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Stability of quadratic functional equations in tempered distributions

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Abstract

We reformulate the following quadratic functional equation:

$$f(kx+y)+f(kx-y)=2k^2f(x)+2f(y)$$

as the equation for generalized functions. Using the fundamental solution of the heat equation, we solve the general solution of this equation and prove the Hyers-Ulam stability in the spaces of tempered distributions and Fourier hyperfunctions.

Keywords: quadratic functional equation; stability; tempered distribution; heat kernel; Gauss transform

1 Introduction

In 1940, Ulam [31] raised a question concerning the stability of group homomorphisms as follows:

> Let G_1 be a group and let G_2 be a metric group with the metric $d(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if a function $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$?

The case of approximately additive mappings was solved by Hyers [16] under the assumption that G_2 is a Banach space. In 1978, Rassias [25] generalized Hyers' result to the unbounded Cauchy difference.

During the last decades, stability problems of various functional equations have been extensively studied and generalized by a number of authors (see [13, 14, 17, 19, 24, 27, 30]). In particular, the stability problem of the following quadratic functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{1.1}$$

was proved by Skof [29]. Thereafter, many authors studied the stability problems of (1.1) in various settings (see [3, 4, 12, 18]). Usually, quadratic functional equations are used to characterize the inner product spaces. Note that a square norm on an inner product space satisfies the parallelogram equality

$$\|x+y\|^2+\|x-y\|^2=2\|x\|^2+2\|y\|^2$$

for all vectors x, y. By virtue of this equality, the quadratic functional equation (1.1) is induced. It is well known that a function f between real vector spaces satisfies (1.1) if and only if there exists a unique symmetric biadditive function B such that $f(x) = B(x,x)$ (see [1, 13, 17, 19, 27]). The biadditive function B is given by

$$B(x,y) = \frac{1}{4}\big(f(x+y) - f(x-y)\big).$$

Recently, Lee *et al.* [21] introduced the following quadratic functional equation which is equivalent to (1.1):

$$f(kx+y) + f(kx-y) = 2k^2 f(x) + 2f(y), \tag{1.2}$$

where k is a fixed positive integer. They proved the Hyers-Ulam-Rassias stability of this equation in Banach spaces. Wang [32] considered the intuitionistic fuzzy stability of (1.2) by using the fixed-point alternative. Saadati and Park [26] proved the Hyers-Ulam-Rassias stability of (1.2) in non-Archimedean $\mathcal{L}$-fuzzy normed spaces.

In this paper, we solve the general solution and the stability problem of (1.2) in the spaces of generalized functions such as $\mathcal{S}'$ of tempered distributions and $\mathcal{F}'$ of Fourier hyperfunctions. Using pullbacks, Chung and Lee [8] reformulated (1.1) as the equation for generalized functions and proved that every solution of (1.1) in $\mathcal{S}'$ (or $\mathcal{F}'$, resp.) is a quadratic form. Also, Chung [7, 11] proved the stability problem of (1.1) in the spaces $\mathcal{S}'$ and $\mathcal{F}'$. Making use of the similar methods as in [7–11, 22], we reformulate (1.2) and the related inequality in the spaces of generalized functions as follows:

$$u \circ A + u \circ B = 2k^2 u \circ P + 2u \circ Q, \tag{1.3}$$

$$\big\| u \circ A + u \circ B - 2k^2 u \circ P - 2u \circ Q \big\| \le \epsilon, \tag{1.4}$$

where A, B, P, and Q are the functions defined by

$$A(x,y) = kx + y, \qquad B(x,y) = kx - y, \qquad P(x,y) = x, \qquad Q(x,y) = y.$$

Here, $\circ$ denotes the pullback of generalized functions and the inequality $\|v\| \le \epsilon$ in (1.4) means that $|\langle v, \varphi\rangle| \le \epsilon \|\varphi\|_{L^1}$ for all test functions φ. We refer to [15] for pullbacks and to [2, 7–11] for more details of the spaces of generalized functions.

As results, we shall prove that every solution u in $\mathcal{S}'$ (or $\mathcal{F}'$, resp.) of Eq. (1.3) is a quadratic form

$$u = \sum_{1 \le i \le j \le n} a_{ij} x_i x_j,$$

where $a_{ij} \in \mathbb{C}$. Also, we shall prove that every solution u in $\mathcal{S}'$ (or $\mathcal{F}'$, resp.) of the inequality (1.4) can be written uniquely in the form

$$u = \sum_{1 \le i \le j \le n} a_{ij} x_i x_j + \mu(x),$$

where μ is a bounded measurable function such that

$$\|\mu\|_{L^\infty} \le \begin{cases} \frac{\epsilon}{2}, & k = 1, \\ \frac{(k^2+1)\epsilon}{2k^2(k^2-1)}, & k \ge 2. \end{cases}$$

2 Preliminaries

In this section, we introduce the spaces of tempered distributions and Fourier hyperfunctions. Here, we use the n-dimensional notations. If $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$, where $\mathbb{N}_0^n$ is the set of nonnegative integers, then $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\alpha! = \alpha_1! \cdots \alpha_n!$. For $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, we denote $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $\partial^\alpha = (\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_n)^{\alpha_n}$.

2.1 Tempered distributions

We present a very useful space of test functions for the tempered distributions as follows.

Definition 2.1 ([15, 28]) An infinitely differentiable function φ in $\mathbb{R}^n$ is called rapidly decreasing if

$$\|\varphi\|_{\alpha,\beta} = \sup_{x\in\mathbb{R}^n} \left|x^\alpha \partial^\beta \varphi(x)\right| < \infty \tag{2.1}$$

for all $\alpha, \beta \in \mathbb{N}_0^n$. The vector space of such functions is denoted by $\mathcal{S}(\mathbb{R}^n)$. A linear functional u on $\mathcal{S}(\mathbb{R}^n)$ is said to be a tempered distribution if there exists the constant $C \ge 0$ and the nonnegative integer N such that

$$\left|\langle u, \varphi\rangle\right| \le C \sum_{|\alpha|,|\beta|\le N} \sup_{x\in\mathbb{R}^n} \left|x^\alpha \partial^\beta \varphi\right|$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$. The set of all tempered distributions is denoted by $\mathcal{S}'(\mathbb{R}^n)$.

We note that, if $\varphi \in \mathcal{S}(\mathbb{R}^n)$, then each derivative of φ decreases faster than $|x|^{-N}$ for all $N > 0$ as $|x| \to \infty$. It is easy to see that the function $\varphi(x) = \exp(-a|x|^2)$, where $a > 0$ belongs to $\mathcal{S}(\mathbb{R}^n)$, but $\psi(x) = (1 + |x|^2)^{-1}$ is not a member of $\mathcal{S}(\mathbb{R}^n)$. It is known from [5] that (2.1) is equivalent to

$$\sup_{x\in\mathbb{R}^n} \left|x^\alpha \varphi(x)\right| < \infty, \qquad \sup_{\xi\in\mathbb{R}^n} \left|\xi^\beta \hat{\varphi}(\xi)\right| < \infty$$

for all $\alpha, \beta \in \mathbb{N}_0^n$, where $\hat{\varphi}$ is the Fourier transform of φ.

For example, every polynomial $p(x) = \sum_{|\alpha|\le m} a_\alpha x^\alpha$, where $a_\alpha \in \mathbb{C}$, defines a tempered distribution by

$$\langle p(x), \varphi\rangle = \int_{\mathbb{R}^n} p(x)\varphi(x)\,dx, \quad \varphi \in \mathcal{S}(\mathbb{R}^n).$$

Note that tempered distributions are generalizations of L^p-functions. These are very useful for the study of Fourier transforms in generality, since all tempered distributions have a Fourier transform.

2.2 Fourier hyperfunctions

Imposing the growth condition on $\|\cdot\|_{\alpha,\beta}$ in (2.1) Sato and Kawai introduced the new space of test functions for the Fourier hyperfunctions as follows.

Definition 2.2 ([6]) We denote by $\mathcal{F}(\mathbb{R}^n)$ the set of all infinitely differentiable functions φ in $\mathbb{R}^n$ such that

$$\|\varphi\|_{A,B} = \sup_{x,\alpha,\beta} \frac{|x^\alpha \partial^\beta \varphi(x)|}{A^{|\alpha|}B^{|\beta|}\alpha!\beta!} < \infty \tag{2.2}$$

for some positive constants A, B depending only on φ. The strong dual of $\mathcal{F}(\mathbb{R}^n)$, denoted by $\mathcal{F}'(\mathbb{R}^n)$, is called the Fourier hyperfunction.

It can be verified that the seminorm (2.2) is equivalent to

$$\|\varphi\|_{h,k} = \sup_{x,\alpha} \frac{|\partial^\alpha \varphi(x)| \exp k|x|}{h^{|\alpha|}\alpha!} < \infty$$

for some constants $h, k > 0$. Furthermore, it is shown in [6] that (2.2) is equivalent to

$$\sup_{x\in\mathbb{R}^n} \left|\varphi(x)\right| \exp k|x| < \infty, \qquad \sup_{\xi\in\mathbb{R}^n} \left|\hat{\varphi}(\xi)\right| \exp h|\xi| < \infty$$

for some $h, k > 0$.

Fourier hyperfunctions were introduced by Sato in 1958. The space $\mathcal{F}'(\mathbb{R}^n)$ is a natural generalization of the space $\mathcal{S}'(\mathbb{R}^n)$ and can be thought informally as distributions of a infinite order. Observing the above growth conditions, we can easily see the following topological inclusions:

$$\mathcal{F}(\mathbb{R}^n) \hookrightarrow \mathcal{S}(\mathbb{R}^n), \qquad \mathcal{S}'(\mathbb{R}^n) \hookrightarrow \mathcal{F}'(\mathbb{R}^n).$$

3 General solution in generalized functions

In order to solve the general solution of (1.3), we employ the n-dimensional heat kernel, fundamental solution of the heat equation,

$$E_t(x) = \begin{cases} (4\pi t)^{-n/2} \exp(-|x|^2/4t), & x \in \mathbb{R}^n, t > 0, \\ 0, & x \in \mathbb{R}^n, t \le 0. \end{cases}$$

Since for each $t > 0$, $E_t(\cdot)$ belongs to the space $\mathcal{F}(\mathbb{R}^n)$, the convolution

$$\tilde{u}(x,t) = (u * E_t)(x) = \langle u_y, E_t(x-y)\rangle$$

is well defined for all u in $\mathcal{F}'(\mathbb{R}^n)$, which is called the Gauss transform of u. Subsequently, the semigroup property

$$(E_t * E_s)(x) = E_{t+s}(x)$$

of the heat kernel is very useful to convert Eq. (1.3) into the classical functional equation defined on upper-half plane. We also use the following famous result, the so-called heat kernel method, which is stated as follows.

Theorem 3.1 ([23]) *Let $u \in \mathcal{S}'(\mathbb{R}^n)$. Then its Gauss transform $\tilde{u}$ is a C^∞-solution of the heat equation*

$$(\partial/\partial t - \Delta)\tilde{u}(x,t) = 0$$

satisfying

(i) There exist positive constants C, M, and N such that

$$\left|\tilde{u}(x,t)\right| \le Ct^{-M}\left(1+|x|\right)^N \quad \text{in } \mathbb{R}^n \times (0,\delta). \tag{3.1}$$

(ii) $\tilde{u}(x,t) \to u$ as $t \to 0^+$ in the sense that for every $\varphi \in \mathcal{S}(\mathbb{R}^n)$,

$$\langle u, \varphi\rangle = \lim_{t\to 0^+} \int \tilde{u}(x,t)\varphi(x)\,dx.$$

Conversely, every C^∞-solution $U(x,t)$ of the heat equation satisfying the growth condition (3.1) can be uniquely expressed as $U(x,t) = \tilde{u}(x,t)$ for some $u \in \mathcal{S}'(\mathbb{R}^n)$.

Similarly, we can represent Fourier hyperfunctions as a special case of the results as in [20]. In this case, the estimate (3.1) is replaced by the following:

For every $\epsilon > 0$, there exists a positive constant C_ϵ such that

$$\left|\tilde{u}(x,t)\right| \le C_\epsilon \exp\left(\epsilon\left(|x| + 1/t\right)\right) \quad \text{in } \mathbb{R}^n \times (0,\delta).$$

Here, we need the following lemma to solve the general solution of (1.3).

Lemma 3.2 *Suppose that $f : \mathbb{R}^n \times (0,\infty) \to \mathbb{C}$ is a continuous function satisfying the equation*

$$f\left(kx+y, k^2t+s\right) + f\left(kx-y, k^2t+s\right) = 2k^2 f(x,t) + 2f(y,s) \tag{3.2}$$

for all $x, y \in \mathbb{R}^n$, $t, s > 0$. Then the solution f is the quadratic-additive function

$$f(x,t) = \sum_{1\le i\le j\le n} a_{ij}x_ix_j + bt$$

for some $a_{ij}, b \in \mathbb{C}$.

Proof Define a function $h : \mathbb{R}^n \times (0,\infty) \to \mathbb{C}$ as $h(x,t) := f(x,t) - f(0,t)$. We immediately have $h(0,t) = 0$ and

$$h\left(kx+y, k^2t+s\right) + h\left(kx-y, k^2t+s\right) = 2k^2 h(x,t) + 2h(y,s) \tag{3.3}$$

for all $x, y \in \mathbb{R}^n$, $t, s > 0$. Putting $y = 0$ in (3.3) yields

$$h\left(kx, k^2t+s\right) = k^2 h(x,t) \tag{3.4}$$

for all $x \in \mathbb{R}^n$, $t,s > 0$. Letting $s \to 0^+$ in (3.4) gives

$$h(kx, k^2t) = k^2 h(x,t) \tag{3.5}$$

for all $x \in \mathbb{R}^n$, $t > 0$. Replacing s by $k^2 s$ in (3.4) and then using (3.5), we obtain

$$h(x, t+s) = h(x,t)$$

for all $x \in \mathbb{R}^n$, $t,s > 0$. This shows that $h(x,t)$ is independent with respect to the second variable. Thus, we see that $H(x) := h(x,t)$ satisfies (1.2). Using the induction argument on the dimension n, we verify that every continuous solution of (1.2) in $\mathbb{R}^n$ is a quadratic form

$$H(x) = h(x,t) = \sum_{1 \le i \le j \le n} a_{ij} x_i x_j,$$

where $a_{ij} \in \mathbb{C}$.

On the other hand, putting $x = y = 0$ in (3.2) yields

$$f(0, k^2t + s) = k^2 f(0,t) + f(0,s) \tag{3.6}$$

for all $t,s > 0$. In view of (3.6), we verify that $\lim_{s\to 0^+} f(0,s) = 0$ and

$$f(0, k^2t) = k^2 f(0,t) \tag{3.7}$$

for all $t > 0$. It follows from (3.6) and (3.7) that we see that $f(0,t)$ satisfies the Cauchy functional equation

$$f(0, t+s) = f(0,t) + f(0,s)$$

for all $t,s > 0$. Given the continuity, we have

$$f(0,t) = bt$$

for some $b \in \mathbb{C}$. Therefore, we finally obtain

$$f(x,t) = h(x,t) + f(0,t) = \sum_{1 \le i \le j \le n} a_{ij} x_i x_j + bt$$

for all $x \in \mathbb{R}^n$, $t > 0$. □

As a direct consequence of the above lemma, we present the general solution of the quadratic functional equation (1.3) in the spaces of generalized functions.

Theorem 3.3 *Every solution u in $\mathcal{S}'(\mathbb{R}^n)$ (or $\mathcal{F}'(\mathbb{R}^n)$, resp.) of Eq. (1.3) is the quadratic form*

$$u = \sum_{1 \le i \le j \le n} a_{ij} x_i x_j$$

for some $a_{ij} \in \mathbb{C}$.

Proof Convolving the tensor product $E_t(\xi)E_s(\eta)$ of n-dimensional heat kernels in both sides of (1.3), we have

$$\begin{aligned}\bigl[(u\circ A)*\bigl(E_t(\xi)E_s(\eta)\bigr)\bigr](x,y) &= \langle u\circ A, E_t(x-\xi)E_s(y-\eta)\rangle\\ &= \left\langle u_\xi, k^{-n}\int E_t\left(x-\frac{\xi-\eta}{k}\right)E_s(y-\eta)\,d\eta\right\rangle\\ &= \left\langle u_\xi, k^{-n}\int E_t\left(\frac{kx+y-\xi-\eta}{k}\right)E_s(\eta)\,d\eta\right\rangle\\ &= \left\langle u_\xi, \int E_{k^2t}(kx+y-\xi-\eta)E_s(\eta)\,d\eta\right\rangle\\ &= \langle u_\xi, (E_{k^2t}*E_s)(kx+y-\xi)\rangle\\ &= \langle u_\xi, E_{k^2t+s}(kx+y-\xi)\rangle\\ &= \tilde{u}\bigl(kx+y, k^2t+s\bigr)\end{aligned}$$

and similarly we get

$$\begin{aligned}&\bigl[(u\circ B)*\bigl(E_t(\xi)E_s(\eta)\bigr)\bigr](x,y) = \tilde{u}\bigl(kx-y, k^2t+s\bigr),\\ &\bigl[(u\circ P)*\bigl(E_t(\xi)E_s(\eta)\bigr)\bigr](x,y) = \tilde{u}(x,t),\\ &\bigl[(u\circ Q)*\bigl(E_t(\xi)E_s(\eta)\bigr)\bigr](x,y) = \tilde{u}(y,s).\end{aligned}$$

Thus, (1.3) is converted into the classical functional equation

$$\tilde{u}\bigl(kx+y,k^2t+s\bigr)+\tilde{u}\bigl(kx-y,k^2t+s\bigr) = 2k^2\tilde{u}(x,t)+2\tilde{u}(y,s)$$

for all $x,y\in\mathbb{R}^n$, $t,s>0$. We note that the Gauss transform $\tilde{u}$ is a C^∞ function and so, by Lemma 3.2, the solution $\tilde{u}$ is of the form

$$\tilde{u}(x,t) = \sum_{1\le i\le j\le n} a_{ij}x_ix_j + bt \tag{3.8}$$

for some $a_{ij}, b\in\mathbb{C}$. By the heat kernel method, we obtain

$$u = \sum_{1\le i\le j\le n} a_{ij}x_ix_j$$

as $t\to 0^+$ in (3.8). □

4 Stability in generalized functions

In this section, we are going to solve the stability problem of (1.4). For the case of $k=1$ in (1.4), the result is known as follows.

Theorem 4.1 ([7, 10]) *Suppose that u in* $\mathcal{S}'(\mathbb{R}^n)$ *(or* $\mathcal{F}'(\mathbb{R}^n)$*, resp.) satisfies the inequality*

$$\|u\circ A+u\circ B-2u\circ P-2u\circ Q\|\le\epsilon.$$

Then there exists a unique quadratic form

$$T(x) = \sum_{1\le i\le j\le n} a_{ij}x_ix_j$$

such that

$$\|u - T(x)\| \le \frac{\epsilon}{2}.$$

We here need the following lemma to solve the stability problem of (1.4).

Lemma 4.2 *Let k be a fixed positive integer with $k \ge 2$. Suppose that $f : \mathbb{R}^n \times (0,\infty) \to \mathbb{C}$ is a continuous function satisfying the inequality*

$$\|f(kx + y, k^2t + s) + f(kx - y, k^2t + s) - 2k^2f(x,t) - 2f(y,s)\|_{L^\infty} \le \epsilon. \tag{4.1}$$

Then there exist a unique function $g(x,t)$ satisfying the quadratic-additive functional equation

$$g(kx + y, k^2t + s) + g(kx - y, k^2t + s) = 2k^2g(x,t) + 2g(y,s)$$

such that

$$\|f(x,t) - g(x,t)\|_{L^\infty} \le \frac{k^2 + 1}{2k^2(k^2 - 1)}\epsilon.$$

Proof Putting $x = y = 0$ in (4.1) yields

$$|f(0,k^2t + s) - k^2f(0,t) - f(0,s)| \le \frac{\epsilon}{2} \tag{4.2}$$

for all $t, s > 0$. In view of (4.2), we see that

$$c := \limsup_{t\to 0^+} f(0,t)$$

exists. Letting $t = t_n \to 0^+$ so that $f(0,t_n) \to c$ in (4.2) gives

$$|c| \le \frac{\epsilon}{2k^2}. \tag{4.3}$$

Putting $y = 0$ and letting $s = s_n \to 0^+$ so that $f(0,s_n) \to c$ in (4.1) we have

$$|f(kx, k^2t) - k^2f(x,t) - c| \le \frac{\epsilon}{2} \tag{4.4}$$

for all $x \in \mathbb{R}^n$, $t > 0$. Using (4.3), we can rewrite (4.4) as

$$\left|\frac{f(kx, k^2t)}{k^2} - f(x,t)\right| \le \frac{k^2 + 1}{2k^4}\epsilon$$

for all $x \in \mathbb{R}^n$, $t > 0$. By the induction argument yields

$$\left|\frac{f(k^n x, k^{2n} t)}{k^{2n}} - f(x,t)\right| \leq \frac{k^2+1}{2k^2(k^2-1)}\epsilon \tag{4.5}$$

for all $n \in \mathbb{N}$, $x \in \mathbb{R}^n$, $t > 0$. We claim that the sequence $\{k^{-2n} f(k^n x, k^{2n} t)\}$ converges. Replacing x by $k^m x$ and t by $k^{2m} t$ in (4.5), respectively, where $m \geq n$, we get

$$\left|\frac{f(k^{m+n} x, k^{2(m+n)} t)}{k^{2(m+n)}} - \frac{f(k^m x, k^{2m} t)}{k^{2m}}\right| \leq \frac{k^2+1}{2k^{2(m+1)}(k^2-1)}\epsilon.$$

Letting $n \to \infty$, by Cauchy convergence criterion, we see that the sequence $\{k^{-2n} f(k^n x, k^{2n} t)\}$ is a Cauchy sequence. We can now define a function $h : \mathbb{R}^n \times (0,\infty) \to \mathbb{C}$ by

$$g(x,t) := \lim_{n\to\infty} \frac{f(k^n x, k^{2n} t)}{k^{2n}}.$$

Letting $n \to \infty$ in (4.5) we obtain

$$\left\|f(x,t) - g(x,t)\right\|_{L^\infty} \leq \frac{k^2+1}{2k^2(k^2-1)}\epsilon. \tag{4.6}$$

Replacing x, y, t, s by $k^n x$, $k^n y$, $k^{2n} t$, $k^{2n} s$ in (4.1), dividing both sides by k^{2n} and letting $n \to \infty$ we have

$$g(kx+y, k^2t+s) + g(kx-y, k^2t+s) = 2k^2 g(x,t) + 2g(y,s) \tag{4.7}$$

for all $x, y \in \mathbb{R}^n$, $t, s > 0$. Next, we shall prove that g is unique. Suppose that there exists another function $h : \mathbb{R}^n \times (0,\infty) \to \mathbb{C}$ such that h satisfies (4.6) and (4.7). Since g and h satisfy (4.7), we see from Lemma 3.2 that

$$g(k^n x, k^{2n} t) = k^{2n} g(x,t), \qquad h(k^n x, k^{2n} t) = k^{2n} h(x,t)$$

for all $n \in \mathbb{N}$, $x \in \mathbb{R}^n$, $t > 0$. One gets from (4.6) that

$$\begin{aligned}
&\left|g(x,t) - h(x,t)\right| \\
&\quad = k^{-2n}\left|g(k^n x, k^{2n} t) - h(k^n x, k^{2n} t)\right| \\
&\quad \leq k^{-2n}\left(\left|g(k^n x, k^{2n} t) - f(k^n x, k^{2n} t)\right| + \left|f(k^n x, k^{2n} t) - h(k^n x, k^{2n} t)\right|\right) \\
&\quad \leq \frac{k^2+1}{k^{2(n+1)}(k^2-1)}\epsilon
\end{aligned}$$

for all $n \in \mathbb{N}$, $x \in \mathbb{R}^n$, $t > 0$. Taking the limit as $n \to \infty$, we conclude that $g(x,t) = h(x,t)$ for all $x \in \mathbb{R}^n$, $t > 0$. □

We now state and prove the main theorem of this paper.

Theorem 4.3 *Suppose that u in $\mathcal{S}'(\mathbb{R}^n)$ (or $\mathcal{F}'(\mathbb{R}^n)$, resp.) satisfies the inequality (1.4). Then there exists a unique quadratic form*

$$T(x) = \sum_{1\le i\le j\le n} a_{ij}x_ix_j$$

such that

$$\|u - T(x)\| \le \begin{cases} \frac{\epsilon}{2}, & k = 1, \\ \frac{(k^2+1)\epsilon}{2k^2(k^2-1)}, & k \ge 2. \end{cases}$$

Proof As discussed above, it is done for the case of $k = 1$. We assume that k is a fixed-positive integer with $k \ge 2$. Convolving the tensor product $E_t(\xi)E_s(\eta)$ of n-dimensional heat kernels in both sides of (1.4), we have

$$\|\tilde{u}(kx + y, k^2t + s) + \tilde{u}(kx - y, k^2t + s) - 2k^2\tilde{u}(x, t) - 2\tilde{u}(y, s)\|_{L^\infty} \le \epsilon.$$

By Lemma 4.2, there exists a unique function $g(x, t)$ satisfying the quadratic-additive functional equation

$$g(kx + y, k^2t + s) + g(kx - y, k^2t + s) = 2k^2g(x, t) + 2g(y, s)$$

such that

$$\|\tilde{u}(x, t) - g(x, t)\|_{L^\infty} \le \frac{k^2 + 1}{2k^2(k^2 - 1)}\epsilon. \tag{4.8}$$

It follows from Lemma 3.2 that $g(x, t)$ is of the form

$$g(x, t) = \sum_{1\le i\le j\le n} a_{ij}x_ix_j + bt$$

for some $a_{ij}, b \in \mathbb{C}$. Letting $t \to 0^+$ in (4.8), we have

$$\left\|u - \sum_{1\le i\le j\le n} a_{ij}x_ix_j\right\| \le \frac{k^2 + 1}{2k^2(k^2 - 1)}\epsilon.$$

This completes the proof. □

Remark 4.4 The resulting inequality in Theorem 4.3 implies that $u - T(x)$ is a measurable function. Thus, all of the solution u in $\mathcal{S}'(\mathbb{R}^n)$ (or $\mathcal{F}'(\mathbb{R}^n)$, resp.) can be written uniquely in the form

$$u = T(x) + \mu(x),$$

where

$$\|\mu\|_{L^\infty} \le \begin{cases} \frac{\epsilon}{2}, & k = 1, \\ \frac{(k^2+1)\epsilon}{2k^2(k^2-1)}, & k \ge 2. \end{cases}$$

Competing interests

The author declares that they have no competing interests.

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