Özdemir et al. *Journal of Inequalities and Applications* 2012, **2012**:21
http://www.journalofinequalitiesandapplications.com/content/2012/1/21

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

On some Hadamard-type inequalities for product of two *s*-convex functions on the co-ordinates

Muhamet Emin Özdemir[1], Muhammad Amer Latif[2] and Ahmet Ocak Akdemir[3*]

* Correspondence: ahmetakdemir@agri.edu.tr
[3]Department of Mathematics, Faculty of Science and Letters, Ağri İbrahim Çeçen University, Ağri 04100, Turkey
Full list of author information is available at the end of the article

Abstract

In this article, Hadamard-type inequalities for product of *s*-convex in the second sense on the co-ordinates in a rectangle from the plane are established.

1. Introduction

A function $f : I \to \mathbb{R}$, $I \subseteq \mathbb{R}$ is an interval, is said to be a convex function on I if

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y) \tag{1.1}$$

holds for all $x, y \in I$ and $t \in [0, 1]$. If the reversed inequality in (1.1) holds, then f is concave. Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function and $a, b \in I$ with $a < b$. Then the following double inequality:

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2} \tag{1.2}$$

is known as Hermite-Hadamard inequality for convex mappings. For particular choice of the function f in (1.2) yields some classical inequalities of means. Both inequalities in (1.2) hold in reversed direction if f is concave.

Some basic definitions can be given as followings:

Definition 1. (See [1], [2, p. 410]) *We say that $f : I \to \mathbb{R}$ is a Godunova-Levin function or that f belongs to the class $Q(I)$ if f is non-negative and for all $x, y \in I$ and $t \in (0, 1)$ we have*

$$f(tx + (1-t)y) \le \frac{f(x)}{t} + \frac{f(y)}{1-t}.$$

Definition 2. (See [3]) *We say that $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is a P-function or that f belongs to the class $P(I)$ if f is non-negative and for all $x, y \in I$ and $t \in [0, 1]$, we have*

$$f(tx + (1-t)y) \le f(x) + f(y).$$

Definition 3. (See [4]) *Let $s \in (0, 1]$. A function $f : [0, \infty) \to [0, \infty)$ is said to be s-convex in the second sense if*

$$f(tx + (1-t)y) \le t^s f(x) + (1-t)^s f(y)$$

for all $x, y \in (0, b)$ and $t \in [0, 1]$.

In [5], Hudzik and Maligranda considered among others the class of functions which are s-convex in the first sense. This class is defined in the following way:

Definition 4. *A function* $f : \mathbb{R}^+ \to \mathbb{R}$, *where* $\mathbb{R}^+ = [0, \infty)$, *is said to be s-convex in the first sense if*

$$f(\alpha x + \beta y) \leq \alpha^s f(x) + \beta^s f(y)$$

for all $x, y \in [0, \infty)$, $\alpha, \beta \geq 0$ *with* $\alpha^s + \beta^s = 1$ *and for some fixed* $s \in (0, 1]$. *We denote by* K_s^1 *the class of all s-convex functions.*

In 1978, Breckner introduced s-convex functions as a generalization of convex functions in [4]. Also, he proved the important fact that the set valued map is s-convex only if the associated support function is s-convex function [6]. Of course, s-convexity means just convexity when $s = 1$. The definition of s-convexity of real valued functions are very important for Orlicz spaces and Banach normed spaces (see [7-9]). A number of properties of s-convex functions are discussed in articles [5,10-13].

In article [14] the following generalization of the previously described functions was given.

Definition 5. (See [14,15]) *Let I, J be intervals* $\mathbb{R}$, $(0, 1) \subseteq J$ *and let* $h : J \to \mathbb{R}$. *A function* $f : I \to \mathbb{R}$ *is called an h-convex function, or that f belongs to the class* $SX(h, I)$, *if for all* $x, y \in I$ *and* $t \in (0, 1)$ *we have*

$$f(tx + (1-t)y) \leq h(t)f(x) + h(1-t)f(y). \tag{1.3}$$

If inequality in (1.3) is reversed, then f is said to be h-concave.

Obviously, if $h(t) = t$, for all $t \in [0, 1] \subseteq J$, then all convex functions belong to $SX(h, I)$ and all concave functions are h-concave; if $h(t) = \frac{1}{t}$, for all $t \in (0, 1)$, then $SX(h, I) = Q(I)$; if $h(t) = 1$, $SX(h, I) \supseteq P(I)$; and if $h(t) = t^s$, where $s \in (0, 1)$, then $SX(h, I) \supseteq K_s^2$. For some recent results about h-convex functions we refer the reader to articles [15-18].

In [10], Dragomir and Fitzpatrick proved the following variant of Hermite-Hadamard inequality which hold for s-convex functions in the second sense:

Theorem 1. *Suppose that* $f : [0, \infty) \to [0, \infty)$ *is an s-convex function in the second sense, where* $s \in (0, 1)$ *and let* $a, b \in [0, \infty)$, $a < b$. *If* $f \in L_1([a, b])$, *then the following inequalities hold:*

$$2^{s-1} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{s+1} \tag{1.4}$$

The constant $k = \frac{1}{s+1}$ *is the best possible in the second inequality in (1.4).*

Again in [10], Dragomir and Fitzpatrick also proved the following Hadamard-type inequality for s-convex functions in the first sense:

Theorem 2. *Suppose that* $f : [0, \infty) \to [0, \infty)$ *is an s-convex function in the first sense, where* $s \in (0, 1)$ *and let* $a, b \in [0, \infty)$. *If* $f \in L_1([a, b])$ *then the following inequalities hold:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+sf(b)}{s+1} \tag{1.5}$$

The above inequalities are sharp.

A modification for convex functions which is also known as co-ordinated convex functions was introduced as following by Dragomir [19]:

Let us consider a bidimensional interval $\Delta =: [a, b] \times [c, d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$. A mapping $f : \Delta \to \mathbb{R}$ is said to be convex on Δ if the following inequality:

$$f(\alpha x + (1-\alpha)z, \alpha y + (1-\alpha)w) \le \alpha f(x, y) + (1-\alpha) f(z, w)$$

holds, for all $(x, y), (z, w) \in \Delta$ and $\alpha \in [0, 1]$.

A function $f : \Delta \to \mathbb{R}$ is said to be convex on the co-ordinates on Δ if the partial mappings $f_y : [a, b] \to \mathbb{R}$, $f_y(u) = f(u, y)$ and $f_x : [c, d] \to \mathbb{R}$, $f_x(v) = f(x, v)$ are convex where defined for all $x \in [a, b]$, $y \in [c, d]$.

In the same article, Dragomir established the following Hadamard-type inequalities for convex functions on the co-ordinates in a rectangle from the plane $\mathbb{R}^2$:

Theorem 3. *Suppose $f : \Delta = [a, b] \times [c, d] \subseteq [0, \infty) \to \mathbb{R}$ is convex function on the co-ordinates on Δ. Then one has the inequalities:*

$$\begin{aligned} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) &\le \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y)\,dy\,dx \\ &\le \frac{f(a, c) + f(b, c) + f(a, d) + f(b, d)}{4} \end{aligned} \tag{1.6}$$

The concept of s-convex functions on the co-ordinates in both sense was introduced by Alomari and Darus [20,21]:

Definition 6. *Consider the bidimensional interval $\Delta =: [a, b] \times [c, d]$ in $[0, \infty)^2$ with $a < b$ and $c < d$. The mapping $f : \Delta \to \mathbb{R}$ is s-convex in the first sense (in the second sense) on Δ if*

$$f(\alpha x + \beta z,\ \alpha y + \beta w) \le \alpha^s f(x,\ y) + \beta^s f(z,\ w)$$

holds for all $(x, y), (z, w) \in \Delta$, $\alpha, \beta \ge 0$ with $\alpha^s + \beta^s = 1$ $(\alpha + \beta = 1)$ and for some fixed $s \in (0, 1]$. We write $f \in K_s^i (i = 1, 2)$ which means that f is s-convex in the first sense when $i = 1$, (in the second sense when $i = 2$).

A function $f : \Delta =: [a, b] \times [c, d] \subseteq [0, \infty)^2 \to \mathbb{R}$ is called s-convex in first sense (in the second sense) on the co-ordinates on Δ if the partial mappings $f_y : [a, b] \to \mathbb{R}$, $f_y(u) = f(u, y)$ and $f_x : [c, d] \to \mathbb{R}$, $f_x(v) = f(x, v)$, are s-convex in the first sense (in the second sense) for all $y \in [c, d]$, $x \in [a, b]$, and $s \in (0, 1]$, i.e, the partial mappings f_y and f_x are s-convex in the first sense (second sense) with some fixed $s \in (0, 1]$.

In [20], Alomari and Darus proved the following inequalities for s-convex functions (in the second sense) on the co-ordinates in a rectangle from the plane $\mathbb{R}^2$:

Theorem 4. *Suppose $f : \Delta = [a, b] \times [c, d] \subseteq [0, \infty) \to \mathbb{R}$ is s-convex function (in the second sense) on the co-ordinates on Δ. Then one has the inequalities:*

$$\begin{aligned} 4^{s-1} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) &\le \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y)\,dy\,dx \\ &\le \frac{f(a, c) + f(b, c) + f(a, d) + f(b, d)}{(s+1)^2} \end{aligned} \tag{1.7}$$

Also in [21] (see also [22]), Alomari and Darus established the following inequalities for s-convex functions (in the first sense) on the co-ordinates in a rectangle from the plane $\mathbb{R}^2$:

Theorem 5. *Suppose $f : \Delta = [a, b] \times [c, d] \subseteq [0, \infty) \to \mathbb{R}$ is s-convex function (in the first sense) on the co-ordinates on Δ. Then one has the inequalities:*

$$\begin{aligned} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) &\leq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)dydx \\ &\leq \frac{f(a,c) + sf(b,c) + sf(a,d) + s^2 f(b,d)}{(s+1)^2} \end{aligned} \tag{1.8}$$

The above inequalities are sharp.

In [23], Sarikaya et al. proved some Hadamard-type inequalities for co-ordinated convex functions as followings:

Theorem 6. *Let $f : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ be a partial differentiable mapping on $\Delta := [a, b] \times [c, d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$. If $\left|\frac{\partial^2 f}{\partial t \partial s}\right|$ is a convex function on the co-ordinates on Δ, then one has the inequalities:*

$$\begin{aligned} &\left| \frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4} \right. \\ &\left. \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)dxdy - A \right| \\ &\leq \frac{(b-a)(d-c)}{16} \left(\frac{\left|\frac{\partial^2 f}{\partial t \partial s}\right|(a,c) + \left|\frac{\partial^2 f}{\partial t \partial s}\right|(a,d) + \left|\frac{\partial^2 f}{\partial t \partial s}\right|(b,c) + \left|\frac{\partial^2 f}{\partial t \partial s}\right|(b,d)}{4} \right) \end{aligned}$$

where

$$A = \frac{1}{2}\left[\frac{1}{(b-a)} \int_a^b [f(x,c) + f(x,d)]dx + \frac{1}{(d-c)} \int_c^d [f(a,y)dy + f(b,y)]dy\right].$$

Theorem 7. *Let $f : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ be a partial differentiable mapping on $\Delta := [a, b] \times [c, d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$. If $\left|\frac{\partial^2 f}{\partial t \partial s}\right|^q$, $q > 1$, is a convex function on the co-ordinates on Δ, then one has the inequalities:*

$$\begin{aligned} &\left| \frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4} \right. \\ &\left. \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d (x,y)dxdy - A \right| \\ &\leq \frac{(b-a)(d-c)}{4(p+1)^{\frac{2}{p}}} \left(\frac{\left|\frac{\partial^2 f}{\partial t \partial s}\right|^q(a,c) + \left|\frac{\partial^2 f}{\partial t \partial s}\right|^q(a,d) + \left|\frac{\partial^2 f}{\partial t \partial s}\right|^q(b,c) + \left|\frac{\partial^2 f}{\partial t \partial s}\right|^q(b,d)}{4} \right)^{\frac{1}{q}} \end{aligned}$$

where

$$A = \frac{1}{2}\left[\frac{1}{(b-a)} \int_a^b [f(x,c) + f(x,d)]dx + \frac{1}{(d-c)} \int_c^d [f(a,y)dy + f(b,y)]dy\right]$$

and $\frac{1}{p} + \frac{1}{q} = 1$.

Theorem 8. *Let* $f : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ *be a partial differentiable mapping on* $\Delta := [a, b] \times [c, d]$ *in* $\mathbb{R}^2$ *with* $a < b$ *and* $c < d$. *If* $\left|\frac{\partial^2 f}{\partial t \partial s}\right|^q$, $q \geq 1$, *is a convex function on the co-ordinates on* Δ, *then one has the inequalities:*

$$\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\right.$$
$$\left.\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dxdy - A\right|$$
$$\leq \frac{(b-a)(d-c)}{16}\left(\frac{\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q(a,c)+\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q(a,d)+\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q(b,c)+\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q(b,d)}{4}\right)^{\frac{1}{q}}$$

where

$$A = \frac{1}{2}\left[\frac{1}{(b-a)}\int_a^b [f(x,c)+f(x,d)]dx + \frac{1}{(d-c)}\int_c^d [f(a,y)dy + f(b,y)]dy\right].$$

For refinements, counterparts, generalizations and new Hadamard-type inequalities see the articles [3,5,10-13,19-28]. In [25] (see also [22]), Alomari and Darus introduced new classes of *s*-convex functions on the co-ordinates as following:

Definition 7. *Consider the bidimensional interval* $\Delta =: [a, b] \times [c, d]$ *in* $[0, \infty)^2$ *with* $a < b$ *and* $c < d$. *The mapping* $f : \Delta \to \mathbb{R}$ *is s-convex in the first sense on* Δ *if there exist* $s_1, s_2 \in (0, 1]$ *such that* $s = \frac{s_1+s_2}{2}$,

$$f(\alpha x + \beta z, \alpha y + \beta w) \leq \alpha^{s_1} f(x,y) + \beta^{s_2} f(z,w)$$

holds for all $(x, y), (z, w) \in \Delta$ *with* $\alpha, \beta \geq 0$ *with* $\alpha^{s_1} + \beta^{s_2} = 1$*and for some fixed* $s_1, s_2 \in (0, 1]$. *We denote this class of functions by* $MWO^1_{s_1,s_2}$.

Definition 8. *Consider the bidimensional interval* $\Delta =: [a, b] \times [c, d]$ *in* $[0, \infty)^2$ *with* $a < b$ *and* $c < d$. *The mapping* $f : \Delta \to \mathbb{R}$ *is s-convex in the second sense on* Δ *if there exist* $s_1, s_2 \in (0, 1]$ *such that* $s = \frac{s_1+s_2}{2}$,

$$f(\alpha x + \beta z, \alpha y + \beta w) \leq \alpha^{s_1} f(x,y) + \beta^{s_2} f(z,w)$$

holds for all $(x, y), (z, w) \in \Delta$ *with* $\alpha, \beta \geq 0$ *with* $\alpha + \beta = 1$ *and for all fixed* $s_1, s_2 \in (0, 1]$. *We denote this class of functions by* $MWO^2_{s_1,s_2}$.

In [26], Pachpatte established some inequalities for product of convex functions as follows:

Theorem 9. *Let* $f, g : [a, b] \subseteq \mathbb{R} \to [0, \infty)$ *be convex functions on* $[a, b]$, $a < b$. *Then*

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx \leq \frac{1}{3}M(a,b) + \frac{1}{6}N(a,b) \tag{1.9}$$

and

$$2f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)g(x)dx + \frac{1}{6}M(a,b) + \frac{1}{3}N(a,b) \tag{1.10}$$

where $M(a, b) = f(a)g(a) + f(b)g(b)$ *and* $N(a, b) = f(a)g(b) + f(b)g(a)$.

Similar results for s-convex functions is due to Kirmaci et al. [13] as follows:

Theorem 10. *Let* $f, g : [a, b] \subseteq \mathbb{R} \to \mathbb{R}$ $a, b \in [0, \infty)$, $a < b$, *be functions such that g and fg are in* $L_1([a, b])$. *If f is convex and non-negative on* $[a, b]$ *and if g is s-convex on* $[a, b]$, *for some* $s \in (0, 1)$, *then*

$$\begin{aligned} &2^s f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)g(x)dx \\ &\le \frac{1}{(s+1)(s+2)}M(a,b) + \frac{1}{s+2}N(a,b) \end{aligned} \tag{1.11}$$

where $M(a, b) = f(a)g(a) + f(b)g(b)$ *and* $N(a, b) = f(a)g(b) + f(b)g(a)$.

Theorem 11. *Let* $f, g : [a, b] \subseteq \mathbb{R} \to \mathbb{R}$ $a, b \in [0, \infty)$, $a < b$, *be functions such that g and fg are in* $L_1([a, b])$. *If f is convex and non-negative on* $[a, b]$ *and if g is s-convex on* $[a, b]$ *for some* $s \in (0, 1)$, *then*

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx \le \frac{1}{s+2}M(a,b) + \frac{1}{(s+1)(s+2)}N(a,b) \tag{1.12}$$

where $M(a, b) = f(a)g(a) + f(b)g(b)$ *and* $N(a, b) = f(a)g(b) + f(b)g(a)$.

Theorem 12. *Let* $f, g : [a, b] \subseteq \mathbb{R} \to \mathbb{R}$ $a, b \in [0, \infty)$, $a < b$, *be functions such that f, g and fg are in* $L_1([a, b])$. *If f is* s_1*-convex and g is* s_2*-convex on* $[a, b]$ *for some fixed* s_1, $s_2 \in (0, 1)$, *then*

$$\begin{aligned} \frac{1}{b-a}\int_a^b f(x)g(x)dx &\le \frac{1}{s_1+s_2+1}M(a,b) + B(s_1+1, s_2+1)N(a,b) \\ &= \frac{1}{s_1+s_2+1}\left[M(a,b) + s_1 s_2 \frac{\Gamma(s_1)\Gamma(s_2)}{\Gamma(s_1+s_2+1)}N(a,b)\right] \end{aligned} \tag{1.13}$$

where $M(a, b) = f(a)g(a) + f(b)g(b)$ *and* $N(a, b) = f(a)g(b) + f(b)g(a)$.

In the last theorem Beta function of Euler type, defined by

$$B(x,y) = \int_0^t t^{x-1}(1-t)^{y-1}dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

has been used.

The main purpose of the present article is to establish new Hadamard-type inequalities similar to the above inequalities, but now for product of s-convex functions (in the second sense) on the co-ordinates in a rectangle from the plane $\mathbb{R}^2$.

2. Main results

We will start with the following theorem.

Theorem 13. *Let* $f, g : \Delta = [a, b] \times [c, d] \subseteq [0, \infty)^2 \to \mathbb{R}$, $a < b$, $c < d$, *be functions such that f, g and fg are in* $L^2(\Delta)$. *If f is non-negative and convex on the coordinates on* Δ *and if g is s-convex in the second sense on the co-ordinates on* Δ, *for all* s_1, $s_2 \in (0, 1)$, *such that* $s = \frac{s_1+s_2}{2}$, *then one has the inequality;*

$$\begin{aligned}&\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)g(x,y)dydx\\&\leq \frac{1}{2}(p^2+r^2)L(a,b,c,d)+\frac{1}{2}(pq+rt)M(a,b,c,d)+\frac{1}{2}(q^2+t^2)N(a,b,c,d)\end{aligned} \tag{2.1}$$

where

$$\begin{aligned}L(a,\ b,\ c,\ d)&=f(a,\ c)g(a,\ c)+f(b,\ c)g(b,\ c)+f(a,\ d)g(a,\ d)+f(b,\ d)g(b,\ d),\\ M(a,\ b,\ c,\ d)&=f(a,\ c)g(a,\ d)+f(a,\ d)g(a,\ c)+f(b,\ c)g(b,\ d)+f(b,\ d)g(b,\ c)\\&\quad+f(b,\ c)g(a,\ c)+f(b,\ d)g(a,\ d)+f(a,\ c)g(b,\ c)+f(a,\ d)g(b,\ d),\\ N(a,\ b,\ c,\ d)&=f(b,\ c)g(a,\ d)+f(b,\ d)g(a,\ c)+f(a,\ c)g(b,\ d)+f(a,\ d)g(b,\ c)\end{aligned}$$

and

$$p=\frac{1}{s_2+2},\quad q=\frac{1}{(s_2+1)(s_2+2)},\quad r=\frac{1}{s_1+2},\quad t=\frac{1}{(s_1+1)(s_1+2)}.$$

Proof. Since f is convex and g is s-convex in the second sense on the co-ordinates on Δ. Therefore the partial mappings

$$f_y:[a,b]\to[0,\infty),\quad f_y(x)=f(x,y)$$

and

$$f_x:[c,d]\to[0,\infty),\quad f_x(y)=f(x,y)$$

are convex and non-negative on $[a, b]$ and $[c, d]$, respectively. The partial mappings

$$g_y:[a,b]\to[0,\infty),\quad g_y(x)=g(x,y)$$

and

$$g_x:[c,d]\to[0,\infty),\quad g_x(y)=g(x,y)$$

are s_1-, s_2-convex on $[a, b]$ and $[c, d]$, respectively, for all $x \in [a, b]$, $y \in [c, d]$, for all $s_1, s_2 \in (0, 1]$, such that $s = \frac{s_1+s_2}{2}$. Now by applying $f_x(y)g_x(y)$ to (1.12) on $[c, d]$, we get

$$\begin{aligned}\frac{1}{d-c}\int_c^d f_x(y)g_x(y)dy &\leq p[f_x(c)g_x(c)+f_x(d)g_x(d)]\\&\quad+\ q[f_x(c)g_x(d)+f_x(d)g_x(c)].\end{aligned}$$

That is

$$\begin{aligned}\frac{1}{d-c}\int_c^d f(x,\ y)g(x,\ y)dy &\leq p[f(x,\ c)g(x,\ c)+f(x,\ d)g(x,\ d)]\\&\quad+\ q[f(x,\ c)g(x,\ d)+f(x,\ d)g(x,\ c)].\end{aligned}$$

Integrating over $[a, b]$ with respect to x and dividing both sides by $b - a$, we have

$$\begin{aligned}&\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)g(x,y)dydx\\&\leq p\left[\frac{1}{b-a}\int_a^b f(x,c)g(x,c)dx+\frac{1}{b-a}\int_a^b f(x,d)g(x,d)dx\right]\\&+q\left[\frac{1}{b-a}\int_a^b f(x,c)g(x,d)dx+\frac{1}{b-a}\int_a^b f(x,d)g(x,c)dx\right].\end{aligned} \tag{2.2}$$

Now by applying (1.12) to each integral on right-hand side of (2.2) again, we get

$$\frac{1}{b-a}\int_a^b f(x,\ c)g(x,\ c)dx \le p[f(a,\ c)g(a,\ c)+f(b,\ c)g(b,\ c)] + q[f(a,\ c)g(b,\ c)+f(b,\ c)g(a,\ c)].$$

$$\frac{1}{b-a}\int_a^b f(x,\ d)g(x,\ d)dx \le p[f(a,\ d)g(a,\ d)+f(b,\ d)g(b,\ d)] + q[f(a,\ d)g(b,\ d)+f(b,\ d)g(a,\ d)].$$

$$\frac{1}{b-a}\int_a^b f(x,\ c)g(x,\ d)dx \le p[f(a,\ c)g(a,\ d)+f(b,\ c)g(b,\ d)] + q[f(a,\ c)g(b,\ d)+f(b,\ c)g(a,\ d)].$$

$$\frac{1}{b-a}\int_a^b f(x,\ d)g(x,\ c)dx \le p[f(a,\ d)g(a,\ c)+f(b,\ d)g(b,\ c)] + q[f(a,\ d)g(b,\ c)+f(b,\ d)g(a,\ c)].$$

On substitution of these inequalities in (2.2), we obtain

$$\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,\ y)g(x,\ y)dydx \le p^2L(a,\ b,\ c,\ d)+pqM(a,\ b,\ c,\ d)+q^2N(a,\ b,\ c,\ d). \tag{2.3}$$

Similarly, if we apply $f_y(x)g_y(x)$ to (1.12) on $[a, b]$, we get the following result:

$$\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,\ y)g(x,\ y)dydx \le r^2L(a,\ b,\ c,\ d)+rtM(a,\ b,\ c,\ d)+t^2N(a,\ b,\ c,\ d) \tag{2.4}$$

Adding the inequalities (2.3), (2.4) and dividing by 2, we get (2.1). □

Theorem 14. *Let $f, g : \Delta = [a, b] \times [c, d] \subseteq \mathbb{R}^2 \to \mathbb{R}$, $a < b$, $c < d$, be functions such that f, g, and fg are in $L^2(\Delta)$. If f is non-negative and convex on the co-ordinates on Δ and if g is s-convex on the co-ordinates on Δ, for all $s_1, s_2 \in (0, 1)$, such that $s = \frac{s_1+s_2}{2}$, then one has the inequality;*

$$2^{s_1+s_2}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)g\left(\frac{a+b}{2},\frac{c+d}{2}\right) \le \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,\ y)g(x,\ y)dydx + \frac{1}{2}(q^2+t^2+2pt+2qr)L(a,\ b,\ c,\ d) + \frac{1}{2}(pq+rt+2qt+2rp)M(a,\ b,\ c,\ d) + \frac{1}{2}(p^2+r^2+2pt+2rq)N(a,\ b,\ c,\ d) \tag{2.5}$$

where $L(a, b, c, d)$, $M(a, b, c, d)$, $N(a, b, c, d)$, p, q, r and t as in Theorem 13.

Proof. Applying $2^{s_1}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)g\left(\frac{a+b}{2},\frac{c+d}{2}\right)$ to (1.11) and multiplying both sides by 2^{s_2}, we get

$$
\begin{aligned}
&2^{s_1+s_2} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) g\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\
&\leq \frac{2^{s_2}}{b-a} \int_a^b f\left(x, \frac{c+d}{2}\right) g\left(x, \frac{c+d}{2}\right) dx \\
&+ t\left[2^{s_2} f\left(a, \frac{c+d}{2}\right) g\left(a, \frac{c+d}{2}\right) + 2^{s_2} f\left(b, \frac{c+d}{2}\right) g\left(b, \frac{c+d}{2}\right)\right] \\
&+ r\left[2^{s_2} f\left(a, \frac{c+d}{2}\right) g\left(b, \frac{c+d}{2}\right) + 2^{s_2} f\left(b, \frac{c+d}{2}\right) g\left(a, \frac{c+d}{2}\right)\right].
\end{aligned} \tag{2.6}
$$

Similarly, by applying $2^{s_2} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) g\left(\frac{a+b}{2}, \frac{c+d}{2}\right)$ to (1.11) and multiplying both sides by 2^{s_1}, we get

$$
\begin{aligned}
&2^{s_1+s_2} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) g\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\
&\leq \frac{2^{s_1}}{d-c} \int_c^d f\left(\frac{a+b}{2}, y\right) g\left(\frac{a+b}{2}, y\right) dy \\
&+ q\left[2^{s_1} f\left(\frac{a+b}{2}, c\right) g\left(\frac{a+b}{2}, c\right) + 2^{s_1} f\left(\frac{a+b}{2}, d\right) g\left(\frac{a+b}{2}, d\right)\right] \\
&+ p\left[2^{s_1} f\left(\frac{a+b}{2}, c\right) g\left(\frac{a+b}{2}, d\right) + 2^{s_1} f\left(\frac{a+b}{2}, d\right) g\left(\frac{a+b}{2}, c\right)\right].
\end{aligned} \tag{2.7}
$$

Adding (2.6) and (2.7), we have

$$
\begin{aligned}
&2^{s_1+s_2+1} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) g\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\
&\leq \frac{2^{s_2}}{b-a} \int_a^b f\left(x, \frac{c+d}{2}\right) g\left(x, \frac{c+d}{2}\right) dx + \frac{2^{s_1}}{d-c} \int_c^d f\left(\frac{a+b}{2}, y\right) g\left(\frac{a+b}{2}, y\right) dy \\
&+ t\left[2^{s_2} f\left(a, \frac{c+d}{2}\right) g\left(a, \frac{c+d}{2}\right) + 2^{s_2} f\left(b, \frac{c+d}{2}\right) g\left(b, \frac{c+d}{2}\right)\right] \\
&+ r\left[2^{s_2} f\left(a, \frac{c+d}{2}\right) g\left(b, \frac{c+d}{2}\right) + 2^{s_2} f\left(b, \frac{c+d}{2}\right) g\left(a, \frac{c+d}{2}\right)\right] \\
&+ q\left[2^{s_1} f\left(\frac{a+b}{2}, c\right) g\left(\frac{a+b}{2}, c\right) + 2^{s_1} f\left(\frac{a+b}{2}, d\right) g\left(\frac{a+b}{2}, d\right)\right] \\
&+ p\left[2^{s_1} f\left(\frac{a+b}{2}, c\right) g\left(\frac{a+b}{2}, d\right) + 2^{s_1} f\left(\frac{a+b}{2}, d\right) g\left(\frac{a+b}{2}, c\right)\right].
\end{aligned} \tag{2.8}
$$

Applying (1.11) to each term within the brackets, we get

$$
\begin{aligned}
2^{s_2} f\left(a, \frac{c+d}{2}\right) g\left(a, \frac{c+d}{2}\right) \leq{}& \frac{1}{d-c} \int_c^d f(a, y) g(a, y) dy \\
&+ t\left[f(a, c) g(a, c) + f(a, d) g(a, d)\right] \\
&+ r\left[f(a, c) g(a, d) + f(a, d) g(a, c)\right].
\end{aligned}
$$

$$
\begin{aligned}
2^{s_2} f\left(b, \frac{c+d}{2}\right) g\left(b, \frac{c+d}{2}\right) \leq{}& \frac{1}{d-c} \int_c^d f(b, y) g(b, y) dy \\
&+ t\left[f(b, c) g(b, c) + f(b, d) g(b, d)\right] \\
&+ r\left[f(b, c) g(b, d) + f(b, d) g(b, c)\right].
\end{aligned}
$$

$$
\begin{aligned}
2^{s_2} f\left(a, \frac{c+d}{2}\right) g\left(b, \frac{c+d}{2}\right) \leq{}& \frac{1}{d-c} \int_c^d f(a, y) g(b, y) dy \\
&+ t\left[f(a, c) g(b, c) + f(a, d) g(b, d)\right] \\
&+ r\left[f(a, c) g(b, d) + f(a, d) g(b, c)\right].
\end{aligned}
$$

$$
\begin{aligned}
2^{s_2} f\left(b, \frac{c+d}{2}\right) g\left(a, \frac{c+d}{2}\right) \leq{}& \frac{1}{d-c} \int_c^d f(b, y) g(a, y) dy \\
&+ t\left[f(b, c) g(a, c) + f(b, d) g(a, d)\right] \\
&+ r\left[f(b, c) g(a, d) + f(b, d) g(a, c)\right].
\end{aligned}
$$

$$2^{s_1} f\left(\frac{a+b}{2}, c\right) g\left(\frac{a+b}{2}, c\right) \leq \frac{1}{b-a}\int_a^b f(x, c)g(x, c)dx + q[f(a, c)g(a, c) + f(b, c)g(b, c)] + p[f(a, c)g(b, c) + f(b, c)g(a, c)].$$

$$2^{s_1} f\left(\frac{a+b}{2}, d\right) g\left(\frac{a+b}{2}, d\right) \leq \frac{1}{b-a}\int_a^b f(x, d)g(x, d)dx + q[f(a, d)g(a, d) + f(b, d)g(b, d)] + p[f(a, d)g(b, d) + f(b, d)g(a, d)].$$

$$2^{s_1} f\left(\frac{a+b}{2}, c\right) g\left(\frac{a+b}{2}, d\right) \leq \frac{1}{b-a}\int_a^b f(x, c)g(x, d)dx + q[f(a, c)g(a, d) + f(b, c)g(b, d)] + p[f(a, c)g(b, d) + f(b, c)g(a, d)].$$

$$2^{s_1} f\left(\frac{a+b}{2}, d\right) g\left(\frac{a+b}{2}, c\right) \leq \frac{1}{b-a}\int_a^b f(x, d)g(x, c)dx + q[f(a, d)g(a, c) + f(b, d)g(b, c)] + p[f(a, d)g(b, c) + f(b, d)g(a, c)].$$

Substituting these inequalities in (2.8) and simplifying, we obtain

$$\begin{aligned}
&2^{s_1+s_2+1} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) g\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\
&\leq \frac{2^{s_2}}{b-a}\int_a^b f\left(x, \frac{c+d}{2}\right) g\left(x, \frac{c+d}{2}\right) dx \\
&+\frac{2^{s_1}}{d-c}\int_c^d f\left(\frac{a+b}{2}, y\right) g\left(\frac{a+b}{2}, y\right) dy \\
&+t\frac{1}{d-c}\int_c^d f(a, y)g(a, y)dy + t\frac{1}{d-c}\int_c^d f(b, y)g(b, y)dy \\
&+r\frac{1}{d-c}\int_c^d f(a, y)g(b, y)dy + r\frac{1}{d-c}\int_c^d f(b, y)g(a, y)dy \\
&+q\frac{1}{b-a}\int_a^b f(x, c)g(x, c)dx + q\frac{1}{b-a}\int_a^b f(x, d)g(x, d)dx \\
&+p\frac{1}{b-a}\int_a^b f(x, c)g(x, d)dx + p\frac{1}{b-a}\int_a^b f(x, d)g(x, c)dx \\
&+(q^2+t^2)L(a, b, c, d) + (pq+rt)M(a, b, c, d) + (p^2+r^2)N(a, b, c, d).
\end{aligned} \tag{2.9}$$

Now by applying $2^{s_1} f\left(\frac{a+b}{2}, y\right) g\left(\frac{a+b}{2}, y\right)$ to (1.11), integrating over $[c, d]$ and dividing both sides by $d - c$, we get

$$\begin{aligned}
&\frac{2^{s_1}}{d-c}\int_c^d f\left(\frac{a+b}{2}, y\right) g\left(\frac{a+b}{2}, y\right) dy \\
&\leq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x, y)g(x, y)dxdy \\
&+t\frac{1}{d-c}\int_c^d f(a, y)g(a, y)dy + t\frac{1}{d-c}\int_c^d f(b, y)g(b, y)dy \\
&+r\frac{1}{d-c}\int_c^d f(a, y)g(b, y)dy + r\frac{1}{d-c}\int_c^d f(b, y)g(a, y)dy.
\end{aligned} \tag{2.10}$$

Again by applying $2^{s_2} f\left(x, \frac{c+d}{2}\right) g\left(x, \frac{c+d}{2}\right)$ to (1.11), integrating over $[a, b]$ and dividing both sides by $b - a$, we get

$$\begin{aligned}
&\frac{2^{s_2}}{b-a}\int_a^b f\left(x, \frac{c+d}{2}\right) g\left(x, \frac{c+d}{2}\right) dx \\
&\leq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x, y)g(x, y)dydx \\
&+q\frac{1}{b-a}\int_a^b f(x, c)g(x, c)dx + q\frac{1}{b-a}\int_a^b f(x, d)g(x, d)dx \\
&+p\frac{1}{b-a}\int_a^b f(x, c)g(x, d)dx + p\frac{1}{b-a}\int_a^b f(x, d)g(x, c)dx.
\end{aligned} \tag{2.11}$$

Adding (2.10) and (2.11), we have

$$\begin{aligned}
&\frac{2^{s_2}}{b-a}\int_c^d f\left(x, \frac{c+d}{2}\right) g\left(x, \frac{c+d}{2}\right) dx + \frac{2^{s_1}}{d-c}\int_c^d f\left(\frac{a+b}{2}, y\right) g\left(\frac{a+b}{2}, y\right) dy \\
&\leq \frac{2}{(b-a)(d-c)}\int_a^b\int_c^d f(x, y)g(x, y)dydx \\
&+t\frac{1}{d-c}\int_c^d f(a, y)g(a, y)dy + t\frac{1}{d-c}\int_c^d f(b, y)g(b, y)dy \\
&+r\frac{1}{d-c}\int_c^d f(a, y)g(b, y)dy + r\frac{1}{d-c}\int_c^d f(b, y)g(a, y)dy \\
&+q\frac{1}{b-a}\int_a^b f(x, c)g(x, c)dx + q\frac{1}{b-a}\int_a^b f(x, d)g(x, d)dx \\
&+p\frac{1}{b-a}\int_a^b f(x, c)g(x, d)dx + p\frac{1}{b-a}\int_a^b f(x, d)g(x, c)dx.
\end{aligned} \tag{2.12}$$

Therefore from (2.9) and (2.12), we obtain

$$\begin{aligned}
&2^{s_1+s_2+1} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) g\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\
&\leq \frac{2}{(b-a)(d-c)}\int_a^b f(x, y)g(x, y)dxdy \\
&+2t\frac{1}{d-c}\int_c^d f(a, y)g(a, y)dy + 2t\frac{1}{d-c}\int_c^d f(b, y)g(b, y)dy \\
&+2r\frac{1}{d-c}\int_c^d f(a, y)g(b, y)dy + 2r\frac{1}{d-c}\int_c^d f(b, y)g(a, y)dy \\
&+2q\frac{1}{b-a}\int_a^b f(x, c)g(x, c)dx + 2q\frac{1}{b-a}\int_a^b f(x, d)g(x, d)dx \\
&+2p\frac{1}{b-a}\int_a^b f(x, c)g(x, d)dx + 2p\frac{1}{b-a}\int_a^b f(x, d)g(x, c)dx \\
&+(q^2+t^2)L(a, b, c, d) + (pq+rt)M(a, b, c, d) + (p^2+r^2)N(a, b, c, d).
\end{aligned} \tag{2.13}$$

By applying (1.12) to each of the integral in (2.13) and simplifying, we get

$$\begin{aligned}
&2^{s_1+s_2+1} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) g\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\\
&\le \frac{2}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) g(x, y) dy dx\\
&+(q^2+t^2+2pt+2qr)L(a, b, c, d)+(pq+rt+2qt+2rp)M(a, b, c, d)\\
&+(p^2+r^2+2pt+2rq)N(a, b, c, d)
\end{aligned}$$

Dividing both sides by 2, we get required result. □

Theorem 15. *Let f, g* : $\Delta = [a, b] \times [c, d] \subseteq \mathbb{R}^2 \to \mathbb{R}$, $a < b$, $c < d$, *be functions such that f, g, and fg are in* $L_2(\Delta)$. *If f is* s_1*-convex on the co-ordinates on* Δ *and g is* s_2*-convex on the co-ordinates on* Δ, *for some fixed* $s_{11}, s_{12}, s_{21}, s_{22} \in (0, 1)$, *such that* $s_1 = \frac{s_{11}+s_{12}}{2}$, $s_2 = \frac{s_{21}+s_{22}}{2}$, *then one has the following inequality;*

$$\begin{aligned}
&\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) g(x, y) dy dx\\
&\le \frac{1}{2}(p_1^2+r_1^2)L(a, b, c, d)+\frac{1}{2}(p_1 q_1+r_1 t_1)M(a, b, c, d)+\frac{1}{2}(q_1^2+t_1^2)N(a, b, c, d)
\end{aligned} \tag{2.14}$$

where $L(a, b, c, d)$, $M(a, b, c, d)$, *and* $N(a, b, c, d)$ *as defined in Theorem 13 and* $p_1 = \frac{1}{s_{12}+s_{22}+1}$, $q_1 = B(s_{12}+1, s_{22}+1)$, $r_1 = \frac{1}{s_{11}+s_{21}+1}$, $t_1 = B(s_{11}+1, s_{21}+1)$.

Proof. By a similar way to Theorem 13 with $p_1 = \frac{1}{s_{12}+s_{22}+1}$, $q_1 = B(s_{12}+1, s_{22}+1)$, $r_1 = \frac{1}{s_{11}+s_{21}+1}$, $t_1 = B(s_{11}+1, s_{21}+1)$ and thus (2.14) is established. □

Theorem 16. *Let f, g* : $\Delta = [a, b] \times [c, d] \subseteq \mathbb{R}^2 \to \mathbb{R}$, $a < b$, $c < d$, *be functions such that f, g, and fg are in* $L_2(\Delta)$. *If f is* s_1*-convex on the co-ordinates on* Δ *and g is* s_2*-convex on the co-ordinates on* Δ, *for some fixed* $s_{11}, s_{12}, s_{21}, s_{22} \in (0, 1)$, *such that* $s_1 = \frac{s_{11}+s_{12}}{2}$, $s_2 = \frac{s_{21}+s_{22}}{2}$, *then one has the following inequality;*

$$\begin{aligned}
&2^{s_{11}+s_{21}+s_{12}+s_{22}-2} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) g\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\\
&\le \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) g(x, y) dy dx\\
&+\frac{1}{2}(q_1^2+t_1^2+2p_1 t_1+2q_1 r_1)L(a, b, c, d)\\
&+\frac{1}{2}(p_1 q_1+r_1 t_1+2q_1 t_1+2r_1 p_1)M(a, b, c, d)\\
&+\frac{1}{2}(p_1^2+r_1^2+2p_1 t_1+2r_1 q_1)N(a, b, c, d)
\end{aligned} \tag{2.15}$$

where $L(a, b, c, d)$, $M(a, b, c, d)$, *and* $N(a, b, c, d)$ *as defined in Theorem 13 and* $p_1 = \frac{1}{s_{12}+s_{22}+1}$, $q_1 = B(s_{12}+1, s_{22}+1)$, $r_1 = \frac{1}{s_{11}+s_{21}+1}$, $t_1 = B(s_{11}+1, s_{21}+1)$.

Proof. By a similar way to Theorem 13 with $p_1 = \frac{1}{s_{12}+s_{22}+1}$, $q_1 = B(s_{12}+1, s_{22}+1)$, $r_1 = \frac{1}{s_{11}+s_{21}+1}$, $t_1 = B(s_{11}+1, s_{21}+1)$ the proof is completed □

Author details

[1]Department of Mathematics, K.K. Education Faculty, Ataturk University, Erzurum 25240, Turkey [2]Department of Mathematics, College of Science, University of Hail, Hail 2440, Saudi Arabia [3]Department of Mathematics, Faculty of Science and Letters, Ağri İbrahim Çeçen University, Ağri 04100, Turkey

Authors' contributions

MA and AOA carried out the design of the study and performed the analysis. MEO (adviser) participated in its design and coordination. All authors read and approved the final manuscript.

Competing interests

The authors declare that they have no competing interests.

Received: 4 May 2011 Accepted: 5 February 2012 Published: 5 February 2012

References

1. Godunova, EK, Levin, VI: Neravenstva dlja funkcii širokogo klassa, soderžašcego vypuklye, monotonnye i nekotorye drugie vidy funkcii. Vycislitel Mat i Mat Fiz Mežvuzov Sb Nauc Trudov, MGPI, Moskva. 138–142 (1985)
2. Mitrinović, DS, Pečarić, J, Fink, AM: Classical and New Inequalities in Analysis. Kluwer Academic Publishers, Dordrecht (1993)
3. Dragomir, SS, Pečarić, J, Persson, LE: Some inequalities of Hadamard type. Soochow J Math. **21**, 335–341 (1995)
4. Breckner, WW: Stetigkeitsaussagen für eine klasse verallgemeinerter konvexer funktionen in topologischen linearen Raumen. Publ Inst Math. **23**, 13–20 (1978)
5. Hudzik, H, Maligranda, L: Some remarks on *s*-convex functions. Aequationes Math. **48**, 100–111 (1994). doi:10.1007/BF01837981
6. Breckner, WW: Continuity of generalized convex and generalized concave set-valued functions. Rev Anal Numér Theor Approx. **22**, 39–51 (1993)
7. Orlicz, W: A note on modular spaces. I Bull Acad Polon Sci Math Astronom Phys. **9**, 157–162 (1961)
8. Matuszewska, W, Orlicz, W: A note on the theory of *s*-normed spaces of *φ*-integrable functions. Studia Math. **21**, 107–115 (1981)
9. Musielak, J: Orlicz spaces and modular spaces. In Lecture Notes in Mathematics, vol. 1034,Springer-Verlag, Berlin (1983)
10. Dragomir, SS, Fitzpatrick, S: The Hadamard's inequality for *s*-convex functions in the second sense. Demonstratio Math. **32**(4):687–696 (1999)
11. Dragomir, SS, Pearce, CEM: Selected topics on Hermite-Hadamard inequalities and applications. RGMIA monographs, Victoria University. http://www.staff.vu.edu.au/RGMIA/monographs/hermite-hadamard.html (2000)
12. Hussain, S, Bhatti, MI, Iqbal, M: Hadamard-type inequalities for *s*-convex functions. Punjab Univ J Math. **41**, 51–60 (2009)
13. Kirmaci, US, Bakula, MK, Özdemir, ME, Pečarić, J: Hadamard-type inequalities for *s*-convex functions. Appl Math Comput. **193**, 26–35 (2007). doi:10.1016/j.amc.2007.03.030
14. Varošanec, S: On *h*-convexity. J Math Anal Appl. **326**, 303–311 (2007). doi:10.1016/j.jmaa.2006.02.086
15. Bombardelli, M, Varošanec, S: Properties of *h*-convex functions related to the Hermite-Hadamard-Fejér inequalities. Comput Math Appl. **58**, 1869–1877 (2009). doi:10.1016/j.camwa.2009.07.073
16. Burai, P, Hazy, A: On approximately *h*-convex functions. J Convex Anal. **18**(2):447–454 (2011)
17. Sarikaya, MZ, Sağlam, A, Yildirim, H: On some Hadamard-type inequalities for *h*-convex functions. J Math Inequal. **2**(3):335–341 (2008)
18. Sarikaya, MZ, Set, E, Özdemir, ME: On some new inequalities of Hadamard type involving *h*-convex functions. Acta Math Univ Comenianae. **LXXIX**(2):265–272 (2010)
19. Dragomir, SS: On Hadamard's inequality for convex functions on the co-ordinates in a rectangle from the plane. Taiwanese J Math. **5**, 775–788 (2001)
20. Alomari, M, Darus, M: The Hadamard's inequality for *s*-convex function of 2-variables on the co-ordinates. Int J Math Anal. **2**(13):629–638 (2008)
21. Alomari, M, Darus, M: Co-ordinated *s*-convex function in the first sense with some Hadamard-type inequalities. Int J Contemp Math Sci. **32**, 1557–1567 (2008)
22. Alomari, M, Darus, M: Hadamard-type inequalities for *s*-convex functions. Int Math Forum. **3**(40):1965–1975 (2008)
23. Sarikaya, MZ, Set, E, Özdemir, ME, Dragomir, SS: New some Hadamard's type inequalities for co-ordinated convex functions. Tamsui Oxford J Math. (2011, in press)
24. Alomari, M, Darus, M: The Hadamard's inequality for *s*-convex function. Int J Math Anal. **2**(13):639–646 (2008)
25. Alomari, M, Darus, M: On co-ordinated *s*-convex functions. Int Math Forum. **3**(40):1977–1989 (2008)
26. Pachpatte, BG: On some inequalities for convex functions. RGMIA Res Rep Coll. **6**(E)http://ajmaa.org/RGMIA/monographs/hermite_hadamard.html (2003)
27. Özdemir, ME, Set, E, Sarikaya, MZ: Some new Hadamard's type inequalities for co-ordinated *m*-convex and (*α*, *m*)-convex functions. Hacettepe J Math Ist. **40**, 219–229 (2011)
28. Akdemir, AO, Ozdemir, ME: Some Hadamard-type inequalities for co-ordinated *P*-convex functions and Godunova-Levin functions. In AIP Conference Proceedings, vol. 1309, pp. 7–15.American Institute of Physics, Melville, New York (2010)

doi:10.1186/1029-242X-2012-21
Cite this article as: Özdemir *et al.*: On some Hadamard-type inequalities for product of two *s*-convex functions on the co-ordinates. *Journal of Inequalities and Applications* 2012 **2012**:21.