemma 2.1 ([17, Lemma 2.1] or [18, Lemma 2.1]) *Let $f : [a,b] \subseteq \mathbb{R} \to \mathbb{R}$ be an n-time differentiable function such that $f^{(n-1)}(x)$ for $n \in \mathbb{N}$ is absolutely continuous on $[a,b]$. Then the identity*

$$\begin{aligned} \int_a^b f(x)\,\mathrm{d}x = \sum_{k=0}^{n-1} \frac{(b-t)^{k+1} + (-1)^k (t-a)^{k+1}}{(k+1)!} f^{(k)}(t) \\ + (-1)^n \int_a^b K_n(t,x) f^{(n)}(x)\,\mathrm{d}x \end{aligned} \tag{2.1}$$

holds for all $t \in [a,b]$, where the kernel $K_n : [a,b] \times [a,b] \to \mathbb{R}$ is defined by

$$K_n(t,x) = \begin{cases} \frac{(x-a)^n}{n!}, & x \in [a,t], \\ \frac{(x-b)^n}{n!}, & x \in [t,b]. \end{cases} \tag{2.2}$$

3 Hermite-Hadamard type inequalities for (α, m)-convex functions

We now set off to establish some new integral inequalities of Hermite-Hadamard type for n-time differentiable (α, m)-convex functions.

Theorem 3.1 *Let $f : \mathbb{R}_0 \to \mathbb{R}$ be an n-time differentiable function for $n \in \mathbb{N}$ and let $0 \le a < b < \infty$ and $\alpha, m \in (0,1]$. If $f^{(n)}(x) \in L[a, \frac{b}{m}]$ and $|f^{(n)}(x)|^q$ for $q \ge 1$ is (α, m)-convex on $[0, \frac{b}{m}]$, then*

$$\begin{aligned} &\left| \frac{1}{b-a} \int_a^b f(x)\,\mathrm{d}x - \frac{1}{b-a} \sum_{k=0}^{n-1} \frac{(b-t)^{k+1} + (-1)^k (t-a)^{k+1}}{(k+1)!} f^{(k)}(t) \right| \\ &\quad \le \frac{1}{(b-a)(n+1)!} \Bigg\{ (t-a)^{n+1} \bigg[\alpha B(n+2,\alpha) \big| f^{(n)}(a) \big|^q \\ &\qquad + m\big(1 - \alpha B(n+2,\alpha)\big) \left| f^{(n)}\left(\frac{t}{m}\right) \right|^q \bigg]^{1/q} \\ &\qquad + (b-t)^{n+1} \left[\frac{1}{n+\alpha+1} \left((n+1) \big| f^{(n)}(t) \big|^q + \alpha m \left| f^{(n)}\left(\frac{b}{m}\right) \right|^q \right) \right]^{1/q} \Bigg\}, \end{aligned} \tag{3.1}$$

where $t \in [a,b]$ and $B(\alpha, \beta)$ is the beta function

$$B(\alpha, \beta) = \int_0^1 t^{\alpha-1} (1-t)^{\beta-1}\,\mathrm{d}t, \quad \alpha, \beta > 0. \tag{3.2}$$

Proof If $a < t < b$, by Lemma 2.1, Hölder's integral inequality, and the (α, m)-convexity of $|f^{(n)}(x)|^q$, we have

$$\begin{aligned} &\left| \frac{1}{b-a} \int_a^b f(x)\,\mathrm{d}x - \frac{1}{b-a} \sum_{k=0}^{n-1} \frac{(b-t)^{k+1} + (-1)^k (t-a)^{k+1}}{(k+1)!} f^{(k)}(t) \right| \\ &\quad \le \frac{1}{(b-a)n!} \left[\int_a^t (x-a)^n \big| f^{(n)}(x) \big|\,\mathrm{d}x + \int_t^b (b-x)^n \big| f^{(n)}(x) \big|\,\mathrm{d}x \right] \end{aligned}$$

$$
\begin{aligned}
&\le \frac{1}{(b-a)n!}\biggl\{\biggl[\int_a^t (x-a)^n\,\mathrm{d}x\biggr]^{1-1/q}\biggl[\int_a^t (x-a)^n\bigl|f^{(n)}(x)\bigr|^q\,\mathrm{d}x\biggr]^{1/q} \\
&\quad + \biggl[\int_t^b (b-x)^n\,\mathrm{d}x\biggr]^{1-1/q}\biggl[\int_t^b (b-x)^n\bigl|f^{(n)}(x)\bigr|^q\,\mathrm{d}x\biggr]^{1/q}\biggr\} \\
&= \frac{1}{(b-a)n!}\biggl\{\biggl[\frac{(t-a)^{n+1}}{n+1}\biggr]^{1-1/q}\biggl[\int_a^t (x-a)^n\biggl|f^{(n)}\biggl(\frac{t-x}{t-a}a \\
&\quad + m\frac{x-a}{t-a}\times\frac{t}{m}\biggr)\biggr|^q\,\mathrm{d}x\biggr]^{1/q} + \biggl[\frac{(b-t)^{n+1}}{n+1}\biggr]^{1-1/q} \\
&\quad \times \biggl[\int_t^b (b-x)^n\biggl|f^{(n)}\biggl(\frac{b-x}{b-t}t + m\frac{x-t}{b-t}\times\frac{b}{m}\biggr)\biggr|^q\,\mathrm{d}x\biggr]^{1/q}\biggr\} \\
&\le \frac{1}{(b-a)n!}\biggl\{\biggl[\frac{(t-a)^{n+1}}{n+1}\biggr]^{1-1/q}\biggl(\int_a^t (x-a)^n\biggl[\biggl(\frac{t-x}{t-a}\biggr)^\alpha\bigl|f^{(n)}(a)\bigr|^q \\
&\quad + m\biggl(1-\biggl(\frac{t-x}{t-a}\biggr)^\alpha\biggr)\biggl|f^{(n)}\biggl(\frac{t}{m}\biggr)\biggr|^q\biggr]\,\mathrm{d}x\biggr)^{1/q} + \biggl[\frac{(b-t)^{n+1}}{n+1}\biggr]^{1-1/q} \\
&\quad \times \biggl(\int_t^b (b-x)^n\biggl[\biggl(\frac{b-x}{b-t}\biggr)^\alpha\bigl|f^{(n)}(t)\bigr|^q \\
&\quad + m\biggl(1-\biggl(\frac{b-x}{b-t}\biggr)^\alpha\biggr)\biggl|f^{(n)}\biggl(\frac{b}{m}\biggr)\biggr|^q\biggr]\,\mathrm{d}x\biggr)^{1/q}\biggr\}.
\end{aligned}
$$

Substituting

$$
\begin{aligned}
&\int_a^t (x-a)^n\biggl\{\biggl(\frac{t-x}{t-a}\biggr)^\alpha\bigl|f^{(n)}(a)\bigr|^q + m\biggl[1-\biggl(\frac{t-x}{t-a}\biggr)^\alpha\biggr]\biggl|f^{(n)}\biggl(\frac{t}{m}\biggr)\biggr|^q\biggr\}\,\mathrm{d}x \\
&\quad = \frac{(t-a)^{n+1}}{n+1}\biggl[\alpha B(n+2,\alpha)\bigl|f^{(n)}(a)\bigr|^q + m\bigl(1-\alpha B(n+2,\alpha)\bigr)\biggl|f^{(n)}\biggl(\frac{t}{m}\biggr)\biggr|^q\biggr]
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_t^b (b-x)^n\biggl\{\biggl(\frac{b-x}{b-t}\biggr)^\alpha\bigl|f^{(n)}(t)\bigr|^q + m\biggl[1-\biggl(\frac{b-x}{b-t}\biggr)^\alpha\biggr]\biggl|f^{(n)}\biggl(\frac{b}{m}\biggr)\biggr|^q\biggr\}\,\mathrm{d}x \\
&\quad = \frac{(b-t)^{n+1}}{(n+1)(n+\alpha+1)}\biggl[(n+1)\bigl|f^{(n)}(t)\bigr|^q + \alpha m\biggl|f^{(n)}\biggl(\frac{b}{m}\biggr)\biggr|^q\biggr]
\end{aligned}
$$

into the above inequality leads to the inequality (3.1) for $t \in (a,b)$.

If $t = a$ or $t = b$, by virtue of Lemma 2.1 and the property that $|f^{(n)}(x)|^q$ is (α,m)-convex on $[0,\frac{b}{m}]$, we have

$$
\begin{aligned}
&\biggl|\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x - \sum_{k=0}^{n-1}\frac{(b-a)^k}{(k+1)!}f^{(k)}(a)\biggr| \\
&\quad \le \frac{1}{(b-a)n!}\biggl[\frac{(b-a)^{n+1}}{n+1}\biggr]^{1-1/q}\biggl\{\int_a^b (b-x)^n\biggl[\biggl(\frac{b-x}{b-a}\biggr)^\alpha\bigl|f^{(n)}(a)\bigr|^q \\
&\quad + m\biggl(1-\biggl(\frac{b-x}{b-a}\biggr)^\alpha\biggr)\biggl|f^{(n)}\biggl(\frac{b}{m}\biggr)\biggr|^q\biggr]\,\mathrm{d}x\biggr\}^{1/q}
\end{aligned}
$$

and

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x-\sum_{k=0}^{n-1}\frac{(a-b)^k}{(k+1)!}f^{(k)}(b)\right|\\&\quad\le\frac{1}{(b-a)n!}\left[\frac{(b-a)^{n+1}}{n+1}\right]^{1-1/q}\left\{\int_a^b(x-a)^n\left[\left(\frac{b-x}{b-a}\right)^{\alpha}\left|f^{(n)}(a)\right|^q\right.\right.\\&\qquad\left.\left.+m\left(1-\left(\frac{b-x}{b-a}\right)^{\alpha}\right)\left|f^{(n)}\left(\frac{b}{m}\right)\right|^q\right]\mathrm{d}x\right\}^{1/q}.\end{aligned}$$

The inequality (3.1) for $t=a$ or $t=b$ follows. Theorem 3.1 is thus proved. □

Corollary 3.1 *Under the conditions of Theorem* 3.1,

(1) *when* $q=1$, *we have*

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x-\frac{1}{b-a}\sum_{k=0}^{n-1}\frac{(b-t)^{k+1}+(-1)^k(t-a)^{k+1}}{(k+1)!}f^{(k)}(t)\right|\\&\quad\le\frac{1}{(b-a)(n+1)!}\left\{(t-a)^{n+1}\left[\alpha B(n+2,\alpha)\left|f^{(n)}(a)\right|\right.\right.\\&\qquad\left.+m\bigl(1-\alpha B(n+2,\alpha)\bigr)\left|f^{(n)}\left(\frac{t}{m}\right)\right|\right]\\&\qquad\left.+(b-t)^{n+1}\left[\frac{1}{n+\alpha+1}\left((n+1)\left|f^{(n)}(t)\right|+\alpha m\left|f^{(n)}\left(\frac{b}{m}\right)\right|\right)\right]\right\};\end{aligned}$$

(2) *when* $\alpha=1$, *we have*

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x-\frac{1}{b-a}\sum_{k=0}^{n-1}\frac{(b-t)^{k+1}+(-1)^k(t-a)^{k+1}}{(k+1)!}f^{(k)}(t)\right|\\&\quad\le\frac{1}{(b-a)(n+1)!}\left(\frac{1}{n+2}\right)^{1/q}\left\{(t-a)^{n+1}\left[\left|f^{(n)}(a)\right|^q+m(n+1)\left|f^{(n)}\left(\frac{t}{m}\right)\right|^q\right]^{1/q}\right.\\&\qquad\left.+(b-t)^{n+1}\left[\left((n+1)\left|f^{(n)}(t)\right|^q+m\left|f^{(n)}\left(\frac{b}{m}\right)\right|^q\right)\right]^{1/q}\right\};\end{aligned}$$

(3) *when* $m=1$, *we have*

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x-\frac{1}{b-a}\sum_{k=0}^{n-1}\frac{(b-t)^{k+1}+(-1)^k(t-a)^{k+1}}{(k+1)!}f^{(k)}(t)\right|\\&\quad\le\frac{1}{(b-a)(n+1)!}\left\{(t-a)^{n+1}\left[\alpha B(n+2,\alpha)\left|f^{(n)}(a)\right|^q\right.\right.\\&\qquad\left.+\bigl(1-\alpha B(n+2,\alpha)\bigr)\left|f^{(n)}(t)\right|^q\right]^{1/q}\\&\qquad\left.+(b-t)^{n+1}\left[\frac{1}{n+\alpha+1}\left((n+1)\left|f^{(n)}(t)\right|^q+\alpha\left|f^{(n)}(b)\right|^q\right)\right]^{1/q}\right\};\end{aligned}$$

(4) *when* $m = \alpha = q = 1$, *we have*

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x - \frac{1}{b-a}\sum_{k=0}^{n-1}\frac{(b-t)^{k+1}+(-1)^k(t-a)^{k+1}}{(k+1)!}f^{(k)}(t)\right| \\ &\quad\le \frac{1}{(b-a)(n+2)!}\bigl\{(t-a)^{n+1}\bigl[\bigl|f^{(n)}(a)\bigr| + (n+1)\bigl|f^{(n)}(t)\bigr|\bigr] \\ &\qquad + (b-t)^{n+1}\bigl[(n+1)\bigl|f^{(n)}(t)\bigr| + \bigl|f^{(n)}(b)\bigr|\bigr]\bigr\}.\end{aligned}$$

Corollary 3.2 *Under the conditions of Theorem* 3.1,

(1) *when* $t = a$, *we have*

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x - \sum_{k=0}^{n-1}\frac{(b-a)^k}{(k+1)!}f^{(k)}(a)\right| \\ &\quad\le \frac{(b-a)^n}{(n+1)!}\left\{\frac{1}{n+\alpha+1}\left[(n+1)\bigl|f^{(n)}(a)\bigr|^q + \alpha m\left|f^{(n)}\left(\frac{b}{m}\right)\right|^q\right]\right\}^{1/q};\end{aligned} \tag{3.3}$$

(2) *when* $t = \frac{a+b}{2}$, *we have*

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x - \sum_{k=0}^{n-1}\frac{[1+(-1)^k](b-a)^k}{2^{k+1}(k+1)!}f^{(k)}\left(\frac{a+b}{2}\right)\right| \\ &\quad\le \frac{(b-a)^n}{2^{n+1}(n+1)!}\left\{\left[\alpha B(n+2,\alpha)\bigl|f^{(n)}(a)\bigr|^q\right.\right. \\ &\qquad \left. + m\bigl(1-\alpha B(n+2,\alpha)\bigr)\left|f^{(n)}\left(\frac{a+b}{2m}\right)\right|^q\right]^{1/q} \\ &\qquad \left. + \left[\frac{1}{n+\alpha+1}\left((n+1)\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|^q + \alpha m\left|f^{(n)}\left(\frac{b}{m}\right)\right|^q\right)\right]^{1/q}\right\};\end{aligned}$$

(3) *when* $t = b$, *we have*

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x - \sum_{k=0}^{n-1}\frac{(a-b)^k}{(k+1)!}f^{(k)}(b)\right| \\ &\quad\le \frac{(b-a)^n}{(n+1)!}\left[\alpha B(n+2,\alpha)\bigl|f^{(n)}(a)\bigr|^q + m\bigl(1-\alpha B(n+2,\alpha)\bigr)\left|f^{(n)}\left(\frac{b}{m}\right)\right|^q\right]^{1/q}.\end{aligned}$$

Theorem 3.2 *Let* $t \in [a,b]$ *and* $f : \mathbb{R}_0 \to \mathbb{R}$ *be an n-time differentiable function for* $n \in \mathbb{N}$, *and let* $0 \le a < b < \infty$ *and* $\alpha, m \in (0,1]$. *If* $f^{(n)}(x) \in L[a, \frac{b}{m}]$, $|f^{(n)}(x)|^q$ *for* $q > 1$ *is* (α, m)-*convex on* $[0, \frac{b}{m}]$, *and* $nq \ge p \ge 0$, *then*

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x - \frac{1}{b-a}\sum_{k=0}^{n-1}\frac{(b-t)^{k+1}+(-1)^k(t-a)^{k+1}}{(k+1)!}f^{(k)}(t)\right| \\ &\quad\le \frac{1}{(b-a)n!}\left(\frac{q-1}{nq+q-p-1}\right)^{1-1/q}\left(\frac{1}{p+1}\right)^{1/q}\left\{(t-a)^{n+1}\right.\end{aligned}$$

$$\times \left[\alpha B(p+2,\alpha)\left|f^{(n)}(a)\right|^q + m\bigl(1-\alpha B(p+2,\alpha)\bigr)\left|f^{(n)}\left(\frac{t}{m}\right)\right|^q\right]^{1/q}$$
$$+ (b-t)^{n+1}\left[\frac{1}{p+\alpha+1}\left((p+1)\left|f^{(n)}(t)\right|^q + \alpha m\left|f^{(n)}\left(\frac{b}{m}\right)\right|^q\right)\right]^{1/q}\Bigg\}. \tag{3.4}$$

Proof When $a < t < b$, by Lemma 2.1 and Hölder's integral inequality, we have

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x - \frac{1}{b-a}\sum_{k=0}^{n-1}\frac{(b-t)^{k+1}+(-1)^k(t-a)^{k+1}}{(k+1)!}f^{(k)}(t)\right| \\
&\quad\le \frac{1}{(b-a)n!}\left[\int_a^t (x-a)^n\left|f^{(n)}(x)\right|\,\mathrm{d}x + \int_t^b (b-x)^n\left|f^{(n)}(x)\right|\,\mathrm{d}x\right] \\
&\quad\le \frac{1}{(b-a)n!}\left\{\left[\int_a^t (x-a)^{(nq-p)/(q-1)}\,\mathrm{d}x\right]^{1-1/q}\left[\int_a^t (x-a)^p\left|f^{(n)}(x)\right|^q\,\mathrm{d}x\right]^{1/q}\right. \\
&\qquad\left. + \left[\int_t^b (b-x)^{(nq-p)/(q-1)}\,\mathrm{d}x\right]^{1-1/q}\left[\int_t^b (b-x)^p\left|f^{(n)}(x)\right|^q\,\mathrm{d}x\right]^{1/q}\right\},
\end{aligned} \tag{3.5}$$

where

$$\int_a^t (x-a)^{(nq-p)/(q-1)}\,\mathrm{d}x = \frac{q-1}{nq+q-p-1}(t-a)^{(nq+q-p-1)/(q-1)} \tag{3.6}$$

and

$$\int_t^b (b-x)^{(nq-p)/(q-1)}\,\mathrm{d}x = \frac{q-1}{nq+q-p-1}(b-t)^{(nq+q-p-1)/(q-1)}. \tag{3.7}$$

Since $|f^{(n)}(x)|^q$ is (α, m)-convex on $[0, \frac{b}{m}]$, we have

$$\begin{aligned}
&\int_a^t (x-a)^p\left|f^{(n)}(x)\right|^q\,\mathrm{d}x \\
&\quad\le \int_a^t (x-a)^p\left\{\left(\frac{t-x}{t-a}\right)^\alpha\left|f^{(n)}(a)\right|^q + m\left[1-\left(\frac{t-x}{t-a}\right)^\alpha\right]\left|f^{(n)}\left(\frac{t}{m}\right)\right|^q\right\}\mathrm{d}x \\
&\quad= \frac{(t-a)^{p+1}}{p+1}\left[\alpha B(p+2,\alpha)\left|f^{(n)}(a)\right|^q + m\bigl(1-\alpha B(p+2,\alpha)\bigr)\left|f^{(n)}\left(\frac{t}{m}\right)\right|^q\right]
\end{aligned}$$

and

$$\begin{aligned}
&\int_t^b (b-x)^p\left|f^{(n)}(x)\right|^q\,\mathrm{d}x \\
&\quad\le \int_t^b (b-x)^p\left\{\left(\frac{b-x}{b-t}\right)^\alpha\left|f^{(n)}(t)\right|^q + m\left[1-\left(\frac{b-x}{b-t}\right)^\alpha\right]\left|f^{(n)}\left(\frac{b}{m}\right)\right|^q\right\}\mathrm{d}x \\
&\quad= \frac{(b-t)^{p+1}}{(p+1)(p+\alpha+1)}\left[(p+1)\left|f^{(n)}(t)\right|^q + \alpha m\left|f^{(n)}\left(\frac{b}{m}\right)\right|^q\right].
\end{aligned}$$

Hence, the inequality (3.4) follows.

When $t = a$ or $t = b$, the proof of the inequality (3.4) is similar to the above argument. The proof of Theorem 3.2 is complete. □

Corollary 3.3 *Under the conditions of Theorem* 3.2,

(1) *if* $\alpha = 1$, *then*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x - \frac{1}{b-a}\sum_{k=0}^{n-1}\frac{(b-t)^{k+1}+(-1)^k(t-a)^{k+1}}{(k+1)!}f^{(k)}(t)\right|$$
$$\leq \frac{1}{(b-a)n!}\left(\frac{q-1}{nq+q-p-1}\right)^{1-1/q}\left[\frac{1}{(p+1)(p+2)}\right]^{1/q}$$
$$\times\left\{(t-a)^{n+1}\left[\left|f^{(n)}(a)\right|^q + m(p+1)\left|f^{(n)}\left(\frac{t}{m}\right)\right|^q\right]^{1/q}\right.$$
$$\left.+(b-t)^{n+1}\left[\left((p+1)\left|f^{(n)}(t)\right|^q + m\left|f^{(n)}\left(\frac{b}{m}\right)\right|^q\right)\right]^{1/q}\right\};$$

(2) *if* $m = 1$, *then*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x - \frac{1}{b-a}\sum_{k=0}^{n-1}\frac{(b-t)^{k+1}+(-1)^k(t-a)^{k+1}}{(k+1)!}f^{(k)}(t)\right|$$
$$\leq \frac{1}{(b-a)n!}\left(\frac{q-1}{nq+q-p-1}\right)^{1-1/q}\left(\frac{1}{p+1}\right)^{1/q}\left\{(t-a)^{n+1}\right.$$
$$\times\left[\alpha B(p+2,\alpha)\left|f^{(n)}(a)\right|^q + \left(1-\alpha B(p+2,\alpha)\right)\left|f^{(n)}(t)\right|^q\right]^{1/q}$$
$$\left.+(b-t)^{n+1}\left[\frac{1}{p+\alpha+1}\left((p+1)\left|f^{(n)}(t)\right|^q + \alpha\left|f^{(n)}(b)\right|^q\right)\right]^{1/q}\right\};$$

(3) *if* $m = \alpha = 1$, *we have*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x - \frac{1}{b-a}\sum_{k=0}^{n-1}\frac{(b-t)^{k+1}+(-1)^k(t-a)^{k+1}}{(k+1)!}f^{(k)}(t)\right|$$
$$\leq \frac{1}{(b-a)n!}\left(\frac{q-1}{nq+q-p-1}\right)^{1-1/q}\left[\frac{1}{(p+1)(p+2)}\right]^{1/q}\left\{(t-a)^{n+1}\left[\left|f^{(n)}(a)\right|^q\right.\right.$$
$$\left.\left.+(p+1)\left|f^{(n)}(t)\right|^q\right]^{1/q} + (b-t)^{n+1}\left[(p+1)\left|f^{(n)}(t)\right|^q + \left|f^{(n)}(b)\right|^q\right]^{1/q}\right\}.$$

Corollary 3.4 *Under the conditions of Theorem* 3.2,

(1) *if* $t = a$, *then*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x - \sum_{k=0}^{n-1}\frac{(b-a)^k}{(k+1)!}f^{(k)}(a)\right|$$
$$\leq \frac{(b-a)^n}{n!}\left(\frac{q-1}{nq+q-p-1}\right)^{1-1/q}\left[\frac{1}{(p+1)(p+\alpha+1)}\right]^{1/q}$$
$$\times\left[(p+1)\left|f^{(n)}(a)\right|^q + \alpha m\left|f^{(n)}\left(\frac{b}{m}\right)\right|^q\right]^{1/q}; \tag{3.8}$$

(2) *if* $t = \frac{a+b}{2}$, *then*

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x - \sum_{k=0}^{n-1}\frac{[1+(-1)^k](b-a)^k}{2^{k+1}(k+1)!}f^{(k)}\left(\frac{a+b}{2}\right)\right| \\
&\quad\le \frac{(b-a)^n}{2^{n+1}n!}\left(\frac{q-1}{nq+q-p-1}\right)^{1-1/q}\left(\frac{1}{p+1}\right)^{1/q} \\
&\qquad\times\left\{\left[\alpha B(p+2,\alpha)\left|f^{(n)}(a)\right|^q + m\big(1-\alpha B(p+2,\alpha)\big)\left|f^{(n)}\left(\frac{a+b}{2m}\right)\right|^q\right]^{1/q}\right. \\
&\qquad\left. + \left[\frac{1}{p+\alpha+1}\left((p+1)\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|^q + \alpha m\left|f^{(n)}\left(\frac{b}{m}\right)\right|^q\right)\right]^{1/q}\right\};
\end{aligned}$$

(3) *if* $t = b$, *then*

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x - \sum_{k=0}^{n-1}\frac{(a-b)^k}{(k+1)!}f^{(k)}(b)\right| \\
&\quad\le \frac{(b-a)^n}{n!}\left(\frac{q-1}{nq+q-p-1}\right)^{1-1/q}\left(\frac{1}{p+1}\right)^{1/q}\left[\alpha B(p+2,\alpha)\left|f^{(n)}(a)\right|^q\right. \\
&\qquad\left. + m\big(1-\alpha B(p+2,\alpha)\big)\left|f^{(n)}\left(\frac{b}{m}\right)\right|^q\right]^{1/q}.
\end{aligned}$$

Corollary 3.5 *Under the conditions of Theorem* 3.2,
(1) *if* $p = 0$, *then*

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x - \frac{1}{b-a}\sum_{k=0}^{n-1}\frac{(b-t)^{k+1}+(-1)^k(t-a)^{k+1}}{(k+1)!}f^{(k)}(t)\right| \\
&\quad\le \frac{1}{(b-a)n!}\left(\frac{q-1}{nq+q-1}\right)^{1-1/q}\left[\frac{1}{(\alpha+1)(\alpha+2)}\right]^{1/q}\left\{(t-a)^{n+1}\left[\left|f^{(n)}(a)\right|^q\right.\right. \\
&\qquad\left.\left. + \alpha m\left|f^{(n)}\left(\frac{t}{m}\right)\right|^q\right]^{1/q} + (b-t)^{n+1}\left[\left|f^{(n)}(t)\right|^q + \alpha m\left|f^{(n)}\left(\frac{b}{m}\right)\right|^q\right]^{1/q}\right\};
\end{aligned}$$

(2) *if* $p = q$, *then*

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x - \frac{1}{b-a}\sum_{k=0}^{n-1}\frac{(b-t)^{k+1}+(-1)^k(t-a)^{k+1}}{(k+1)!}f^{(k)}(t)\right| \\
&\quad\le \frac{1}{(b-a)n!}\left(\frac{q-1}{nq-1}\right)^{1-1/q}\left(\frac{1}{q+1}\right)^{1/q}\left\{(t-a)^{n+1}\right. \\
&\qquad\times\left[\alpha B(q+2,\alpha)\left|f^{(n)}(a)\right|^q + m\big(1-\alpha B(q+2,\alpha)\big)\left|f^{(n)}\left(\frac{t}{m}\right)\right|^q\right]^{1/q} \\
&\qquad\left. + (b-t)^{n+1}\left[\frac{1}{q+\alpha+1}\left((q+1)\left|f^{(n)}(t)\right|^q + \alpha m\left|f^{(n)}\left(\frac{b}{m}\right)\right|^q\right)\right]^{1/q}\right\};
\end{aligned}$$

(3) *if* $p = nq$, *then*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x - \frac{1}{b-a}\sum_{k=0}^{n-1}\frac{(b-t)^{k+1}+(-1)^k(t-a)^{k+1}}{(k+1)!}f^{(k)}(t)\right|$$
$$\le \frac{1}{(b-a)n!}\left(\frac{1}{nq+1}\right)^{1/q}\Biggl\{(t-a)^{n+1}$$
$$\times\left[\alpha B(nq+2,\alpha)\bigl|f^{(n)}(a)\bigr|^q + m\bigl(1-\alpha B(nq+2,\alpha)\bigr)\left|f^{(n)}\left(\frac{t}{m}\right)\right|^q\right]^{1/q}$$
$$+ (b-t)^{n+1}\left[\frac{1}{nq+\alpha+1}\left((nq+1)\bigl|f^{(n)}(t)\bigr|^q + \alpha m\left|f^{(n)}\left(\frac{b}{m}\right)\right|^q\right)\right]^{1/q}\Biggr\}.$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the manuscript and read and approved the final manuscript.

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References

1. Toader, G: Some generalizations of the convexity. In: Proceedings of the Colloquium on Approximation and Optimization, pp. 329-338. Univ. Cluj-Napoca, Cluj-Napoca (1985)
2. Bakula, MK, Özdemir, ME, Pečarić, J: Hadamard type inequalities for *m*-convex and (α, m)-convex functions. J. Inequal. Pure Appl. Math. **9**(4), Art. 96 (2008). Available online at http://www.emis.de/journals/JIPAM/article1032.html
3. Dragomir, SS, Agarwal, RP: Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula. Appl. Math. Lett. **11**(5), 91-95 (1998). Available online at http://dx.doi.org/10.1016/S0893-9659(98)00086-X
4. Dragomir, SS, Toader, G: Some inequalities for *m*-convex functions. Stud. Univ. Babeş-Bolyai, Math. **38**(1), 21-28 (1993)
5. Dragomir, SS, Pearce, CEM: Selected Topics on Hermite-Hadamard Type Inequalities and Applications. RGMIA Monographs. Victoria University, Melbourne (2000). Available online at http://rgmia.org/monographs/hermite_hadamard.html
6. Bai, R-F, Qi, F, Xi, B-Y: Hermite-Hadamard type inequalities for the *m*- and (α, m)-logarithmically convex functions. Filomat **27**(1), 1-7 (2013).
7. Chun, L, Qi, F: Integral inequalities of Hermite-Hadamard type for functions whose 3rd derivatives are *s*-convex. Appl. Math. **3**(11), 1680-1685 (2012). Available online at http://dx.doi.org/10.4236/am.2012.311232
8. Jiang, W-D, Niu, D-W, Hua, Y, Qi, F: Generalizations of Hermite-Hadamard inequality to *n*-time differentiable functions which are *s*-convex in the second sense. Analysis (Munich) **32**(3), 209-220 (2012). Available online at http://dx.doi.org/10.1524/anly.2012.1161
9. Qi, F, Wei, Z-L, Yang, Q: Generalizations and refinements of Hermite-Hadamard's inequality. Rocky Mt. J. Math. **35**(1), 235-251 (2005). Available online at http://dx.doi.org/10.1216/rmjm/1181069779
10. Wang, S-H, Xi, B-Y, Qi, F: On Hermite-Hadamard type inequalities for (α, m)-convex functions. Int. J. Open Probl. Comput. Sci. Math. **5**(4), 47-56 (2012)
11. Wang, S-H, Xi, B-Y, Qi, F: Some new inequalities of Hermite-Hadamard type for *n*-time differentiable functions which are *m*-convex. Analysis (Munich) **32**(3), 247-262 (2012). Available online at http://dx.doi.org/10.1524/anly.2012.1167
12. Xi, B-Y, Bai, R-F, Qi, F: Hermite-Hadamard type inequalities for the *m*- and (α, m)-geometrically convex functions. Aequ. Math. **84**(3), 261-269 (2012). Available online at http://dx.doi.org/10.1007/s00010-011-0114-x
13. Xi, B-Y, Qi, F: Some Hermite-Hadamard type inequalities for differentiable convex functions and applications. Hacet. J. Math. Stat. (2013, in press)
14. Xi, B-Y, Qi, F: Some integral inequalities of Hermite-Hadamard type for convex functions with applications to means. J. Funct. Spaces Appl. **2012**, Article ID 980438 (2012). Available online at http://dx.doi.org/10.1155/2012/980438
15. Xi, B-Y, Wang, S-H, Qi, F: Some inequalities of Hermite-Hadamard type for functions whose 3rd derivatives are *P*-convex. Appl. Math. **3**(12) 1898-1902 (2012). Available online at http://dx.doi.org/10.4236/am.2012.312260
16. Zhang, T-Y, Ji, A-P, Qi, F: On integral inequalities of Hermite-Hadamard type for *s*-geometrically convex functions. Abstr. Appl. Anal. **2012**, Article ID 560586 (2012). Available online at http://dx.doi.org/10.1155/2012/560586

17. Cerone, P, Dragomir, SS, Roumeliotis, J: Some Ostrowski type inequalities for n-time differentiable mappings and applications. RGMIA Res. Rep. Collect. **1**(1), Art. 6 (1998). Available online at http://rgmia.org/v1n1.php
18. Cerone, P, Dragomir, SS, Roumelotis, J: Some Ostrowski type inequalities for n-time differentiable mappings and applications. Demonstr. Math. **32**(4), 697-712 (1999)

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