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Existence results of a kind of Sturm-Liouville type singular boundary value problem with non-linear boundary conditions

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Abstract

By using the Leggett-Williams fixed-point theorem, a series of Sturm-Liouville type boundary value problems are studied. We impose sufficient conditions on f and get the existence of at least three positive solutions. The interesting point is that the boundary conditions are non-linear.

Keywords: Sturm-Liouville type; positive solutions; boundary value problem; fixed point theorem; p-Laplacian

1 Introduction

In this paper, we consider the existence of triple positive solutions of a kind of four-point boundary value problem

$$(\phi_p(u'(t)))' + h(t)f(t, u(t)) = 0, \quad 0 < t < 1, \tag{1.1}$$

subject to one of the following boundary conditions:

$$g_1(u'(0)) - u(\xi) = 0, \qquad \beta u'(1) + u(\eta) = 0, \tag{1.2}$$

$$\alpha u'(0) - u(\xi) = 0, \qquad g_2(u'(1)) + u(\eta) = 0, \tag{1.3}$$

where $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $\alpha \geq \xi$, $\beta \geq 1 - \eta$, $0 < \xi < \eta < 1$, and g_1, g_2 satisfy

(H_1) (i) $g_1 \in C((-\infty, +\infty), (-\infty, +\infty))$, and $g_1(0) = 0$;
(ii) $g_1(x) \geq \xi x$ for $x \in [0, +\infty)$;
(iii) g_1 is increasing on x.

(H_2) (i) $g_2 \in C((-\infty, +\infty), (-\infty, +\infty))$ and g_2 is odd;
(ii) $g_2(x) \geq (1 - \eta)x$ for $x \in [0, +\infty)$;
(iii) g_2 is increasing on x.

We will assume throughout:

(C_1) $h(t) \in C((0, 1), (0, +\infty))$, which implies that $h(t)$ may be singular at $t = 0$ or/and $t = 1$. $0 < \int_0^1 h(t)\,dt < +\infty$, and $h(t)$ is not identically zero on any subinterval of $[0, 1]$.

(C_2) $f \in C([0, 1] \times [0, +\infty), [0, +\infty))$, $f \not\equiv 0$.

Boundary value problems (BVPs) for ordinary differential equations play a very important role in both theory and application. They are used to describe a large number of physical, biological and chemical phenomena. And in analyzing these non-linear phenomena, many mathematical models give rise to the problems for which only nonnegative solutions make sense. Therefore, since the publication of the monograph *Positive Solutions of Operator Equations* in 1964 by the academician Krasnosel'skii, many research articles on the theory of positive solutions of nonlinear problems have appeared [1–5]. In this vast field of research, particular attention has been focused on the BVPs with linear boundary conditions. However, BVPs with non-linear boundary conditions have not received much attention. Obviously, studying such kind of BVPs is more practical but more challenging. The main challenge is that it is not easy to construct an operator and, in turn, to get its solutions.

In [6], Kong *et al.* studied

$$\big(\Phi_p\big(u'(t)\big)\big)' + q(t)f\big(u(t)\big) = 0, \quad 0 < t < 1 \tag{1.4}$$

subject to one of the following non-linear boundary conditions:

$$\begin{aligned} &u(0) - g_1\big(u'(0)\big) = 0, \qquad u(1) = 0, \\ &u'(0) = 0, \qquad u(1) + g_2\big(u'(1)\big) = 0, \end{aligned} \tag{1.5}$$

where $\Phi_p(v) = |v|^{p-2}v$, $p > 1$, $q(t)$ is a nonnegative measurable function, $f(u)$ is a nonnegative continuous function on $[0, +\infty)$, and $g_1(v)$, $g_2(v)$ are all continuous functions defined on $(-\infty, +\infty)$. By using the theory of fixed point index, the authors get the existence of at least two positive solutions. For more references about the classical Sturm-Liouville BVP, we refer the readers to [7–17].

BC '$u'(0) - \alpha(u(\xi)) = 0$, $u'(1) + \beta(u(\eta)) = 0$ $(*)$' which is the special case of BC (1.2) and (1.3) can be called Sturm-Liouville type BC, the reason is that when $\xi = 0$, $\eta = 1$, BC $(*)$ become the classic Sturm-Liouville BC. To the best knowledge of the authors, few papers have dealt with equation (1.1) subject to the non-linear BC (1.2) and (1.3). So to fill this gap, motivated by the results mentioned above, in this paper we consider BVP (1.1), (1.2) and (1.1), (1.3). At the end of this paper, we also give its generalization which is more universal.

On the other hand, the main tool used in this paper is the Leggett-Williams fixed point theorem, which is different from [6]. In order to apply this fixed point theorem to our paper, we need to make use of some techniques such as the definition of the cone and the zooming of inequalities *etc.* The interesting point in this paper is that the boundary condition is the non-linear Sturm-Liouville type boundary condition.

2 Background material and the fixed point theorem

In this section, for convenience, we present the main definitions and the theorem that will be used in this paper.

Definition 2.1 Let E be a real Banach space. A nonempty closed convex set $P \subset E$ is called a cone if it satisfies the following two conditions:

(i) $\lambda u \in P$ for all $\lambda \geq 0$,
(ii) $u, -u \in P$ implies $u = 0$.

Every cone $P \subset E$ induces an ordering in E given by $x \geq y$ if and only if $y - x \in P$.

Definition 2.2 The map α is said to be a nonnegative concave continuous function provided that $\alpha : P \to [0, \infty)$ is continuous and

$$\alpha\big(\lambda x + (1-\lambda)y\big) \geq \lambda\alpha(x) + (1-\lambda)\alpha(y)$$

for all $x, y \in P$ and $0 \leq \lambda \leq 1$.

Let a, b, r be constants, $P_r = \{u \in P : \|u\| < r\}$, $P(\alpha, b, d) = \{u \in P : a \leq \alpha(u), \|u\| \leq b\}$.

Theorem 2.1 (Leggett-Williams [18]) *Let $A : \overline{P}_c \to \overline{P}_c$ be a completely continuous map and α be a nonnegative continuous concave functional on P such that for $\forall u \in \overline{P}_c$, there holds $\alpha u \leq \|u\|$. Suppose there exist a, b, d with $0 < a < b < d \leq c$ such that*

(i) $\{u \in P(\alpha, b, d) : \alpha u > b\} \neq \emptyset$ and $\alpha(Au) > b$ for all $u \in P(\alpha, b, d)$;

(ii) $\|Au\| < a$ for all $u \in \overline{P}_a$;

(iii) $\alpha(Au) > b$ for all $u \in P(\alpha, b, d)$ with $\|Au\| > d$.

Then A has at least three fixed points u_1, u_2, u_3 satisfying

$$\|u_1\| < a, \qquad b < \alpha(u_2), \qquad \|u_3\| > a, \quad \textit{and} \quad \|u_3\| < b.$$

In what follows, by ϕ_q we denote the inverse to ϕ_p, where $\frac{1}{p} + \frac{1}{q} = 1$.

3 Existence results for BVP (1.1), (1.2)

3.1 Some useful lemmas

Lemma 3.1 *Suppose that $v \in C[0,1]$, $v(t) \geq 0$, $v(t) \not\equiv 0$ on any subinterval of $[0,1]$. Then BVP*

$$\begin{cases} (\phi_p(u'(t)))' + v(t) = 0, & 0 < t < 1, \\ g_1(u'(0)) - u(\xi) = 0, & \beta u'(1) + u(\eta) = 0, \end{cases} \tag{3.1}$$

has the unique solution

$$u(t) = \begin{cases} g_1(\phi_q(\int_0^\sigma v(s)\,\mathrm{d}s)) + \int_\xi^t \phi_q(\int_s^\sigma v(\tau)\,\mathrm{d}\tau)\,\mathrm{d}s, & (\diamondsuit) \quad t \in [0,\sigma], \\ \beta\phi_q(\int_\sigma^1 v(s)\,\mathrm{d}s) + \int_t^\eta \phi_q(\int_\sigma^s v(\tau)\,\mathrm{d}\tau)\,\mathrm{d}s, & (\heartsuit) \quad t \in [\sigma,1], \end{cases} \tag{3.2}$$

where σ is a solution of the equation

$$A_1(t) - A_2(t) = 0, \quad t \in [0,1]. \tag{3.3}$$

And $A_1(t)$, $A_2(t)$ are defined as follows:

$$A_1(t) = g_1\left(\phi_q\left(\int_0^t v(s)\,\mathrm{d}s\right)\right) + \int_\xi^t \phi_q\left(\int_s^t v(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s,$$

$$A_2(t) = \beta\phi_q\left(\int_t^1 v(s)\,\mathrm{d}s\right) + \int_t^\eta \phi_q\left(\int_t^s v(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s.$$

Proof At the beginning, we show that (3.3) has a unique solution.

$$\begin{aligned}
&A_1(t) - A_2(t) \\
&\quad = g_1\left(\phi_q\left(\int_0^t \nu(s)\,\mathrm{d}s\right)\right) + \int_t^{\xi} \phi_q\left(\int_t^s \nu(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s \\
&\qquad - \beta\phi_q\left(\int_t^1 \nu(s)\,\mathrm{d}s\right) - \int_t^{\eta} \phi_q\left(\int_t^s \nu(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s \\
&\quad = g_1\left(\phi_q\left(\int_0^t \nu(s)\,\mathrm{d}s\right)\right) + \int_{\xi}^{\eta} \phi_q\left(\int_s^t \nu(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s + \beta\phi_q\left(\int_1^t \nu(s)\,\mathrm{d}s\right).
\end{aligned}$$

Obviously, $A_1(t) - A_2(t)$ is strictly increasing on $t \in [0,1]$. At the same time, we can notice that $A_1(0) - A_2(0) < 0$, and $A_1(1) - A_2(1) > 0$. Further, $A_1(t) - A_2(t)$ is continuous, thus there must exist one point $\sigma \in (0,1)$ such that $A_1(t) - A_1(t) = 0$, which means that $A_1(t)$, $A_2(t)$ must intersect at a unique point σ.

Next, we intend to prove that the solution to BVP (3.1) can be expressed as (3.2). Let u be a solution of BVP (3.1). $(\phi_p(u'(t)))' = -\nu(t) \leq 0$ implies $u''(t) \leq 0$, together with the boundary condition, we claim that there must exists one point $\sigma \in (0,1)$ such that $u'(\sigma) = 0$. (It can be easily checked that σ cannot be 0 or 1. In fact, if $u'(0) = 0$, we have $u(\xi) = 0$, $u'(1) < 0$, $u(\eta) > 0$, which deduces a contradiction.) If not, without loss of generality, we suppose that $x'(t) < 0$ for all $t \in (0,1)$. Then $u(t)$ must be decreasing on $t \in (0,1)$, using the boundary condition in equation (3.1), we have $u(\xi) = g_1(u'(0)) < 0$, $u(\eta) = -\beta u'(1) > 0$, which is a contradiction

$$-\left(\phi_p\left(u'(t)\right)\right)' = \nu(t). \tag{3.4}$$

Integrate both sides of (3.4) from t to σ to conclude

$$\phi_p\left(u'(t)\right) = \int_t^{\sigma} \nu(s)\,\mathrm{d}s.$$

Then

$$u'(t) = \phi_q\left(\int_t^{\sigma} \nu(s)\,\mathrm{d}s\right). \tag{3.5}$$

Integrate both sides of (3.5) from 0 to t to conclude

$$u(t) = u(0) + \int_0^t \phi_q\left(\int_s^{\sigma} \nu(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s. \tag{3.6}$$

By (3.5) and (3.6), we have

$$\begin{aligned}
&u'(0) = \phi_q\left(\int_0^{\sigma} \nu(s)\,\mathrm{d}s\right), \\
&u(\xi) = u(0) + \int_0^{\xi} \phi_q\left(\int_s^{\sigma} \nu(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s.
\end{aligned}$$

Considering the BC in (3.1), we arrive at

$$u(0) = g_1\left(\phi_q\left(\int_0^\sigma \nu(s)\,\mathrm{d}s\right)\right) - \int_0^\xi \phi_q\left(\int_s^\sigma \nu(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s. \tag{3.7}$$

Substitute (3.7) into (3.6), we get

$$u(t) = g_1\left(\phi_q\left(\int_0^\sigma \nu(s)\,\mathrm{d}s\right)\right) + \int_\xi^t \phi_q\left(\int_\sigma^s \nu(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s. \tag{3.8}$$

By the similar argument, we can obtain

$$u(t) = \beta\phi_q\left(\int_\sigma^1 \nu(s)\,\mathrm{d}s\right) + \int_\sigma^\eta \phi_q\left(\int_\sigma^s \nu(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s. \tag{3.9}$$

Hence, for $t \in [0,1]$, the solution of (3.1) can be expressed as (3.2), which completes our proof. □

Remark 3.1 In fact, for $t \in [0,1]$, the solution of (3.1) can be expressed both by ($\diamondsuit$) and ($\heartsuit$), but just for convenience, we write it into two parts.

Lemma 3.2 *Let $\nu(t)$ satisfy all the conditions in Lemma 3.1, then the solution $u(t)$ of BVP (3.1) is concave on $t \in [0,1]$. What is more, $u(t) \geq 0$.*

Proof $(\phi_p(u'(t)))' = -\nu(t) \leq 0$ means $u''(t) \leq 0$, so $u(t)$ is concave on $t \in [0,1]$.

Next, we try to prove that $u(t) \geq 0$. By Lemma 3.1, we know that $u(t)$ can be expressed as (3.2). When $t \in [0,\sigma]$, since (H_1)(ii), we have

$$\begin{aligned}
(\diamondsuit) &= g_1\left(\phi_q\left(\int_0^\sigma \nu(s)\,\mathrm{d}s\right)\right) + \int_\xi^t \phi_q\left(\int_s^\sigma \nu(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s \\
&\geq \xi\phi_q\left(\int_0^\sigma \nu(s)\,\mathrm{d}s\right) - \int_0^\xi \phi_q\left(\int_s^\sigma \nu(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s + \int_0^t \phi_q\left(\int_s^\sigma \nu(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s \\
&\geq \int_0^\xi \phi_q\left(\int_0^\sigma \nu(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s - \int_0^\xi \phi_q\left(\int_0^\sigma \nu(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s + \int_0^t \phi_q\left(\int_s^\sigma \nu(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s \\
&\geq \int_0^t \phi_q\left(\int_s^\sigma \nu(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s \geq 0.
\end{aligned}$$

Further, when $t \in [\sigma,1]$, $\beta \geq 1-\eta$, similarly, we can get $(\heartsuit) \geq 0$. Thus $u(t) \geq 0$ for all $t \in [0,1]$. The proof is complete. □

3.2 Existence of positive solutions to BVP (1.1), (1.2)

In this subsection, we try to impose some growth conditions on the non-linear term f which allows us to apply Theorem 2.1 to establish the existence of triple positive solutions to BVP (1.1), (1.2).

Let $E = C^+[0,1] = \{u \in C[0,1], u(t) \geq 0\}$, and let it be endowed with the maximum norm $\|u\| = \max_{0\leq t\leq 1}|u(t)|$.

The cone $P \subset E$ is defined as

$$P = \{u \in E : u \text{ is concave on } t \in [0,1], \text{ there exists one point } \sigma \in (0,1) \\ \text{such that } u'(\sigma) = 0\}.$$

In our main results, we will make use of the following lemmas.

Lemma 3.3 *For $u \in P$, then $u(t) \geq \min\{t, 1-t\}\|u\|$.*

Proof For $t \in [0, \sigma]$, we have

$$\frac{u(t)}{t} \geq \frac{u(\sigma)}{\sigma} \geq \frac{\max_{0 \leq t \leq 1} |u(t)|}{1} \quad \Rightarrow \quad u(t) \geq t\|u\|.$$

Similarly, $u(t) \geq (1-t)\|u\|$ when $t \in [\sigma, 1]$. Thus $u(t) \geq \min\{t, 1-t\}\|u\|$, which completes our proof. □

Remark 3.2 From Lemma 3.3, we know that when $k \geq 3$, there holds $\min_{1/k \leq t \leq 1-1/k} u(t) \geq \frac{1}{k}\|u\|$.

Define $T_1 : P \to E$ as follows:

$$(T_1 u)(t) = \begin{cases} g_1(\phi_q(\int_0^\sigma h(s)f(s,u(s))\,\mathrm{d}s)) \\ \quad + \int_\xi^t \phi_q(\int_s^\sigma h(\tau)f(\tau,u(\tau))\,\mathrm{d}\tau)\,\mathrm{d}s, \quad t \in [0,\sigma], \\ \beta\phi_q(\int_\sigma^1 h(s)f(s,u(s))\,\mathrm{d}s) \\ \quad + \int_t^\eta \phi_q(\int_\sigma^s h(\tau)f(\tau,u(\tau))\,\mathrm{d}\tau)\,\mathrm{d}s, \quad t \in [\sigma,1]. \end{cases} \tag{3.10}$$

Lemma 3.4 *Suppose that (C_1) and (C_2) hold, then $T_1 : P \to P$ is completely continuous.*

Proof To justify this, we first show that $T_1 : P \to P$ is well defined. Let $u \in P$. Easily, we have $(T_1 u)(t) \geq 0$, $\beta(T_1 u)'(1) + (T_1 u)(\eta) = 0$. Next, we show that T_1 is concave on $t \in [0,1]$. By (3.10), we have

$$(T_1 x)'(t) = \begin{cases} \phi_q(\int_t^\sigma h(s)f(s,u(s)u)\,\mathrm{d}s), & t \in [0,\sigma], \\ -\phi_q(\int_\sigma^t h(s)f(s,u(s))\,\mathrm{d}s), & t \in [\sigma,1]. \end{cases} \tag{3.11}$$

Obviously, $(T_1 x)''(t) \leq 0$, this implies that T is concave on $t \in [0,1]$. Further, $(T_1 x)'(t) \geq 0$ on $t \in [0,\sigma]$, and $(T_1 x)'(t) \leq 0$ on $t \in [\sigma,1]$, $(T_1 x)'(\sigma) = 0$. Thus $T_1 : P \to P$ is well defined.

It is easy to prove that T_1 is continuous with respect to x. For more details, see [19, p.6].

For any bounded subset Ω of P, it can be easily verified that $T_1(\Omega)$ is bounded and equicontinuous. Thus by the Arzela-Ascoli theorem, $T_1(\Omega)$ is relatively compact, therefore, T_1 is completely continuous. □

Throughout, we suppose that $k \geq 3$ is big enough such that $\xi, \eta \in [\frac{1}{k}, 1-\frac{1}{k}]$. And the nonnegative continuous concave functional α_1 is defined on the cone P by

$$\alpha_1(x) = \min_{\frac{1}{k} \leq t \leq (1-\frac{1}{k})} |x(t)|, \quad \text{for } x \in P.$$

Let

$$M_1 = \beta\phi_q\left(\int_0^1 h(s)\,\mathrm{d}s\right) + \max\left\{\int_0^\eta \phi_q\left(\int_0^s h(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s, \int_\eta^1 \phi_q\left(\int_s^1 h(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s\right\},$$

$$m_1 = \frac{1}{k}\cdot\min\left\{\xi\phi_q\left(\int_{\frac{1}{k}}^{\frac{1}{2}} h(s)\,\mathrm{d}s\right) + \int_\xi^{\frac{1}{2}} \phi_q\left(\int_s^{\frac{1}{2}} h(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s,\right.$$

$$\left.\beta\phi_q\left(\int_{\frac{1}{2}}^{1-\frac{1}{k}} h(s)\,\mathrm{d}s\right) + \int_{\frac{1}{2}}^\eta \phi_q\left(\int_{\frac{1}{2}}^s h(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s\right\}.$$

Theorem 3.5 *Assume that* (H_1) *holds, there exist constants such that* $0 < a_1 < b_1 \le \min\{\frac{1}{k}, \frac{m_1}{M_1}\}c_1$. *Further suppose that*

(S_1) $f(t,u) < \phi_p(a_1/M_1)$, *for* $(t,u) \in [0,1] \times [0,a_1]$;
(S_2) $f(t,u) \le \phi_p(c_1/M_1)$, *for* $(t,u) \in [0,1] \times [0,c_1]$;
(S_3) $f(t,u) \ge \phi_p(b_1/m_1)$, *for* $(t,u) \in [\frac{1}{k}, 1-\frac{1}{k}] \times [b_1, kb_1]$.

Then boundary value problem (1.1), (1.2) has at least three positive solutions u_1, u_2, u_3 *such that*

$$\|u_1\| < a_1, \qquad \alpha_1(u_2) > b_1 \quad \text{and} \quad \|u_3\| > a_1, \qquad \alpha_1(u_3) < b_1. \tag{3.12}$$

Proof Problem (1.1), (1.2) has a solution $u = u(t)$ if and only if it solves the operator equation $u = T_1u$. Thus we set out to verify that the operator T_1 satisfies the Leggett-Williams fixed point Theorem 2.1, which will prove the existence of at least three fixed points of T_1.

Now, we will prove the main theorem step by step.

Step 1. We firstly show that $T_1 : \overline{P}_{a_1} \to \overline{P}_{a_1}$. In view of ($S_1$), we have

$$\begin{aligned}\|T_1u\| &= \left|(T_1u)(\sigma)\right| = \beta\phi_q\left(\int_\sigma^1 h(s)f\left(s,u(s)\right)\mathrm{d}s\right) + \int_\sigma^\eta \phi_q\left(\int_\sigma^s h(\tau)f\left(\tau,u(\tau)\right)\mathrm{d}\tau\right)\mathrm{d}s \\ &\le \max\left\{\beta\phi_q\left(\int_0^1 h(s)f\left(s,u(s)\right)\mathrm{d}s\right) + \int_0^\eta \phi_q\left(\int_0^s h(\tau)f\left(\tau,u(\tau)\right)\mathrm{d}\tau\right)\mathrm{d}s,\right. \\ &\qquad \left.\beta\phi_q\left(\int_0^1 h(s)f\left(s,u(s)\right)\mathrm{d}s\right) + \int_\eta^1 \phi_q\left(\int_s^1 h(\tau)f\left(\tau,u(\tau)\right)\mathrm{d}\tau\right)\mathrm{d}s\right\} \\ &\le \frac{a_1}{M_1}\cdot M_1 = a_1.\end{aligned}$$

Similarly, (S_2) implies that $T_1 : \overline{P}_{c_1} \to \overline{P}_{c_1}$.

Step 2. We try to prove Theorem 2.1(i) is satisfied.

Choose $u(t) = kb_1$. Obviously, $u(t) \in P(\alpha_1, b_1, kb_1)$, thus $\{u(t) \in P(\alpha_1, b_1, kb_1)\} \ne \emptyset$.

If $u \in P(\alpha_1, b_1, kb_1)$, then for $t \in [\frac{1}{k}, 1-\frac{1}{k}]$, $b_1 < u(t) \le kb_1$. In view of (S_3), we have

$$\begin{aligned}\alpha_1(Tu) &= \min_{\frac{1}{k}\le t\le(1-\frac{1}{k})} \left|(Tu)(t)\right| \ge \min\left\{\frac{1}{k}, 1-\frac{1}{k}\right\} \max_{0\le t\le 1} \left|(Tu)(t)\right| = \frac{1}{k}\left|(Tu)(\sigma)\right| \\ &= \frac{1}{k}\left(g_1\left(\phi_q\left(\int_0^\sigma h(s)f\left(s,u(s)\right)\mathrm{d}s\right)\right) + \int_\xi^\sigma \phi_q\left(\int_s^\sigma h(\tau)f\left(\tau,u(\tau)\right)\mathrm{d}\tau\right)\mathrm{d}s\right) \\ &\ge \frac{1}{k}\min\left\{\xi\phi_q\left(\int_0^{\frac{1}{2}} h(s)f\left(s,u(s)\right)\mathrm{d}s\right) + \int_\xi^{\frac{1}{2}} \phi_q\left(\int_s^{\frac{1}{2}} h(\tau)f\left(\tau,u(\tau)\right)\mathrm{d}\tau\right)\mathrm{d}s,\right.\end{aligned}$$

$$\beta\left(\phi_q\left(\int_{\frac{1}{2}}^{1} h(s)f\big(s,u(s)\big)\,\mathrm{d}s\right)\right)+\int_{\frac{1}{2}}^{\eta}\phi_q\left(\int_{\frac{1}{2}}^{s} h(\tau)f\big(\tau,u(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s\Big\}$$

$$\geq \frac{1}{k}\min\Big\{\xi\phi_q\left(\int_{\frac{1}{k}}^{\frac{1}{2}} h(s)f\big(s,u(s)\big)\,\mathrm{d}s\right)+\int_{\xi}^{\frac{1}{2}}\phi_q\left(\int_{s}^{\frac{1}{2}} h(\tau)f\big(\tau,u(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s,$$

$$\beta\left(\phi_q\left(\int_{\frac{1}{2}}^{1-\frac{1}{k}} h(s)f\big(s,u(s)\big)\,\mathrm{d}s\right)\right)+\int_{\frac{1}{2}}^{\eta}\phi_q\left(\int_{\frac{1}{2}}^{s} h(\tau)f\big(\tau,u(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s\Big\}$$

$$\geq \frac{b_1}{m_1}\cdot m_1 = b_1.$$

Thus, Theorem 2.1(i) holds.

Step 3. We try to prove that Theorem 2.1(iii) holds.

If $u \in P(\alpha_1, b_1, c_1)$ and $\|Tu\| > kb_1$, then we have

$$\alpha_1(Tu) = \min_{\frac{1}{k}\leq t\leq 1-\frac{1}{k}}(Tu)(t) \geq \frac{1}{k}\|Tu\| > b_1.$$

Hence, Theorem 2.1(iii) also holds. An application of Theorem 2.1 means that BVP (1.1), (1.2) has at least three positive solutions satisfied (3.12). The proof is complete. □

4 Existence results for BVP(1.1), (1.3)

4.1 Some useful lemmas

Lemma 4.1 *Suppose that $v(t)$ satisfies all the conditions in Lemma 3.1. Then BVP*

$$\begin{cases} (\phi_p(u'(t)))' + v(t) = 0, & 0 < t < 1, \\ g_1(u'(0)) - u(\xi) = 0, & \beta u'(1) + u(\eta) = 0. \end{cases} \tag{4.1}$$

has the unique solution

$$u(t) = \begin{cases} \alpha\phi_q(\int_0^{\sigma} v(s)\,\mathrm{d}s) + \int_{\xi}^{t}\phi_q(\int_s^{\sigma} v(\tau)\,\mathrm{d}\tau)\,\mathrm{d}s, & t\in[0,\sigma], \\ g_2(\phi_q(\int_{\sigma}^{1} v(s)\,\mathrm{d}s)) + \int_t^{\eta}\phi_q(\int_{\sigma}^{s} v(\tau)\,\mathrm{d}\tau)\,\mathrm{d}s, & t\in[\sigma,1], \end{cases} \tag{4.2}$$

where σ is a solution of the equation

$$B_1(t) - B_2(t) = 0, \quad t\in[0,1]. \tag{4.3}$$

And $B_1(t)$, $B_2(t)$ are defined as follows:

$$B_1(t) = \alpha\phi_q\left(\int_0^t v(s)\,\mathrm{d}s\right) + \int_{\xi}^{t}\phi_q\left(\int_s^t v(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s$$

$$B_2(t) = g_2\left(\phi_q\left(\int_t^1 v(s)\,\mathrm{d}s\right)\right) + \int_t^{\eta}\phi_q\left(\int_t^s v(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s.$$

Proof Essentially, we can prove it using the same discussion as in Lemma 3.1. Here, we omit it. □

Lemma 4.2 *Let $v(t)$ satisfy all the conditions in Lemma 3.1, then the solution $u(t)$ of BVP (4.1) is concave on $t\in[0,1]$. What is more, $u(t)\geq 0$.*

4.2 Existence of positive solutions to BVP (1.1), (1.3)

In this subsection, we try to impose some growth conditions on the non-linear term f which allow us to apply Theorem 2.1 to establish the existence of triple positive solutions to problem (1.1).

Let E, cone P and the norm be defined as in Section 3.

Let $k \geq 3$ be big enough so that $\xi, \eta \in [\frac{1}{k}, 1 - \frac{1}{k}]$; and the nonnegative continuous concave functional α_2 be defined on the cone P by

$$\alpha_2(x) = \min_{\frac{1}{k} \leq t \leq 1 - \frac{1}{k}} |x(t)|, \quad \text{for } x \in P.$$

Define $T_2 : P \to E$ as follows:

$$(T_2 u)(t) = \begin{cases} \alpha\phi_q(\int_0^\sigma h(s)f(s,u(s))\,\mathrm{d}s) \\ \quad + \int_\xi^t \phi_q(\int_s^\sigma h(\tau)f(\tau,u(\tau))\,\mathrm{d}\tau)\,\mathrm{d}s, & t \in [0,\sigma], \\ g_2(\phi_q(\int_\sigma^1 h(s)f(s,u(s))\,\mathrm{d}s) \\ \quad + \int_t^\eta \phi_q(\int_\sigma^s h(\tau)f(\tau,u(\tau))\,\mathrm{d}\tau)\,\mathrm{d}s, & t \in [\sigma,1]. \end{cases} \tag{4.4}$$

Lemma 4.3 *Suppose that (C_1) and (C_2) hold, then $T_2 : P \to P$ is completely continuous.*

Let

$$M_2 = \alpha\phi_q\left(\int_0^1 h(s)\,\mathrm{d}s\right) + \max\left\{\int_0^\xi \phi_q\left(\int_0^s h(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s, \int_\xi^1 \phi_q\left(\int_s^1 h(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s\right\}$$

$$m_2 = \frac{1}{k} \cdot \min\left\{\alpha\phi_q\left(\int_{\frac{1}{k}}^{\frac{1}{2}} h(s)\,\mathrm{d}s\right) + \int_\xi^{\frac{1}{2}} \phi_q\left(\int_s^{\frac{1}{2}} h(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s,\right.$$
$$\left.(1-\eta)\phi_q\left(\int_{\frac{1}{2}}^{1-\frac{1}{k}} h(s)\,\mathrm{d}s\right) + \int_{\frac{1}{2}}^\eta \phi_q\left(\int_{\frac{1}{2}}^s h(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s\right\}.$$

Theorem 4.4 *Assume that (H_2) holds, there exist constants such that $0 < a_2 < b_2 \leq \min\{\frac{1}{k}, \frac{m_2}{M_2}\}c_2$. Further suppose that*

(S_4) $f(t,u) < \phi_p(a_2/M_2)$, for $(t,u) \in [0,1] \times [0,a_2]$;
(S_5) $f(t,u) \leq \phi_p(c_2/M_2)$, for $(t,u) \in [0,1] \times [0,c_2]$;
(S_6) $f(t,u) \geq \phi_p(b_2/m_2)$, for $(t,u) \in [\frac{1}{k}, 1-\frac{1}{k}] \times [b_2, kb_2]$.

Then boundary value problem (1.1) (1.3) has at least three positive solutions u_1, u_2, u_3 such that

$$\|u_1\| < a_2, \qquad \alpha_2(u_2) > b_2 \quad \text{and} \quad \|u_3\| > a_2, \qquad \alpha_2(u_3) < b_2. \tag{4.5}$$

Proof By making use of Theorem 2.1, we can prove the result by essentially the same method. Here we omit it. □

5 A generalization result

In this section, we consider the existence results of (1.1) subject to the following non-linear four-point boundary condition:

$$g_1\big(u'(0)\big) - u(\xi) = 0, \qquad g_2\big(u'(1)\big) + u(\eta) = 0. \tag{5.1}$$

In addition to (H_1), (H_2), we assume at least one of the following conditions holds:

(H_3) There exists a positive constant $B_1 \geq \xi$ such that $g_1(x) \leq B_1 x$;
(H_4) There exists a positive constant $B_2 \geq 1 - \eta$ such that $g_1(x) \leq B_2 x$.

Without loss of generality, we suppose (H_4) holds. Essentially by the same method, we can get the multiple existence results for BVP (1.1), (5.1). In what follows, we just list the main results but the proofs are all omitted.

Lemma 5.1 *Suppose that $v(t)$ satisfies all the conditions in Lemma 3.1, then BVP*

$$\begin{cases} (\phi_p(u'(t)))' + v(t) = 0, & 0 < t < 1, \\ g_1(u'(0)) - u(\xi) = 0, & g_2(u'(1)) + u(\eta) = 0 \end{cases} \tag{5.2}$$

has the unique solution

$$u(t) = \begin{cases} g_1(\phi_q(\int_0^\sigma v(s)\,\mathrm{d}s)) + \int_\xi^t \phi_q(\int_s^\sigma v(\tau)\,\mathrm{d}\tau)\,\mathrm{d}s, & (i) \quad t \in [0,\sigma], \\ g_2(\phi_q(\int_\sigma^1 v(s)\,\mathrm{d}s)) + \int_t^\eta \phi_q(\int_\sigma^s v(\tau)\,\mathrm{d}\tau)\,\mathrm{d}s, & (ii) \quad t \in [\sigma,1], \end{cases} \tag{5.3}$$

where σ is a solution of the equation

$$C_1(t) - C_2(t) = 0, \quad t \in [0,1], \tag{5.4}$$

and $C_1(t)$, $C_2(t)$ are defined as follows:

$$C_1(t) = g_1\left(\phi_q\left(\int_0^t v(s)\,\mathrm{d}s\right)\right) + \int_\xi^t \phi_q\left(\int_s^t v(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s,$$
$$C_2(t) = g_2\left(\phi_q\left(\int_t^1 v(s)\,\mathrm{d}s\right)\right) + \int_t^\eta \phi_q\left(\int_t^s v(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s.$$

Lemma 5.2 *Let $v(t)$ satisfy all the conditions in Lemma 3.1, then the solution $u(t)$ of BVP (5.2) is concave on $t \in [0,1]$. What is more, $u(t) \geq 0$.*

Let E, cone P and the norm be defined as in Section 3; and let the nonnegative continuous concave functional α_3 be defined on the cone P by

$$\alpha_3(x) = \min_{\frac{1}{k} \leq t \leq 1-\frac{1}{k}} \left|x(t)\right|, \quad \text{for } x \in P.$$

Define $T_3 : P \to E$ as follows:

$$(T_3 u)(t) = \begin{cases} g_1(\phi_q(\int_0^\sigma h(s) f(s,u(s))\,\mathrm{d}s)) \\ \quad + \int_\xi^t \phi_q(\int_s^\sigma h(\tau) f(\tau,u(\tau))\,\mathrm{d}\tau)\,\mathrm{d}s, & t \in [0,\sigma], \\ g_2(\phi_q(\int_\sigma^1 h(s) f(s,u(s)\,\mathrm{d}s) \\ \quad + \int_t^\eta \phi_q(\int_\sigma^s h(\tau) f(\tau,u(\tau))\,\mathrm{d}\tau)\,\mathrm{d}s, & t \in [\sigma,1]. \end{cases} \tag{5.5}$$

Lemma 5.3 *Suppose that (C_1) and (C_2) hold, then $T_3 : P \to P$ is completely continuous.*

Let

$$M_3 = B_1\phi_q\left(\int_0^1 h(s)\,\mathrm{d}s\right) + \max\left\{\int_0^\xi \phi_q\left(\int_0^s h(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s, \int_\xi^1 \phi_q\left(\int_s^1 h(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s\right\}$$

$$m_3 = \frac{1}{k}\cdot\min\left\{\xi\phi_q\left(\int_{\frac{1}{k}}^{\frac{1}{2}} h(s)\,\mathrm{d}s\right) + \int_\xi^{\frac{1}{2}} \phi_q\left(\int_s^{\frac{1}{2}} h(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s,\right.$$
$$\left.(1-\eta)\phi_q\left(\int_{\frac{1}{2}}^{1-\frac{1}{k}} h(s)\,\mathrm{d}s\right) + \int_{\frac{1}{2}}^\eta \phi_q\left(\int_{\frac{1}{2}}^s h(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s\right\}.$$

Theorem 5.4 *Assume that* (H_1), (H_4) *hold, there exist constants such that* $0 < a_3 < b_3 \leq \min\{\frac{1}{k}, \frac{m_3}{M_3}\}c_3$. *Further suppose that*

(S_7) $f(t,u) < \phi_p(a_3/M_3)$, *for* $(t,u) \in [0,1] \times [0,a_3]$;
(S_8) $f(t,u) \leq \phi_p(c_3/M_3)$, *for* $(t,u) \in [0,1] \times [0,c_3]$;
(S_9) $f(t,u) \geq \phi_p(b_3/m_3)$, *for* $(t,u) \in [\frac{1}{k}, 1-\frac{1}{k}] \times [b_3, kb_3]$.

Then boundary value problem (1.1), (1.4) has at least three positive solutions u_1, u_2, u_3 *such that*

$$\|u_1\| < a_3, \qquad \alpha_3(u_2) > b_3 \quad \textit{and} \quad \|u_3\| > a_3, \qquad \alpha_3(u_3) < b_3. \tag{5.6}$$

6 Example

In this section, we give examples to illustrate our main results.

Example 6.1 Consider the following non-linear four-point boundary value problem:

$$\begin{cases} (\frac{u'}{\sqrt{u'}})' + \frac{1}{t^{1/2}}f(t,u(t)) = 0, & 0 < t < 1, \\ u'^2(0) + u'(0) - u(1/2) = 0, & 2u'(1) + u(2/3) = 0, \end{cases} \tag{6.1}$$

where

$$f(t,u(t)) = \begin{cases} \sin t + (\frac{u}{15})^8, & 0 < u \leq 24.9, \\ \sin t + (\frac{24.9}{15})^8, & u > 24.9. \end{cases}$$

And we can see that $p = 3/2$, $g_1(x) = x^2 + x$, $\xi = 1/2$, $\eta = 2/3$. Choose $k = 5$, $a_1 = 80/9$, $b_1 = 23$, $c_1 = 32{,}000$. By direct calculation, we can get $M_1 = 80/9$, $m_1 = 0.027$. Also the following conditions hold.

(i) $f(t,u) \leq 0.86 < 1 = \phi_p(a_1/M_1)$, for $(t,u) \in [0,1] \times [0,80/9]$;
(ii) $f(t,u) \leq 58.51 < 60 = \phi_p(c_1/M_1)$, for $(t,u) \in [0,1] \times [0,32{,}000]$;
(iii) $f(t,u,v) \geq 30.5 > 29.2 = \phi_p(b_1/m_1)$, for $(t,u) \in [\frac{1}{k}, 1-\frac{1}{k}] \times [23,115]$.

Thus, all the conditions in Theorem 3.5 are satisfied, so BVP (6.1) has at least three positive solutions u_1, u_2, u_3 satisfying

$$\|u_1\| < 80/9, \qquad \alpha(u_2) > 23 \quad \text{and} \quad \|u_3\| > 80/9, \qquad \alpha(u_3) < 23. \tag{6.2}$$

Example 6.2 Consider the following non-linear four-point boundary value problem

$$\begin{cases} u'' + \frac{1}{(1-t)^{1/2}} f(t,u(t)) = 0, & 0 < t < 1, \\ 3u'(0) - u(1/4) = 0, & g_2(u'(1)) + u(1/2) = 0, \end{cases} \tag{6.3}$$

where

$$f(t,u) = \begin{cases} -(t-\frac{1}{2})^2 + \frac{1}{4} + (\frac{u}{10})^3, & 0 \le u \le 3{,}400, \\ -(t-\frac{1}{2})^2 + \frac{1}{4} + (\frac{3400}{10})^3, & u \ge 3{,}400 \end{cases}$$

and

$$g_2(x) = x^3 + 1.$$

Choose $k = 4$, $a_2 = 6.866$, $b_2 = 150$, $c_2 = 2.75 \times 10^8$. Obviously, $p = 2$, $\alpha = 3$, $\xi = 1/4$, $\eta = 1/2$, By direct calculation, we can get that $q = 2$, $M_2 = 6.866$, $m_2 = 0.052$. Also the following conditions hold.

(i) $f(t,u) \le 0.5737 < 1 = \phi_p(a_2/M_2)$, for $(t,u) \in [0,1] \times [0, 6.866]$;
(ii) $f(t,u) \le 3.9305 \times 10^7 < 4.005 \times 10^7 = \phi_p(c_2/M_2)$, for $(t,u) \in [0,1] \times [0, 2.75 \times 10^8]$;
(iii) $f(t,u,v) \ge 3{,}375 > 2{,}884.6 = \phi_p(b_2/m_2)$, for $(t,u) \in [\frac{1}{4}, \frac{3}{4}] \times [150, 600]$.

Thus, all the conditions in Theorem 4.4 are satisfied, so BVP (6.1) has at least three positive solutions u_1, u_2, u_3 satisfying

$$\|u_1\| < 6.866, \qquad \alpha(u_2) > 150 \quad \text{and} \quad \|u_3\| > 6.866, \qquad \alpha(u_3) < 150. \tag{6.4}$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
JZ and WG conceived of the study, and participated in its coordination. JZ drafted the manuscript. All authors read and approved the final manuscript.

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