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Probability inequalities for END sequence and their applications

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Abstract

Some probability inequalities for extended negatively dependent (END) sequence are provided. Using the probability inequalities, we present some moment inequalities, especially the Rosenthal-type inequality for END sequence. At last, we study the asymptotic approximation of inverse moment for nonnegative END sequence with finite first moments, which generalizes and improves the corresponding results of Wu et al. [Stat. Probab. Lett. **79**, 1366-1371 (2009)], Wang et al. [Stat. Probab. Lett. **80**, 452-461 (2010)], and Sung [J. Inequal. Appl. **2010**, Article ID 823767, 13pp. (2010). doi:10.1155/2010/823767].

MSC(2000): 60E15; 62G20.

Keywords: extended negatively dependent sequence, probability inequality, moment inequality, inverse moment

1 Introduction

It is well known that the probability inequality plays an important role in various proofs of limit theorems. In particular, it provides a measure of convergence rate for the strong law of large numbers. The main purpose of the article is to provide some probability inequalities for extended negatively dependent (END) sequence, which contains independent sequence, NA sequence, and NOD sequence as special cases. These probability inequalities for END random variables are mainly inspired by Fakoor and Azarnoosh [1] and Asadian et al. [2]. Using the probability inequalities, we can further study the moment inequalities and asymptotic approximation of inverse moment for END sequence.

First, we will recall the definitions of NOD and END sequences.

Definition 1.1 (cf. Joag-Dev and Proschan [3]). *A finite collection of random variables X_1, X_2, \dots, X_n is said to be negatively upper orthant dependent (NUOD) if for all real numbers x_1, x_2, \dots, x_n ,*

$$P(X_i > x_i, i = 1, 2, \dots, n) \leq \prod_{i=1}^n P(X_i > x_i), \quad (1.1)$$

and negatively lower orthant dependent (NLOD) if for all real numbers x_1, x_2, \dots, x_n ,

$$P(X_i \leq x_i, i = 1, 2, \dots, n) \leq \prod_{i=1}^n P(X_i \leq x_i). \quad (1.2)$$

A finite collection of random variables X_1, X_2, \dots, X_n is said to be negatively orthant dependent (NOD) if they are both NUOD and NLOD.

An infinite sequence $\{X_n, n \geq 1\}$ is said to be NOD if every finite subcollection is NOD.

Definition 1.2 (cf. Liu [4]). We call random variables $\{X_n, n \geq 1\}$ END if there exists a constant $M > 0$ such that both

$$P(X_1 > x_1, X_2 > x_2, \dots, X_n > x_n) \leq M \prod_{i=1}^n P(X_i > x_i) \tag{1.3}$$

and

$$P(X_1 \leq x_1, X_2 \leq x_2, \dots, X_n \leq x_n) \leq M \prod_{i=1}^n P(X_i \leq x_i) \tag{1.4}$$

hold for each $n \geq 1$ and all real numbers x_1, x_2, \dots, x_n .

The concept of END sequence was introduced by Liu [4]. Some applications for END sequence have been found. See, for example, Liu [4] obtained the precise large deviations for dependent random variables with heavy tails. Liu [5] studied the sufficient and necessary conditions of moderate deviations for dependent random variables with heavy tails. It is easily seen that independent random variables and NOD random variables are END. Joag-Dev and Proschan [3] pointed out that NA random variables are NOD. Thus, NA random variables are END. Since END random variables are much weaker than independent random variables, NA random variables and NOD random variables, studying the limit behavior of END sequence is of interest.

Throughout the article, let $\{X_n, n \geq 1\}$ be a sequence of END random variables defined on a fixed probability space (Ω, \mathcal{F}, P) with respective distribution functions F_1, F_2, \dots . Denote $X^+ = \max\{0, X\}$. $c_n \sim d_n$ means $c_n d_n^{-1} \rightarrow 1$ as $n \rightarrow \infty$, and $c_n = o(d_n)$ means $c_n d_n^{-1} \rightarrow 0$ as $n \rightarrow \infty$. Let M and C be positive constants which may be different in various places. Set

$$M_{t,n} = \sum_{i=1}^n E|X_i|^t, \quad S_n = \sum_{i=1}^n X_i, \quad n \geq 1.$$

The following lemma is useful.

Lemma 1.1 (cf. Liu [5]). Let random variables X_1, X_2, \dots, X_n be END.

(i) If f_1, f_2, \dots, f_n are all nondecreasing (or nonincreasing) functions, then random variables $f_1(X_1), f_2(X_2), \dots, f_n(X_n)$ are END.

(ii) For each $n \geq 1$, there exists a constant $M > 0$ such that

$$E \left(\prod_{j=1}^n X_j^+ \right) \leq M \prod_{j=1}^n EX_j^+. \tag{1.5}$$

Lemma 1.2. Let $\{X_n, n \geq 1\}$ be a sequence of END random variables and $\{t_n, n \geq 1\}$ be a sequence of nonnegative numbers (or nonpositive numbers), then for each $n \geq 1$, there exists a constant $M > 0$ such that

$$E \left(\prod_{i=1}^n e^{t_i X_i} \right) \leq M \prod_{i=1}^n E e^{t_i X_i}. \tag{1.6}$$

As a byproduct, for any $t \in \mathbb{R}$,

$$E \left(\prod_{i=1}^n e^{tX_i} \right) \leq M \prod_{i=1}^n E e^{tX_i}. \tag{1.7}$$

Proof. The desired result follows from Lemma 1.1 (i) and (ii) immediately. \square

The organization of this article is as follows: The probability inequalities for END sequence are provided in Section 2, the moment inequalities for END sequence are presented in Section 3, and the asymptotic approximation of inverse moment for non-negative END sequence is studied in Section 4.

2 Probability inequalities for sums of END sequence

In this section, we will give some probability inequalities for END random variables, which can be applied to obtain the moment inequalities and strong law of large numbers. The proofs of the probability inequalities for END random variables are mainly inspired by Fakoor and Azarnoosh [1] and Asadian et al. [2]. Let x, y be arbitrary positive numbers.

Theorem 2.1. *Let $0 < t \leq 1$. Then, there exists a positive constant M such that*

$$P(S_n \geq x) \leq \sum_{i=1}^n P(X_i \geq y) + M \exp \left\{ \frac{x}{y} - \frac{x}{y} \log \left(1 + \frac{xy^{t-1}}{M_{t,n}} \right) \right\}. \tag{2.1}$$

If $xy^{t-1} > M_{t,n}$, then

$$P(S_n \geq x) \leq \sum_{i=1}^n P(X_i \geq y) + M \exp \left\{ \frac{x}{y} - \frac{M_{t,n}}{y^t} - \frac{x}{y} \log \left(\frac{xy^{t-1}}{M_{t,n}} \right) \right\}. \tag{2.2}$$

Proof. For $y > 0$, denote $Y_i = \min(X_i, y)$, $i = 1, 2, \dots, n$ and $T_n = \sum_{i=1}^n Y_i$, $n \geq 1$. It is easy to check that

$$\{S_n \geq x\} \subset \{T_n \neq S_n\} \cup \{T_n \geq x\},$$

which implies that for any positive number h ,

$$P(S_n \geq x) \leq P(T_n \neq S_n) + P(T_n \geq x) \leq \sum_{i=1}^n P(X_i \geq y) + e^{-hx} E e^{hT_n}. \tag{2.3}$$

Lemma 1.1 (i) implies that Y_1, Y_2, \dots, Y_n are still END random variables. It follows from (2.3) and Lemma 1.2 that

$$P(S_n \geq x) \leq \sum_{i=1}^n P(X_i \geq y) + M e^{-hx} \prod_{i=1}^n E e^{hY_i}, \tag{2.4}$$

where M is a positive constant. For $0 < t \leq 1$, the function $(e^{hu} - 1)/u^t$ is increasing on $u > 0$. Thus,

$$\begin{aligned} E e^{hY_i} &= \int_{-\infty}^y (e^{hu} - 1) dF_i(u) + \int_y^{\infty} (e^{hy} - 1) dF_i(u) + 1 \\ &\leq \int_0^y (e^{hu} - 1) dF_i(u) + \int_y^{\infty} (e^{hy} - 1) dF_i(u) + 1 \\ &\leq \frac{e^{hy} - 1}{y^t} \int_0^y u^t dF_i(u) + \frac{e^{hy} - 1}{y^t} \int_y^{\infty} u^t dF_i(u) + 1 \\ &\leq 1 + \frac{e^{hy} - 1}{y^t} E|X_i|^t \leq \exp \left\{ \frac{e^{hy} - 1}{y^t} E|X_i|^t \right\}. \end{aligned}$$

Combining the inequality above and (2.4), we can get that

$$P(S_n \geq x) \leq \sum_{i=1}^n P(X_i \geq y) + M \exp \left\{ \frac{e^{hy} - 1}{y^t} M_{t,n} - hx \right\}. \quad (2.5)$$

Taking $h = \frac{1}{y} \log \left(1 + \frac{xy^{t-1}}{M_{t,n}} \right)$ in the right-hand side of (2.5), we can get (2.1) immediately. If $xy^{t-1} > M_{t,n}$, then the right-hand side of (2.5) attains a minimum value when $h = \frac{1}{y} \log \left(1 + \frac{xy^{t-1}}{M_{t,n}} \right)$. Substitute this value of h to the right-hand side of (2.5), we can get (2.2) immediately. This completes the proof of the theorem.

By Theorem 2.1, we can get the following Theorem 2.2 immediately.

Theorem 2.2. *Let $0 < t \leq 1$. Then, there exists a positive constant M such that*

$$P(|S_n| \geq x) \leq \sum_{i=1}^n P(|X_i| \geq y) + 2M \exp \left\{ \frac{x}{y} - \frac{x}{y} \log \left(1 + \frac{xy^{t-1}}{M_{t,n}} \right) \right\}. \quad (2.6)$$

If $xy^{t-1} > M_{t,n}$, then

$$P(|S_n| \geq x) \leq \sum_{i=1}^n P(X_i \geq y) + 2M \exp \left\{ \frac{x}{y} - \frac{M_{t,n}}{y^t} - \frac{x}{y} \log \left(\frac{xy^{t-1}}{M_{t,n}} \right) \right\}. \quad (2.7)$$

Theorem 2.3. *Assume that $EX_i = 0$ for each $i \geq 1$, then for any $h, x, y > 0$, there exists a positive constant M such that*

$$P(|S_n| \geq x) \leq \sum_{i=1}^n P(|X_i| \geq y) + 2M \exp \left\{ \frac{e^{hy} - 1 - hy}{y^2} M_{2,n} - hx \right\}. \quad (2.8)$$

If we take $h = \frac{1}{y} \log \left(1 + \frac{xy}{M_{2,n}} \right)$, then

$$P(|S_n| \geq x) \leq \sum_{i=1}^n P(|X_i| \geq y) + 2M \exp \left\{ \frac{x}{y} - \frac{x}{y} \log \left(1 + \frac{xy}{M_{2,n}} \right) \right\}. \quad (2.9)$$

Proof. We use the same notations as that in Theorem 2.1. It is easy to see that $(e^{hu} - 1 - hu)/u^2$ is nondecreasing on the real line. Therefore,

$$\begin{aligned} Ee^{hY_i} &\leq 1 + hEX_i + \int_{-\infty}^y (e^{hu} - 1 - hu)dF_i(u) + \int_y^{\infty} (e^{hy} - 1 - hy)dF_i(u) \\ &= 1 + \int_{-\infty}^y \frac{e^{hu} - 1 - hu}{u^2} u^2 dF_i(u) + \int_y^{\infty} (e^{hy} - 1 - hy)dF_i(u) \\ &\leq 1 + \frac{e^{hy} - 1 - hy}{y^2} \left(\int_{-\infty}^y u^2 dF_i(u) + \int_y^{\infty} y^2 dF_i(u) \right) \\ &\leq 1 + \frac{e^{hy} - 1 - hy}{y^2} EX_i^2 \leq \exp \left\{ \frac{e^{hy} - 1 - hy}{y^2} EX_i^2 \right\}, \end{aligned}$$

which implies that

$$P(S_n \geq x) \leq \sum_{i=1}^n P(X_i \geq y) + M \exp \left\{ \frac{e^{hy} - 1 - hy}{y^2} M_{2,n} - hx \right\}.$$

Replacing X_i by $-X_i$, we have

$$P(-S_n \geq x) \leq \sum_{i=1}^n P(-X_i \geq y) + M \exp \left\{ \frac{e^{hy} - 1 - hy}{y^2} M_{2,n} - hx \right\}.$$

Therefore, (2.8) follows from statements above immediately, which yields the desired result (2.9). The proof is completed.

Theorem 2.4. *Assume that $EX_i = 0$ and $|X_i| \leq C$ for each $i \geq 1$, where C is a positive constant. Denote $B_n = \sum_{i=1}^n EX_i^2$ for each $n \geq 1$. Then, for any $x > 0$, there exists a positive constant M such that*

$$P(S_n \geq x) \leq M \exp \left\{ -\frac{x}{2C} \operatorname{arcsinh} h \left(\frac{Cx}{2B_n} \right) \right\} \quad (2.10)$$

and

$$P(|S_n| \geq x) \leq 2M \exp \left\{ -\frac{x}{2C} \operatorname{arcsinh} h \left(\frac{Cx}{2B_n} \right) \right\}. \quad (2.11)$$

Proof. It is easily seen that

$$e^x - x - 1 \leq e^x + e^{-x} - 2 = 2(\cosh x - 1) = 2(\cosh |x| - 1), \quad x \in \mathbb{R}$$

and

$$2(\cosh x - 1) \leq x \sinh x, \quad x \geq 0.$$

Thus, for all $\alpha > 0$ and $i = 1, 2, \dots, n$, we can get that

$$\begin{aligned} E(e^{\alpha X_i} - 1) &= E(e^{\alpha X_i} - \alpha X_i - 1) \leq 2E(\cosh \alpha X_i - 1) \\ &= 2E(\cosh \alpha |X_i| - 1) \leq E(\alpha |X_i| \sinh \alpha |X_i|) \\ &= E\left(\alpha^2 X_i^2 \frac{\sinh \alpha |X_i|}{\alpha |X_i|}\right) \leq \frac{\alpha EX_i^2}{C} \sinh \alpha C. \end{aligned}$$

The last inequality above follows from the fact that the function $\frac{\sinh x}{x}$ is nondecreasing on the half-line $(0, \infty)$.

Since $x = x - 1 + 1 \leq e^{x-1}$ for all $x \in \mathbb{R}$, we have by Lemma 1.2 that

$$E\left(\prod_{i=1}^n e^{\alpha X_i}\right) \leq M \prod_{i=1}^n Ee^{\alpha X_i} \leq M \prod_{i=1}^n \exp(Ee^{\alpha X_i} - 1) \leq M \exp\left(\alpha B_n \frac{\sinh \alpha C}{C}\right),$$

where C is a positive constant. Therefore, for all $\alpha > 0$ and $x > 0$, we have

$$P(S_n \geq x) \leq e^{-\alpha x} Ee^{\alpha S_n} \leq M \exp \left\{ \alpha \left(B_n \frac{\sinh \alpha C}{C} - x \right) \right\}. \quad (2.12)$$

Taking $\alpha = \frac{1}{C} \operatorname{arcsinh} h \left(\frac{Cx}{2B_n} \right)$ in the right-hand side of (2.12), we can see that $B_n \frac{\sinh \alpha C}{C} = \frac{x}{2}$ and (2.10) follows.

Since $\{-X_n, n \geq 1\}$ is still a sequence of END random variables from Lemma 1.1, we have by (2.10) that

$$P(-S_n \geq x) \leq M \exp \left\{ -\frac{x}{2C} \operatorname{arcsinh} h \left(\frac{Cx}{2B_n} \right) \right\}. \quad (2.13)$$

Hence, (2.11) follows from (2.10) and (2.13) immediately. This completes the proof of the theorem.

Theorem 2.5. *Assume that $EX_i = 0$ and $|X_i| \leq C$ for each $i \geq 1$, where C is a positive constant. If $B_n = \sum_{i=1}^n EX_i^2 = O(n)$, then $n^{-1}S_n \rightarrow 0$ completely and in consequence $n^{-1}S_n \rightarrow 0$ a.s.*

Proof. For any $\varepsilon > 0$, we have by Theorem 2.4 that

$$P(S_n \geq n\varepsilon) \leq M \exp \left\{ -\frac{n\varepsilon}{2C} \arcsin h \left(\frac{Cn\varepsilon}{2B_n} \right) \right\} \leq M \exp \{-nD\},$$

where D is a positive constant. Therefore,

$$\sum_{n=1}^{\infty} P(S_n \geq n\varepsilon) < \infty,$$

which implies that $n^{-1}S_n \rightarrow 0$ completely and in consequence $n^{-1}S_n \rightarrow 0$ a.s. by Borel-Cantelli Lemma. The proof is completed.

Theorem 2.6. *Assume that $EX_n^2 < \infty$ and $ES_n \leq 0$ for each $n \geq 1$. Denote $s_n = ES_n^2$. If there exists a nondecreasing sequence of positive numbers $\{c_n, n \geq 1\}$ such that $P(S_n \leq c_n) = 1$, then for any $x > 0$,*

$$P(S_n \geq x) \leq \exp \left\{ -\frac{x^2}{2(s_n + xc_n)} \left[1 + \frac{2}{3} \log \left(1 + \frac{xc_n}{s_n} \right) \right] \right\}. \quad (2.14)$$

In order to prove Theorem 2.6, the following lemma is useful.

Lemma 2.1 (cf. Shao [6]). *For any $x \geq 0$,*

$$\log(1+x) \geq \frac{x}{1+x} + \frac{x^2}{2(1+x)^2} \left[1 + \frac{2}{3} \log(1+x) \right].$$

Proof of Theorem 2.6. Noting that $(e^x - 1 - x)/x^2$ is nondecreasing on the real line, for any $h > 0$ and $n \geq 1$, we have

$$\begin{aligned} Ee^{hS_n} &= 1 + hES_n + E \left[\frac{e^{hS_n} - 1 - hS_n}{(hS_n)^2} (hS_n)^2 \right] \\ &\leq 1 + E \left[\frac{e^{hc_n} - 1 - hc_n}{(hc_n)^2} (hS_n)^2 \right] = 1 + \left(\frac{e^{hc_n} - 1 - hc_n}{c_n^2} \right) s_n \\ &\leq \exp \left\{ \left(\frac{e^{hc_n} - 1 - hc_n}{c_n^2} \right) s_n \right\}. \end{aligned}$$

Hence,

$$P(S_n \geq x) \leq e^{-hx} Ee^{hS_n} \leq \exp \left\{ \left(\frac{e^{hc_n} - 1 - hc_n}{c_n^2} \right) s_n - hx \right\}. \quad (2.15)$$

Taking $h = \frac{1}{c_n} \log \left(1 + \frac{xc_n}{s_n} \right)$ in the right-hand side of (2.15), we can obtain that

$$P(S_n \geq x) \leq \exp \left\{ \frac{x}{c_n} - \frac{x}{c_n} \left(1 + \frac{s_n}{xc_n} \right) \log \left(1 + \frac{xc_n}{s_n} \right) \right\}. \quad (2.16)$$

By Lemma 2.1, we can get that

$$\begin{aligned} &\frac{x}{c_n} \left(1 + \frac{s_n}{xc_n} \right) \log \left(1 + \frac{xc_n}{s_n} \right) \\ &\geq \frac{x}{c_n} \left(1 + \frac{s_n}{xc_n} \right) \left\{ \frac{xc_n}{s_n + xc_n} + \frac{1}{2} \left(\frac{xc_n}{s_n + xc_n} \right)^2 \left[1 + \frac{2}{3} \log \left(1 + \frac{xc_n}{s_n} \right) \right] \right\} \\ &= \frac{x}{c_n} + \frac{x^2}{2(s_n + xc_n)} \left[1 + \frac{2}{3} \log \left(1 + \frac{xc_n}{s_n} \right) \right]. \end{aligned}$$

The desired result (2.14) follows from the above inequality and (2.16) immediately.

3 Moment inequalities for END sequence

In this section, we will present some moment inequalities, especially the Rosenthal-type inequality for END sequence by means of the probability inequalities that obtained in Section 2. The proofs are also inspired by Asadian et al. [2]. The Rosenthal-type inequality can be applied to prove the asymptotic approximation of inverse moment for nonnegative END random variables in Section 4.

Theorem 3.1. *Let $0 < t \leq 1$ and $g(x)$ be a nonnegative even function and nondecreasing on the half-line $[0, \infty)$. Assume that $g(0) = 0$ and $Eg(X_i) < \infty$ for each $i \geq 1$, then for every $r > 0$, there exists a positive constant M such that*

$$Eg(S_n) \leq \sum_{i=1}^n Eg(rX_i) + 2Me^r \int_0^\infty \left(1 + \frac{x^t}{r^{t-1}M_{t,n}}\right)^{-r} dg(x). \quad (3.1)$$

Proof. Taking $y = \frac{x}{r}$ in Theorem 2.2, we have

$$P(|S_n| \geq x) \leq \sum_{i=1}^n P\left(|X_i| \geq \frac{x}{r}\right) + 2Me^r \left(1 + \frac{x^t}{r^{t-1}M_{t,n}}\right)^{-r},$$

which implies that

$$\int_0^\infty P(|S_n| \geq x) dg(x) \leq \sum_{i=1}^n \int_0^\infty P(r|X_i| \geq x) dg(x) + 2Me^r \int_0^\infty \left(1 + \frac{x^t}{r^{t-1}M_{t,n}}\right)^{-r} dg(x).$$

Therefore, the desired result (3.1) follows from the inequality above and Lemma 2.4 in Petrov [7] immediately. This completes the proof of the theorem.

Corollary 3.1. *Let $0 < t \leq 1$, $p \geq t$, and $E|X_i|^p < \infty$ for each $i \geq 1$. Then, there exists a positive constant $C(p, t)$ depending only on p and t such that*

$$E|S_n|^p \leq C(p, t) \left(M_{p,n} + M_{t,n}^{p/t}\right). \quad (3.2)$$

Proof. Taking $g(x) = |x|^p$, $p \geq t$ in Theorem 3.1, we can get that

$$E|S_n|^p \leq r^p \sum_{i=1}^n E|X_i|^p + 2pMe^r \int_0^\infty x^{p-1} \left(1 + \frac{x^t}{r^{t-1}M_{t,n}}\right)^{-r} dx. \quad (3.3)$$

It is easy to check that

$$\begin{aligned} I &\doteq \int_0^\infty x^{p-1} \left(1 + \frac{x^t}{r^{t-1}M_{t,n}}\right)^{-r} dx \\ &= \int_0^\infty x^{p-1} \left(\frac{r^{t-1}M_{t,n}}{r^{t-1}M_{t,n} + x^t}\right)^r dx \\ &= \int_0^\infty x^{p-1} \left(1 - \frac{x^t}{r^{t-1}M_{t,n} + x^t}\right)^r dx. \end{aligned}$$

If we set $\gamma = \frac{x^t}{r^{t-1}M_{t,n} + x^t}$ in the last equality above, then we have for $r > p/t$ that

$$\begin{aligned} I &= \frac{r^{p-p/t}M_{t,n}^{p/t}}{t} \int_0^1 \gamma^{\frac{p}{t}-1} (1-\gamma)^{r-\frac{p}{t}-1} d\gamma \\ &= \frac{r^{p-p/t}M_{t,n}^{p/t}}{t} B\left(\frac{p}{t}, r - \frac{p}{t}\right), \end{aligned}$$

where

$$B(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1} dx, \quad \alpha, \beta > 0$$

is the Beta function. Substitute I to (3.3) and choose

$$C(p, t) = \max \left\{ r^p, 2pMe^r \frac{B\left(\frac{p}{t}, r - \frac{p}{t}\right) r^{p-p/t}}{t} \right\},$$

we can obtain the desired result (3.2) immediately. The proof is completed.

Similar to the proofs of Theorem 3.1 and Corollary 3.1, we can get the following Theorem 3.2 and Corollary 3.2 using Theorem 2.3. The details are omitted.

Theorem 3.2. *Let $EX_i = 0$ for each $i \geq 1$. Assume that the conditions of Theorem 3.1 are satisfied, then for every $r > 0$, there exists a positive constant M such that*

$$Eg(S_n) \leq \sum_{i=1}^n Eg(rX_i) + 2Me^r \int_0^\infty \left(1 + \frac{x^2}{rM_{2,n}}\right)^{-r} dg(x). \quad (3.4)$$

Corollary 3.2 (Rosenthal-type inequality). *Let $p \geq 2$, $EX_i = 0$, and $E|X_i|^p < \infty$ for each $i \geq 1$. Then, there exists a positive constant C_p depending only on p such that*

$$E|S_n|^p \leq C_p \left[\sum_{i=1}^n E|X_i|^p + \left(\sum_{i=1}^n E|X_i|^2 \right)^{p/2} \right]. \quad (3.5)$$

4 Asymptotic approximation of inverse moment for nonnegative END random variables

Recently, Wu et al. [8] studied the asymptotic approximation of inverse moment for nonnegative independent random variables by means of the truncated method and Bernstein's inequality, and obtained the following result:

Theorem A. *Let $\{Z_n, n \geq 1\}$ be a sequence of independent, nonnegative, and non-degenerated random variables. Suppose that*

- (i) $EZ_n^2 < \infty, \forall n \geq 1$;
- (ii) $EX_n \rightarrow \infty$ as $n \rightarrow \infty$, where

$$X_n = \sum_{i=1}^n Z_i/B_n, \quad B_n^2 = \sum_{i=1}^n \text{Var}Z_i;$$

- (iii) there exists a finite positive constant C_1 not depending on n such that $\sup_{1 \leq i \leq n} EZ_i/B_n \leq C_1$;
- (iv) for some $\eta > 0$,

$$B_n^{-2} \sum_{i=1}^n EZ_i^2 I(Z_i > \eta B_n) \rightarrow 0, \quad n \rightarrow \infty. \quad (4.1)$$

Then, for all real numbers $a > 0$ and $\alpha > 0$,

$$E(a + X_n)^{-\alpha} \sim (a + EX_n)^{-\alpha}, \quad n \rightarrow \infty. \quad (4.2)$$

Wang et al. [9] pointed out that the condition (iii) in Theorem A can be removed and extended the result for independent random variables to the case of NOD random variables. Shi et al. [10] obtained (4.2) for $B_n = 1$ and pointed out that the existence of finite second moments is not required. Sung [11] studied the asymptotic approximation of inverse moments for nonnegative random variables satisfying a Rosenthal-type inequality. For more details about asymptotic approximation of inverse moment, one can refer to Garcia and Palacios [12], Kaluszka and Okolewski [13], and Hu et al. [14], and so on.

The main purpose of this section is to show that (4.2) holds under very mild conditions. Our results will extend and improve the results of Wu et al. [8], Wang et al. [9], and Sung [11].

Now, we state and prove the results of asymptotic approximation of inverse moments for nonnegative END random variables.

Theorem 4.1. *Let $\{Z_n, n \geq 1\}$ be a sequence of nonnegative END random variables and $\{B_n, n \geq 1\}$ be a sequence of positive constants. Suppose that*

- (i) $EZ_n < \infty, \forall n \geq 1$;
- (ii) $\mu_n \doteq EX_n \rightarrow \infty$ as $n \rightarrow \infty$, where $X_n = B_n^{-1} \sum_{k=1}^n Z_k$;
- (iii) there exists some $b > 0$ such that

$$\frac{\sum_{k=1}^n EZ_k I(Z_k > bB_n)}{\sum_{k=1}^n EZ_k} \rightarrow 0, \quad n \rightarrow \infty. \tag{4.3}$$

Then, for all real numbers $a > 0$ and $\alpha > 0$, (4.2) holds.

Proof. It is easily seen that $f(x) = (a + x)^{-\alpha}$ is a convex function of x on $[0, \infty)$, by Jensen's inequality, we have

$$E(a + X_n)^{-\alpha} \geq (a + EX_n)^{-\alpha}, \tag{4.4}$$

which implies that

$$\liminf_{n \rightarrow \infty} (a + EX_n)^\alpha E(a + X_n)^{-\alpha} \geq 1. \tag{4.5}$$

To prove (4.2), it is enough to show that

$$\limsup_{n \rightarrow \infty} (a + EX_n)^\alpha E(a + X_n)^{-\alpha} \leq 1. \tag{4.6}$$

In order to prove (4.6), we need only to show that for all $\delta \in (0, 1)$,

$$\limsup_{n \rightarrow \infty} (a + EX_n)^\alpha E(a + X_n)^{-\alpha} \leq (1 - \delta)^{-\alpha}. \tag{4.7}$$

By (iii), we can see that for all $\delta \in (0, 1)$, there exists $n(\delta) > 0$ such that

$$\sum_{k=1}^n EZ_k I(Z_k > bB_n) \leq \frac{\delta}{4} \sum_{k=1}^n EZ_k, \quad n \geq n(\delta). \tag{4.8}$$

Let

$$U_n = B_n^{-1} \sum_{k=1}^n [Z_k I(Z_k \leq bB_n) + bB_n I(Z_k > bB_n)]$$

and

$$E(a + X_n)^{-\alpha} = E(a + X_n)^{-\alpha}I(U_n \geq \mu_n - \delta\mu_n) + E(a + X_n)^{-\alpha}I(U_n < \mu_n - \delta\mu_n) \quad (4.9)$$

$$\doteq Q_1 + Q_2.$$

For Q_1 , since $X_n \geq U_n$, we have

$$Q_1 \leq E(a + X_n)^{-\alpha}I(X_n \geq \mu_n - \delta\mu_n) \leq (a + \mu_n - \delta\mu_n)^{-\alpha}. \quad (4.10)$$

By (4.8), we have for $n \geq n(\delta)$ that

$$|\mu_n - EU_n| = \left| B_n^{-1} \sum_{k=1}^n EZ_k I(Z_k > bB_n) - B_n^{-1} \sum_{k=1}^n bB_n EI(Z_k > bB_n) \right|$$

$$\leq B_n^{-1} \sum_{k=1}^n EZ_k I(Z_k > bB_n) + B_n^{-1} \sum_{k=1}^n bB_n EI(Z_k > bB_n) \quad (4.11)$$

$$\leq B_n^{-1} \sum_{k=1}^n EZ_k I(Z_k > bB_n) + B_n^{-1} \sum_{k=1}^n EZ_k I(Z_k > bB_n)$$

$$= 2B_n^{-1} \sum_{k=1}^n EZ_k I(Z_k > bB_n) \leq \delta\mu_n/2.$$

For each $n \geq 1$, it is easy to see that $\{Z_k I(Z_k \leq bB_n) + bB_n I(Z_k > bB_n), 1 \leq k \leq n\}$ are END random variables by Lemma 1.1. Therefore, by (4.11), Markov's inequality, Corollary 3.2, and C_r 's inequality, for any $p > 2$ and $n \geq n(\delta)$,

$$Q_2 \leq a^{-\alpha} P(U_n < \mu_n - \delta\mu_n)$$

$$= a^{-\alpha} P(EU_n - U_n > \delta\mu_n - (\mu_n - EU_n))$$

$$\leq a^{-\alpha} P(EU_n - U_n > \delta\mu_n/2)$$

$$\leq a^{-\alpha} P(|U_n - EU_n| > \delta\mu_n/2) \leq C\mu_n^{-p} E|U_n - EU_n|^p$$

$$\leq C\mu_n^{-p} \left(B_n^{-2} \sum_{k=1}^n EZ_k^2 I(Z_k \leq bB_n) + B_n^{-2} \sum_{k=1}^n b^2 B_n^2 EI(Z_k > bB_n) \right)^{p/2}$$

$$+ C\mu_n^{-p} \left[B_n^{-p} \sum_{k=1}^n EZ_k^p I(Z_k \leq bB_n) + B_n^{-p} \sum_{k=1}^n b^p B_n^p EI(Z_k > bB_n) \right] \quad (4.12)$$

$$\leq C\mu_n^{-p} \left(B_n^{-1} \sum_{k=1}^n EZ_k I(Z_k \leq bB_n) + B_n^{-1} \sum_{k=1}^n EZ_k I(Z_k > bB_n) \right)^{p/2}$$

$$+ C\mu_n^{-p} B_n^{-1} \sum_{k=1}^n EZ_k I(Z_k \leq bB_n) + C\mu_n^{-p} B_n^{-1} \sum_{k=1}^n EZ_k I(Z_k > bB_n)$$

$$= C\mu_n^{-p} (\mu_n^{p/2} + \mu_n) = C(\mu_n^{-p/2} + \mu_n^{1-p}).$$

Taking $p > \max\{2, 2\alpha, \alpha + 1\}$, we have by (4.9), (4.10), and (4.12) that

$$\limsup_{n \rightarrow \infty} (a + \mu_n)^\alpha E(a + X_n)^{-\alpha}$$

$$\leq \limsup_{n \rightarrow \infty} (a + \mu_n)^\alpha (a + \mu_n - \delta\mu_n)^{-\alpha} + \limsup_{n \rightarrow \infty} (a + \mu_n)^\alpha (C\mu_n^{-p/2} + C\mu_n^{1-p})$$

$$= (1 - \delta)^{-\alpha},$$

which implies (4.7). This completes the proof of the theorem.

Remark 4.1. Theorem 4.1 in this article generalizes and improves the corresponding results of Wu et al. [8], Wang et al. [9], and Sung [11]. First, Theorem 4.1 in this article is based on the condition $EZ_n < \infty, \forall n \geq 1$, which is weaker than the condition $EZ_n^2 < \infty, \forall n \geq 1$ in the above cited references. Second, $\{B_n, n \geq 1\}$ is an arbitrary sequence of positive constants in Theorem 4.1, while $B_n^2 = \sum_{i=1}^n \text{Var}Z_i$ in the above cited references. If we take $B_n \equiv 1$, we can get the asymptotic approximation of inverse moments for the partial sums of nonnegative END random variables. Third, (4.3) is weaker than (4.1). Actually, by the condition (4.1), we can see that

$$B_n^{-1} \sum_{i=1}^n EZ_i I(Z_i > \eta B_n) \leq \eta^{-1} B_n^{-2} \sum_{i=1}^n EZ_i^2 I(Z_i > \eta B_n) \rightarrow 0, \quad n \rightarrow \infty,$$

which implies that

$$\frac{\sum_{i=1}^n EZ_i I(Z_i > \eta B_n)}{\sum_{i=1}^n EZ_i} = \frac{B_n^{-1} \sum_{i=1}^n EZ_i I(Z_i > \eta B_n)}{\mu_n} \rightarrow 0, \quad n \rightarrow \infty,$$

since $\mu_n \rightarrow \infty$, i.e. (4.3) holds.

Acknowledgements

The author was most grateful to the Editor Andrei Volodin and anonymous referee for the careful reading of the manuscript and valuable suggestions which helped in significantly improving an earlier version of this article. The research was supported by the National Natural Science Foundation of China (11171001, 71071002), the Academic Innovation Team of Anhui University (KJTD001B), Provincial Natural Science Research Project of Anhui Colleges (KJ2010A005), Talents youth Fund of Anhui Province Universities (2010SQRL016ZD) and Youth Science Research Fund of Anhui University (2009QN011A).

Authors' contributions

Some probability inequalities and moment inequalities for extended negatively dependent (END) sequence are provided. The asymptotic approximation of inverse moment for nonnegative END sequence with finite first moments is obtained. All authors read and approved the final manuscript.

Competing interests

The author declares that they have no competing interests.

Received: 25 June 2011 Accepted: 27 October 2011 Published: 27 October 2011

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doi:10.1186/1029-242X-2011-98

Cite this article as: Shen: Probability inequalities for END sequence and their applications. *Journal of Inequalities and Applications* 2011 **2011**:98.

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