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Convergence theorems for uniformly quasi-ϕ-asymptotically nonexpansive mappings, generalized equilibrium problems, and variational inequalities

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Abstract

In this article, we introduce an iterative algorithm for finding a common element of the set of common fixed points of an infinite family of closed and uniformly quasi-ϕ-asymptotically nonexpansive mappings, the set of the variational inequality for an α-inverse-strongly monotone operator, and the set of solutions of the generalized equilibrium problems. We obtain a strong convergence theorem for the sequences generated by this process in a 2-uniformly convex and uniformly smooth Banach space. The results presented in this article improve and extend the recent results of Zegeye [Nonlinear Anal. **72**, 2136-2146 (2010)], Wattanawitoon and Kumam [Nonlinear Anal. Hybrid Syst. **3**(1), 11-20 (2009)] and many others.
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1 Introduction and preliminaries

Let C be a nonempty closed convex subset of a real Banach space E with $||\cdot||$ and E^* the dual space of E. Recall that a mapping $T: C \to C$ is said to be *L-Lipschitz continuous* if $||Tx - Ty|| \le L||x - y||$, $\forall x, y \in C$, and a mapping T is said to be *nonexpansive* if $||Tx - Ty|| \le ||x - y||$, $\forall x, y \in C$. A point $x \in C$ is a *fixed point* of T provided $Tx = x$. Denote by $F(T)$ the set of fixed points of T; that is, $F(T) = \{x \in C : Tx = x\}$. Let $A : C \to E^*$ be a mapping. Then, A is called

(i) *monotone* if

$$\langle Ax - Ay, x - y\rangle \ge 0, \quad \forall x, y \in C,$$

(ii) *α-inverse-strongly monotone* if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \ge \alpha||Ax - Ay||^2, \quad \forall x, y \in C.$$

Remark 1.1. It is easy to see that an α-inverse-strongly monotone is monotone and $\frac{1}{\alpha}$-Lipschitz continuous.

Let f be a bifunction of $C \times C$ into $\mathbb{R}$ and $B : C \to E^*$ be a monotone mapping. The *generalized equilibrium problem*, denoted by *GEP*, is to find $x \in C$ such that

$$f(x, y) + \langle Bx, y - x\rangle \geq 0, \quad \forall y \in C. \tag{1.1}$$

The set of solutions for the problem (1.1) is denoted by $GEP(f, B)$, that is,

$$GEP(f, B) := \{x \in C : f(x, y) + \langle Bx, y - x\rangle \geq 0, \forall y \in C\}.$$

If $B \equiv 0$, the problem (1.1) reduce into the *equilibrium problem for f*, denoted by $EP(f)$, is to find $x \in C$ such that

$$f(x, y) \geq 0, \quad \forall y \in C. \tag{1.2}$$

If $f \equiv 0$, the problem (1.1) reduce into the *classical variational inequality problem*, denoted by $VI(B, C)$, is to find $x^* \in C$ such that

$$\langle Bx^*, y - x^*\rangle \geq 0, \quad \forall y \in C. \tag{1.3}$$

The above formulation (1.1) is more general than equilibrium problem (1.2) and cover monotone inclusion problems, saddle point problems, variational inequality problems, minimization problems, vector equilibrium problems, and Nash equilibria in noncooperative games. In addition, there are several other problems, for example, the complementarity problem, fixed point problem, and optimization problem, which can also be written in the form of an $GEP(f, B)$. In other words, the $EP(f)$ is an unifying model for several problems arising in physics, engineering, science, optimization, economics, etc. In the last two decades, many articles have appeared in the literature on the existence of solutions of $EP(f)$; see, for example, [1,2] and references therein. Some solution methods have been proposed to solve the $GEP(f, B)$ and $EP(f)$; see, for example, [1,3-13] and references therein.

Consider the functional defined by

$$\phi(x, y) = ||x||^2 - 2\langle x, Jy\rangle + ||y||^2, \quad \forall x, y \in E. \tag{1.4}$$

As well known that if C is a nonempty closed convex subset of a Hilbert space H and $P_C : H \to C$ is the metric projection of H onto C, then P_C is nonexpansive. This fact actually characterizes Hilbert spaces and consequently, it is not available in more general Banach spaces. It is obvious from the definition of function ϕ that

$$(||x|| - ||y||)^2 \leq \phi(x, y) \leq (||x|| + ||y||)^2, \quad \forall x, y \in E. \tag{1.5}$$

If E is a Hilbert space, then $\phi(x, y) = ||x - y||^2$, for all $x, y \in E$. On the other hand, the *generalized projection* [14] $\Pi_C : E \to C$ is a map that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(x, y)$, that is, $\Pi_C x = \bar{x}$, where $\bar{x}$ is the solution to the minimization problem

$$\phi(\bar{x}, x) = \inf_{y \in C} \phi(y, x), \tag{1.6}$$

existence and uniqueness of the operator Π_C follows from the properties of the functional $\phi(x, y)$ and strict monotonicity of the mapping J (see, for example, [14-18]).

Recall that a point p in C is said to be an *asymptotic fixed point* of T [19] if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty} ||x_n - Tx_n|| = 0$.

The set of asymptotic fixed points of T will be denoted by $\widetilde{F(T)}$. A mapping T is said to be *ϕ-nonexpansive*, if $\phi(Tx, Ty) \le \phi(x, y)$ for $x, y \in C$.

A mapping T from C into itself is said to be *relatively nonexpansive mapping* [20-22] if

(R1) $F(T)$ is nonempty;

(R2) $\phi(p, Tx) \le \phi(p, x)$ for all $x \in C$ and $p \in F(T)$;

(R3) $\widetilde{F(T)} = F(T)$.

A mapping T is said to be *relatively quasi-nonexpansive* (or *quasi-ϕ-nonexpansive*) if the conditions (R1) and (R2) are satisfied. The asymptotic behavior of a relatively nonexpansive mapping was studied in [23-25].

A mapping T is said to be *quasi-ϕ-asymptotically nonexpansive* if $F(T) \neq \emptyset$ and there exists a real sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ such that $\phi(p, T^n x) \le k_{n\phi}(p, x)$ for all $n \ge 1$ $x \in C$ and $p \in F(T)$. We note that the class of relatively quasi-nonexpansive mappings is more general than the class of relatively nonexpansive mappings [23-27] which requires the strong restriction: $F(T) = \widetilde{F(T)}$.

A mapping T is said to be *closed* if for any sequence $\{x_n\} \subset C$ with $x_n \to x$ and $Tx_n \to y$, then $Tx = y$. It is easy to know that each relatively nonexpansive mapping is closed.

A Banach space E is said to be *strictly convex* if $||\frac{x+y}{2}|| < 1$ for all $x, y \in E$ with $||x|| = ||y|| = 1$ and $x \neq y$. Let $U = \{x \in E : ||x|| = 1\}$ be the unit sphere of E. Then, a Banach space E is said to be *smooth* if the limit $\lim_{t\to 0} \frac{\|x+ty\|-\|x\|}{t}$ exists for each $x, y \in U$. It is also said to be *uniformly smooth* if the limit is attained uniformly for $x, y \in U$. Let E be a Banach space. The *modulus of convexity* of E is the function $\delta : [0, 2] \to [0, 1]$ defined by $\delta(\varepsilon) = \inf\{1 - \| \frac{x+y}{2} \| : x, y \in E, \| x \| = \| y \| = 1, \| x - y \| \ge \varepsilon\}$. A Banach space E is *uniformly convex* if and only if $\delta(\varepsilon) > 0$ for all $\varepsilon \in (0, 2]$. Let p be a fixed real number with $p \ge 2$. A Banach space E is said to be *p-uniformly convex* if there exists a constant $c > 0$ such that $\delta(\varepsilon) \ge c\varepsilon^{p}$ for all $\varepsilon \in [0, 2]$; see [28,29] for more details. Observe that every p-uniform convex is uniformly convex. One should note that no a Banach space is p-uniform convex for $1 < p < 2$. It is well known that a Hilbert space is *2-uniformly convex*, uniformly smooth. For each $p > 1$, the *generalized duality mapping* $J_p : E \to 2^{E^*}$ is defined by $J_p>(x) = \{x^* \in E^* : \langle x, x^*\rangle = ||x||^p, ||x^*|| = ||x||^{p-1}\}$ for all $x \in E$. In particular, $J = J_2$ is called the *normalized duality mapping*. If E is a Hilbert space, then $J = I$, where I is the identity mapping.

Remark 1.2. If E is a reflexive, strictly convex, and smooth Banach space, then for $x, y \in E$, $\phi(x, y) = 0$ if and only if $x = y$. It is sufficient to show that if $\phi(x, y) = 0$, then $x = y$. From (1.4), we have $||x|| = ||y||$. This implies that $\langle x, Jy\rangle = ||x||^2 = ||Jy||^2$. From the definition of J, one has $Jx = Jy$. Therefore, we have $x = y$; see [16,18] for more details.

Remark 1.3. The following basic properties can be found in Cioranescu [16].

(i) If E is a uniformly smooth Banach space, then J is uniformly continuous on each bounded subset of E.

(ii) If E is a reflexive and strictly convex Banach space, then J^{-1} is norm-weak*-continuous.

(iii) If E is a smooth, strictly convex, and reflexive Banach space, then the normalized duality mapping $J : E \to 2^{E^*}$ is single-valued, one-to-one, and onto.

(iv) A Banach space E is uniformly smooth if and only if E^* is uniformly convex.

(v) Each uniformly convex Banach space E has the *Kadec-Klee property*, that is, for any sequence $\{x_n\} \subset E$, if $x_n \rightharpoonup x \in E$ and $||x_n|| \to ||x||$, then $x_n \to x$.

In 2004, Matsushita and Takahashi [30] introduced the following iteration: a sequence $\{x_n\}$ is defined by

$$x_{n+1} = \Pi_C J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTx_n), \tag{1.7}$$

where the initial guess element $x_0 \in C$ is arbitrary, $\{\alpha_n\}$ is a real sequence in [0, 1], T is a relatively nonexpansive mapping, and Π_C denotes the generalized projection from E onto a closed convex subset C of E. They proved that the sequence $\{x_n\}$ converges weakly to a fixed point of T . Later, in year 2005, Matsushita and Takahashi [26] proposed the following hybrid iteration method with generalized projection for relatively nonexpansive mapping T in a Banach space E:

$$\begin{cases} x_0 \in C \ \text{chosen arbitrarily}, \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTx_n), \\ C_n = \{z \in C : \phi(z, y_n) \le \phi(z, x_n)\}, \\ Q_n = \{z \in C : \langle x_n - z, Jx_0 - Jx_n\rangle \ge 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x_0. \end{cases} \tag{1.8}$$

They proved that $\{x_n\}$ converges strongly to $\Pi_{F(T)}x_0$, where $\Pi_{F(T)}$ is the generalized projection from C onto $F(T)$.

In 2008, Iiduka and Takahashi [31] introduced the following iterative scheme for finding a solution of the variational inequality problem for an inverse-strongly monotone operator A in a 2-uniformly convex and uniformly smooth Banach space $E : x_1 = x \in C$ and

$$x_{n+1} = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n), \tag{1.9}$$

for every n = 1, 2, 3,..., where Π_C is the generalized metric projection from E onto C, J is the duality mapping from E into E^*, and $\{\lambda_n\}$ is a sequence of positive real numbers. They proved that the sequence $\{x_n\}$ generated by (1.9) converges weakly to some element of $VI(A, C)$.

In [32,33], Takahashi and Zembayashi studied the problem of finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of an equilibrium problem in the framework of Banach spaces. Wattanawitoon and Kumam [34] using the idea of Takahashi and Zembayashi [32] extend the notion from relatively nonexpansive mappings or ϕ-nonexpansive mappings to two relatively quasi-nonexpansive mappings and also proved some strong convergence theorems to approximate a common fixed point of relatively quasi-nonexpansive mappings and the set of solutions of an equilibrium problem in the framework of Banach spaces.

On the other hand, the block iterative method is a method which often used by many authors to solve the convex feasibility problem (see, [11,35,36], etc.). In 2008, Plubtieng and Ungchittrakool [37] established strong convergence theorems of block iterative methods for a finite family of relatively nonexpansive mappings in a Banach space by using the hybrid method in mathematical programming. In 2010, Chang et al. [38] proposed the modified block iterative algorithm for solving the convex feasibility problems for an infinite family of closed and uniformly quasi-ϕ-asymptotically nonexpansive mapping, they obtain the strong convergence theorems in a Banach space.

In this article, motivated and inspired by the study of Chang et al. [38], Qin et al. [9], Takahashi and Zembayashi [32], Wattanawitoon and Kumam [34], and Zegeye [39], we introduce a new modified block hybrid projection algorithm for finding a common element of the set of the variational inequality for an α-inverse-strongly monotone operator, the set of solutions of the generalized equilibrium problems, and the set of common fixed points of an infinite family of closed and uniformly quasi-ϕ-asymptotically nonexpansive mappings which more general than closed quasi-ϕ-nonexpansive mappings in the framework Banach spaces. The results presented in this article improve and generalize the main results of Chang et al. [38], Zegeye [39], Wattanawitoon and Kumam [34], and some well-known results in the literature.

2 Basic results

We also need the following lemmas for the proof of our main results.

Lemma 2.1. (*Beauzamy* [40] and *Xu* [41]). *If E be a 2-uniformly convex Banach space. Then, for all* $x, y \in E$, *we have*

$$\| x - y \| \leq \frac{2}{c^2} \| Jx - Jy \|,$$

where J is the normalized duality mapping of E and $0 < c \leq 1$.

The best constant $\frac{1}{c}$ in lemma is called the *p-uniformly convex constant* of *E*.

Lemma 2.2. (*Beauzamy* [40] and *Zalinescu* [42]). *If E be a p-uniformly convex Banach space and let p be a given real number with* $p \geq 2$. *Then, for all* $x, y \in E$, $j_x \in J_p(x)$, *and* $j_y \in J_p(y)$

$$\langle x - y, j_x - j_y \rangle \geq \frac{c^p}{2^{p-2}p} \| x - y \|^p,$$

where J_p *is the generalized duality mapping of E, and* $\frac{1}{c}$*is the p-uniformly convexity constant of E.*

Lemma 2.3. (*Kamimura and Takahashi* [17]). *Let E be a uniformly convex and smooth Banach space and let* $\{x_n\}$ *and* $\{y_n\}$ *be two sequences of E. If* $\phi(x_n, y_n) \to 0$ *and either* $\{x_n\}$ *or* $\{y_n\}$ *is bounded, then* $||x_n - y_n|| \to 0$.

Lemma 2.4. (*Alber* [14]). *Let C be a nonempty closed convex subset of a smooth Banach space E and* $\times \in E$. *Then,* $x_0 = \Pi_C x$ *if and only if*

$$\langle x_0 - y, Jx - Jx_0 \rangle \geq 0, \quad \forall y \in C.$$

Lemma 2.5. (*Alber* [14]). *Let E be a reflexive, strictly convex, and smooth Banach space, let C be a nonempty closed convex subset of E and let* $\times \in E$. *Then,*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x), \quad \forall y \in C.$$

For solving the equilibrium problem for a bifunction $f : C \times C \to \mathbb{R}$, let us assume that *f* satisfies the following conditions:

(A1) $f(x, x) = 0$ for all $x \in C$;

(A2) f is monotone, i.e., $f(x, y) + f(y, x) \leq 0$ for all $x, y \in C$;

(A3) for each $x, y, z \in C$,

$$\lim_{t \downarrow 0} f(tz + (1 - t)x, y) \leq f(x, y);$$

(A4) for each $x \in C$, $y \alpha f(x, y)$ is convex and lower semi-continuous.

Lemma 2.6. (*Blum and Oettli* [1]). *Let C be a closed convex subset of a smooth, strictly convex, and reflexive Banach space E, let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4), and let r >0 and $\times \in E$. Then, there exists $z \in C$ such that*

$$f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0, \quad \forall y \in C.$$

Replacing x with $J^{-1}(Jx - rBx)$, where B is a monotone mapping from C into E^*, then there exists $z \in C$ such that

$$f(z, y) + \langle Bz, y - z\rangle + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0, \quad \forall y \in C.$$

Lemma 2.7. (*Zegeye* [39]). *Let C be a closed convex subset of a uniformly smooth, strictly convex, and reflexive Banach space E, and let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4), and let B be a monotone mapping from C into E^*. For r >0 and $\times \in E$, define a mapping $T_r : C \to C$ as follows:*

$$T_r x = \left\{ z \in C : f(z, y) + \langle Bx, y - z\rangle + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0, \quad \forall y \in C \right\}$$

for all $\times \in C$. Then, the following hold:

(1) *T_r is single-valued;*

(2) *T_r is a firmly nonexpansive-type mapping, for all $x, y \in E$,*

$$\langle T_r x - T_r y, JT_r x - JT_r y\rangle \leq \langle T_r x - T_r y, Jx - Jy\rangle;$$

(3) *$F(T_r) = GEP(f, B)$;*

(4) *$GEP(f, B)$ is closed and convex.*

Lemma 2.8. (*Zegeye* [39]). *Let C be a closed convex subset of a smooth, strictly convex, and reflexive Banach space E, let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4), and let B be a monotone mapping from C into E^*. For r >0, $\times \in E$, and $q \in F(T_r)$, we have that*

$$\phi(q, T_r x) + \phi(T_r x, x) \leq \phi(q, x).$$

Let E be a reflexive, strictly convex, smooth Banach space and J is the duality mapping from E into E^*. Then, J^{-1} is also single value, one-to-one, surjective, and it is the duality mapping from E^* into E. We make use of the following mapping V studied in Alber [14]

$$V(x, x^*) = \| x\|^2 - 2\langle x, x^*\rangle + \| x^*\|^2 \tag{2.1}$$

for all $x \in E$ and $x^* \in E^*$, that is, $V(x, x^*) = \phi(x, J^{-1}(x^*))$.

Lemma 2.9. (*Alber* [14]). *Let E be a reflexive, strictly convex, smooth Banach space and let V be as in* (2.1). *Then,*

$$V(x, x^*) + 2\langle J^{-1}(x^*) - x, y^*\rangle \leq V(x, x^* + y^*)$$

for all $\times \in E$ and $x^*, y^* \in E^*$.

An operator $\mathcal{M} \subset E \times E^*$ is said to be *monotone* if $\langle x - y, x^* - y^*\rangle \geq 0$ whenever $(x, x^*), (y, y^*) \in T$. We denote the set $\{x \in E : 0 \in Tx\}$ by $\mathcal{M}^{-1}0$. A monotone $\mathcal{M}$ is said to be *maximal* if its graph $G(\mathcal{M}) = \{(x, y) : y \in \mathcal{M}x\}$ is not property contained in the graph of any other monotone operator. If $\mathcal{M}$ is maximal monotone, then the solution set $\mathcal{M}^{-1}0$ is closed and convex. Let E be a reflexive, strictly convex, and smooth Banach space, it is known that $\mathcal{M}$ is a maximal monotone if and only if $R(J + r\mathcal{M}) = E^*$ for all $r > 0$. Define the *resolvent* of $\mathcal{M}$ by $J_r x = x_r$. In other words, $J_r = (J + r\mathcal{M})^{-1}J$ for all $r > 0$. J_r is a single-valued mapping from E to $D(\mathcal{M})$. Also, $\mathcal{M}^{-1}(0) = F(J_r)$ for all $r > 0$, where $F(J_r)$ is the set of all fixed points of J_r. Define, for $r > 0$, the *Yosida approximation* of $\mathcal{M}$ by $\mathcal{M}_r = (J - JJ_r)/r$. We know that $\mathcal{M}_r x \in \mathcal{M}(J_r x)$ for all $r > 0$ and $x \in E$.

Let A be an inverse-strongly monotone mapping of C into E^* which is said to be *hemicontinuous* if for all $x, y \in C$, the mapping F of [0, 1] into E^*, defined by $F(t) = A(tx + (1 - t)y)$, is continuous with respect to the weak* topology of E^*. We define by $N_C(v)$ *the normal cone* for C at a point $v \in C$, that is,

$$N_C(v) = \{x^* \in E^* : \langle v - y, x^* \rangle \geq 0, \forall y \in C\}. \tag{2.2}$$

Lemma 2.10. (*Rockafellar* [43]). *Let C be a nonempty, closed convex subset of a Banach space E, and A is a monotone, hemicontinuous operator of C into E^*. Let $\mathcal{M} \subset E \times E^*$ be an operator defined as follows:*

$$\mathcal{M}v = \begin{cases} Av + N_C(v), & v \in C; \\ \emptyset \text{otherwise}. \end{cases} \tag{2.3}$$

Then, $\mathcal{M}$ is maximal monotone and $\mathcal{M}^{-1}0 = VI(A, C)$.

Lemma 2.11. (*Chang et al.* [38]). *Let E be a uniformly convex Banach space, $r > 0$ be a positive number and $B_r(0)$ be a closed ball of E. Then, for any given sequence $\{x_i\}_{i=1}^{\infty} \subset B_r(0)$ and for any given sequence $\{\lambda_i\}_{i=1}^{\infty}$ of positive number with $\sum_{n=1}^{\infty} \lambda_n = 1$, there exists a continuous, strictly increasing, and convex function $g : [0, 2r) \to [0, \infty)$ with $g(0) = 0$ such that, for any positive integer i, j with $i < j$,*

$$\left\| \sum_{n=1}^{\infty} \lambda_n x_n \right\|^2 \leq \sum_{n=1}^{\infty} \lambda_n \| x_n \|^2 - \lambda_i \lambda_j g(\| x_i - x_j \|). \tag{2.4}$$

Lemma 2.12. (*Chang et al.* [38]). *Let E be a real uniformly smooth and strictly convex Banach space, and C be a nonempty closed convex subset of E. Let $T : C \to C$ be a closed and quasi-ϕ-asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1, \infty)$, $k_n \to 1$. Then, $F(T)$ is a closed convex subset of C.*

3 Main results

Definition 3.1. (Chang et al. [38]) (1) Let $\{S_i\}_{i=1}^{\infty} : C \to C$ be a sequence of mapping. $\{S_i\}_{i=1}^{\infty}$ is said to be *a family of uniformly quasi-ϕ-asymptotically nonexpansive mappings*, if $\mathcal{F} := \cap_{n=1}^{\infty} F(S_n) \neq \emptyset$, and there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ such that for each $i \geq 1$

$$\phi(p, S_i^n x) \leq k_n \phi(p, x), \quad \forall p \in \mathcal{F}, \quad x \in C, \quad \forall n \geq 1. \tag{3.1}$$

(2) A mapping $S : C \to C$ is said to be *uniformly L-Lipschitz continuous*, if there exists a constant $L > 0$ such that

$$\| S^n x - S^n y \| \leq L \| x - y \|, \quad \forall x, y \in C. \tag{3.2}$$

In this section, we prove the new convergence theorems for finding the set of solutions of a general equilibrium problems, the common fixed point set of a family of closed and uniformly quasi-ϕ-asymptotically nonexpansive mappings, and the solution set of variational inequalities for an α-inverse strongly monotone mapping in a 2-uniformly convex and uniformly smooth Banach space.

Theorem 3.2. *Let C be a nonempty closed and convex subset of a 2-uniformly convex and uniformly smooth Banach space E. Let A be an α-inverse-strongly monotone mapping of C into E^* satisfying $||Ay|| \le ||Ay - Au||$, $\forall y \in C$ and $u \in VI(A, C) \neq \emptyset$. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4) and B be a continuous monotone mapping of C into E^*. Let $\{S_i\}_{i=1}^{\infty} : C \to C$be an infinite family of closed uniformly L_i-Lipschitz continuous and uniformly quasi-ϕ-asymptotically nonexpansive mappings with a sequence $\{k_n\} \subset [1, \infty)$, $k_n \to 1$ such that $F := \cap_{i=1}^{\infty} F(S_i) \cap GEP(f, B) \cap VI(A, C)$is a nonempty and bounded subset in C. For an initial point $x_0 \in E$ with $x_1 = \Pi_{C_1} x_0$and $C_1 = C$, we define the sequence $\{x_n\}$ as follows:*

$$\begin{cases} z_n = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n), \\ y_n = J^{-1}(\alpha_{n,0} Jx_n + \sum_{i=1}^{\infty} \alpha_{n,i} JS_i^n z_n), \\ f(u_n, y) + \langle By_n, y - u_n \rangle + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n \rangle \ge 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \le \phi(z, x_n) + \zeta_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \ge 1, \end{cases} \tag{3.3}$$

where $\zeta_n = \sup_{q \in F} (k_n - 1)\phi(q, x_n)$, $\{\alpha_{n,i}\}$ is sequence in $[0, 1]$, $\{r_n\} \subset [d, \infty)$ for some $d > 0$ and $\{\lambda_n\} \subset [a, b]$ for some a,b with $0 < a < b < c^2\alpha/2$, where $\frac{1}{c}$is the 2-uniformly convexity constant of E. If $\sum_{i=0}^{\infty} \alpha_{n,i} = 1$for all $n \ge 0$ and $\liminf_{n\to\infty} \alpha_{n,0}\alpha_{n,i} > 0$ *for all $i \ge 1$, then $\{x_n\}$ converges strongly to $p \in F$, where $p = \Pi_F x_0$.*

Proof. We first show that C_{n+1} is closed and convex for each $n \ge 0$. Clearly, $C_1 = C$ is closed and convex. Suppose that C_n is closed and convex for each $n \in \mathbb{N}$. Since for any $z \in C_n$, we know that $\phi(z, u_n) \le \phi(z, x_n) + \zeta_n$ is equivalent to $2\langle z, Jx_n - Ju_n \rangle \le ||x_n||^2 - ||u_n||^2 + \zeta_n$. Hence, C_{n+1} is closed and convex. Next, we show that $F \subset C_n$ for all $n \ge 0$. Indeed, put $u_n = T_{r_n} y_n$ for all $n \ge 0$. On the other hand, from Lemma 2.7, one has $T_{r_n}$is relatively quasi-nonexpansive mappings and $F \subset C_1 = C$. Suppose $F \subset C_n$ for $n \in \mathbb{N}$, by the convexity of $|| \cdot ||^2$, property of ϕ, Lemma 2.11 and by uniformly quasi-ϕ-asymptotically nonexpansive of S_n for each $q \in F \subset C_n$, we have

$$\begin{aligned} \phi(q, u_n) &= \phi(q, T_{r_n} y_n) \\ &\le \phi(q, y_n) \\ &= \phi\left(q, J^{-1}\left(\alpha_{n,0}Jx_n + \sum_{i=1}^{\infty}\alpha_{n,i}JS_i^n z_n\right)\right) \\ &= \| q\|^2 - 2\left\langle q, \alpha_{n,0}Jx_n + \sum_{i=1}^{\infty}\alpha_{n,i}JS_i^n z_n\right\rangle + \left\|\alpha_{n,0}Jx_n + \sum_{i=1}^{\infty}\alpha_{n,i}JS_i^n z_n\right\|^2 \\ &= \| q\|^2 - 2\alpha_{n,0}\langle q, Jx_n\rangle - 2\sum_{i=1}^{\infty}\alpha_{n,i}\langle q, JS_i^n z_n\rangle + \left\|\alpha_{n,0}Jx_n + \sum_{i=1}^{\infty}\alpha_{n,i}JS_i^n z_n)\right\|^2 \\ &\le \| q\|^2 - 2\alpha_{n,0}\langle q, Jx_n\rangle - 2\sum_{i=1}^{\infty}\alpha_{n,i}\langle q, JS_i^n z_n\rangle + \alpha_{n,0}\| Jx_n\|^2 + \sum_{i=1}^{\infty}\alpha_{n,i}\| JS_i^n z_n\|^2 \\ &\quad - \alpha_{n,0}\alpha_{n,j} g\| Jz_n - JS_j^n z_n\| \\ &= \| q\|^2 - 2\alpha_{n,0}\langle q, Jx_n\rangle + \alpha_{n,0}\| Jx_n\|^2 - 2\sum_{i=1}^{\infty}\alpha_{n,i}\langle q, JS_i^n z_n\rangle \\ &\quad + \sum_{i=1}^{\infty}\alpha_{n,i}\| JS_i^n z_n\|^2 - \alpha_{n,0}\alpha_{n,j} g\| Jz_n - JS_j^n z_n\| \\ &= \alpha_{n,0}\phi(q, x_n) + \sum_{i=1}^{\infty}\alpha_{n,i}\phi(q, S_i^n z_n) - \alpha_{n,0}\alpha_{n,j} g\| Jz_n - JS_j^n z_n\| \\ &\le \alpha_{n,0}\phi(q, x_n) + \sum_{i=1}^{\infty}\alpha_{n,i}k_n\phi(q, z_n) - \alpha_{n,0}\alpha_{n,j} g\| Jz_n - JS_j^n z_n\| . \end{aligned} \tag{3.4}$$

It follows from Lemma 2.9 that

$$\begin{aligned}\phi(q,z_n) &= \phi(q,\Pi_C J^{-1}(Jx_n-\lambda_n Ax_n))\\ &\le \phi(q,J^{-1}(Jx_n-\lambda_n Ax_n))\\ &= V(q,Jx_n-\lambda_n Ax_n)\\ &\le V(q,(Jx_n-\lambda_n Ax_n)+\lambda_n Ax_n)-2\langle J^{-1}(Jx_n-\lambda_n Ax_n)-q,\lambda_n Ax_n\rangle\\ &= V(q,Jx_n)-2\lambda_n\langle J^{-1}(Jx_n-\lambda_n Ax_n)-q,Ax_n\rangle\\ &= \phi(q,x_n)-2\lambda_n\langle x_n-q,Ax_n\rangle+2\langle J^{-1}(Jx_n-\lambda_n Ax_n)-x_n,-\lambda_n Ax_n\rangle.\end{aligned} \tag{3.5}$$

Since $q\in VI(A,C)$ and A is an α-inverse-strongly monotone mapping, we have

$$\begin{aligned}-2\lambda_n\langle x_n-q,Ax_n\rangle &= -2\lambda_n\langle x_n-q,Ax_n-Aq\rangle-2\lambda_n\langle x_n-q,Aq\rangle\\ &\le -2\lambda_n\langle x_n-q,Ax_n-Aq\rangle\\ &= -2\alpha\lambda_n\|Ax_n-Aq\|^2.\end{aligned} \tag{3.6}$$

From Lemma 2.1 and A is an α-inverse-strongly monotone mapping, we also have

$$\begin{aligned}2\langle J^{-1}(Jx_n-\lambda_n Ax_n)-x_n,-\lambda_n Ax_n\rangle &= 2\langle J^{-1}(Jx_n-\lambda_n Ax_n)-J^{-1}(Jx_n),-\lambda_n Ax_n\rangle\\ &\le 2\|J^{-1}(Jx_n-\lambda_n Ax_n)-J^{-1}(Jx_n)\|\|\lambda_n Ax_n\|\\ &\le \tfrac{4}{c^2}\|JJ^{-1}(Jx_n-\lambda_n Ax_n)-JJ^{-1}(Jx_n)\|\|\lambda_n Ax_n\|\\ &= \tfrac{4}{c^2}\|Jx_n-\lambda_n Ax_n-Jx_n\|\|\lambda_n Ax_n\|\\ &= \tfrac{4}{c^2}\|\lambda_n Ax_n\|^2\\ &= \tfrac{4}{c^2}\lambda_n^2\|Ax_n\|^2\\ &\le \tfrac{4}{c^2}\lambda_n^2\|Ax_n-Aq\|^2.\end{aligned} \tag{3.7}$$

Substituting (3.6) and (3.7) into (3.5), we obtain

$$\begin{aligned}\phi(q,z_n) &\le \phi(q,x_n)-2\alpha\lambda_n\|Ax_n-Aq\|^2+\tfrac{4}{c^2}\lambda_n^2\|Ax_n-Aq\|^2\\ &= \phi(q,x_n)+2\lambda_n\left(\tfrac{2}{c^2}\lambda_n-\alpha\right)\|Ax_n-Aq\|^2\\ &\le \phi(q,x_n).\end{aligned} \tag{3.8}$$

Substituting (3.8) into (3.4), we also have

$$\begin{aligned}\phi(q,u_n) &\le \alpha_{n,0}\phi(q,x_n)+\sum_{i=1}^{\infty}\alpha_{n,i}k_n\phi(q,x_n)-\alpha_{n,0}\alpha_{n,j}g\|Jz_n-JS_j^nz_n\|\\ &\le \alpha_{n,0}k_n\phi(q,x_n)+\sum_{i=1}^{\infty}\alpha_{n,i}k_n\phi(q,x_n)-\alpha_{n,0}\alpha_{n,j}g\|Jz_n-JS_j^nz_n\|\\ &\le k_n\phi(q,x_n)-\alpha_{n,0}\alpha_{n,j}g\|Jz_n-JS_j^nz_n\|\\ &\le \phi(q,x_n)+\sup_{q\in F}(k_n-1)\phi(q,x_n)-\alpha_{n,0}\alpha_{n,j}g\|Jz_n-JS_j^nz_n\|\\ &= \phi(q,x_n)+\zeta_n-\alpha_{n,0}\alpha_{n,j}g\|Jz_n-JS_j^nz_n\|\\ &\le \phi(q,x_n)+\zeta_n.\end{aligned} \tag{3.9}$$

This shows that $q\in C_{n+1}$ implies that $F\subset C_{n+1}$ and hence, $F\subset C_n$ for all $n\ge 0$. This implies that the sequence $\{x_n\}$ is well defined. From definition of C_{n+1} that $x_n=\Pi_{C_n}x_0$ and $x_{n+1}=\Pi_{C_{n+1}}x_0\in C_{n+1}\subset C_n$, we have

$$\phi(x_n,x_0)\le\phi(x_{n+1},x_0),\quad \forall n\ge 0. \tag{3.10}$$

By Lemma 2.5, we get

$$\begin{aligned}\phi(x_n, x_0) &= \phi(\Pi_{C_n}x_0, x_0)\\ &\leq \phi(q, x_0) - \phi(q, x_n)\\ &\leq \phi(q, x_0), \quad \forall q \in F.\end{aligned} \tag{3.11}$$

From (3.10) and (3.11), then $\{\phi(x_n, x_0)\}$ are nondecreasing and bounded. Hence, we obtain that $\lim_{n\to\infty} \phi(x_n, x_0)$ exists. In particular, by (1.5), the sequence $\{(||x_n|| - ||x_0||)^2\}$ is bounded. This implies $\{x_n\}$ is also bounded. Denote

$$K = \sup_{n\geq 0}\{\| x_n \|\} < \infty. \tag{3.12}$$

Moreover, by the definition of $\{\zeta_n\}$ and (3.12), it follows that

$$\zeta_n \to 0 \quad \text{as } n \to \infty. \tag{3.13}$$

Next, we show that $\{x_n\}$ is a Cauchy sequence in C. Since $x_m = \Pi_{C_m}x_0 \in C_m \subset C_n$, for $m > n$, by Lemma 2.5, we have

$$\begin{aligned}\phi(x_m, x_n) &= \phi(x_m, \Pi_{C_n}x_0)\\ &\leq \phi(x_m, x_0) - \phi(\Pi_{C_n}x_0, x_0)\\ &= \phi(x_m, x_0) - \phi(x_n, x_0).\end{aligned}$$

Since $\lim_{n\to\infty} \phi(x_n, x_0)$ exists and we taking $m, n \to \infty$, then we get $\phi(x_m, x_n) \to 0$. From Lemma 2.3, we have $\lim_{n\to\infty} ||x_m - x_n|| = 0$. Thus, $\{x_n\}$ is a Cauchy sequence and by the completeness of E and there exist a point $p \in C$ such that $x_n \to p$ as $n \to \infty$.

Now, we claim that $||Ju_n - Jx_n|| \to 0$, as $n \to \infty$. By definition of $\Pi_{C_n}x_0$, we have

$$\begin{aligned}\phi(x_{n+1}, x_n) &= \phi(x_{n+1}, \Pi_{C_n}x_0)\\ &\leq \phi(x_{n+1}, x_0) - \phi(\Pi_{C_n}x_0, x_0)\\ &= \phi(x_{n+1}, x_0) - \phi(x_n, x_0).\end{aligned}$$

Since $\lim_{n\to\infty} \phi(x_n, x_0)$ exists, we also have

$$\lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0. \tag{3.14}$$

Again from Lemma 2.3 that

$$\lim_{n\to\infty} \| x_{n+1} - x_n \| = 0. \tag{3.15}$$

From J is uniformly norm-to-norm continuous on bounded subsets of E, we obtain

$$\lim_{n\to\infty} \| Jx_{n+1} - Jx_n \| = 0. \tag{3.16}$$

Since $x_{n+1} = \Pi_{C_{n+1}}x_0 \in C_{n+1} \subset C_n$ and the definition of C_{n+1}, we have

$$\phi(x_{n+1}, u_n) \leq \phi(x_{n+1}, x_n) + \zeta_n.$$

By (3.14) and (3.13) that

$$\lim_{n\to\infty} \phi(x_{n+1}, u_n) = 0. \tag{3.17}$$

Again applying Lemma 2.3, we have

$$\lim_{n\to\infty} \| x_{n+1} - u_n \| = 0. \tag{3.18}$$

Since

$$\begin{aligned}\| u_n - x_n \| &= \| u_n - x_{n+1} + x_{n+1} - x_n \| \\ &\le \| u_n - x_{n+1} \| + \| x_{n+1} - x_n \| .\end{aligned}$$

It follows that

$$\lim_{n\to\infty} \| u_n - x_n \| = 0. \tag{3.19}$$

Since J is uniformly norm-to-norm continuous on bounded subsets of E, we also have

$$\lim_{n\to\infty} \| Ju_n - Jx_n \| = 0. \tag{3.20}$$

Next, we will show that $p \in F := GEP(f, B) \cap (\cap_{i=1}^{\infty} F(S_i)) \cap VI(A, C)$.

(a) First, we show that $p \in GEP(f, B)$. From (3.4) and (3.8), we get $\phi(p, y_n) \le \phi(p, x_n)$. By Lemma 2.8 and $u_n = T_{r_n} y_n$, we observe that

$$\begin{aligned}\phi(u_n, y_n) &= \phi(T_{r_n} y_n, y_n) \\ &\le \phi(q, y_n) - \phi(q, T_{r_n} y_n) \\ &\le \phi(q, x_n) - \phi(q, T_{r_n} y_n) \\ &= \phi(q, x_n) - \phi(q, u_n) \\ &= \| q\|^2 - 2\langle q, Jx_n\rangle + \| x_n\|^2 - (\| q\|^2 - 2\langle q, Ju_n\rangle + \| u_n\|^2) \\ &= \| x_n\|^2 - \| u_n\|^2 - 2\langle q, Jx_n - Ju_n\rangle \\ &\le \| x_n - u_n \| (\| x_n \| + \| u_n \|) + 2 \| q \| \| Jx_n - Ju_n \| .\end{aligned} \tag{3.21}$$

From (3.19), (3.20), and Lemma 2.3, we have

$$\lim_{n\to\infty} \| u_n - y_n \| = 0. \tag{3.22}$$

Again since J is uniformly norm-to-norm continuous, we also have

$$\lim_{n\to\infty} \| Ju_n - Jy_n \| = 0. \tag{3.23}$$

From (A2), we note that

$$\langle By_n, y - u_n\rangle + \tfrac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \ge -f(u_n, y) \ge f(y, u_n), \quad \forall y \in C,$$

and hence

$$\langle By_n, y - u_n\rangle + \left\langle y - u_n, \frac{Ju_n - Jy_n}{r_n}\right\rangle \ge f(y, u_n), \quad \forall y \in C. \tag{3.24}$$

For t with $0 < t < 1$ and $y \in C$, let $y_t = ty + (1 - t)p$. Then, $y_t \in C$ and hence

$$0 \ge -\langle By_n, y_t - u_n\rangle - \left\langle y_t - u_n, \frac{Ju_n - Jy_n}{r_n}\right\rangle + f(y_t, u_n), \quad \forall y_t \in C.$$

It follows that

$$\begin{aligned}\langle By_t, y_t - u_n\rangle &\ge \langle By_t, y_t - u_n\rangle - \langle By_n, y_t - u_n\rangle - \left\langle y_t - u_n, \frac{Ju_n - Jy_n}{r_n}\right\rangle + f(y_t, u_n), \quad \forall y_t \in C \\ &= \langle By_t, y_t - u_n\rangle - \langle Bu_n, y_t - u_n\rangle + \langle Bu_n, y_t - u_n\rangle \\ &\quad - \langle By_n, y_t - u_n\rangle - \left\langle y_t - u_n, \frac{Ju_n - Jy_n}{r_n}\right\rangle + f(y_t, u_n), \quad \forall y_t \in C \\ &= \langle By_t - Bu_n, y_t - u_n\rangle + \langle Bu_n - By_n, y_t - u_n\rangle - \left\langle y_t - u_n, \frac{Ju_n - Jy_n}{r_n}\right\rangle + f(y_t, u_n), \quad \forall y_t \in C.\end{aligned}$$

By the continuity of B, J, and from (3.22) and (3.23), we obtain that $Bu_n - By_n \to 0$ as $n \to \infty$. From $r_n > 0$ then $\frac{\|Ju_n - Jy_n\|}{r_n} \to 0$ as $n \to \infty$. Since B is monotone, we know that $\langle By_t - Bu_n, y_t - u_n \rangle \geq 0$. Thus, it follows from (A4) that

$$\begin{aligned} f(y_t, p) &\leq \lim_{n\to\infty} inf\, f(y_t, u_n) \\ &\leq \lim_{n\to\infty} \langle By_t, y_t - u_n \rangle \\ &= \langle By_t, y_t - p \rangle. \end{aligned}$$

From the conditions (A1) and (A4) we have

$$\begin{aligned} 0 &= f(y_t, y_t) \\ &\leq tf(y_t, y) + (1-t)f(y_t, p) \\ &\leq tf(y_t, y) + (1-t)\langle By_t, y_t - p \rangle \\ &\leq tf(y_t, y) + (1-t)t\langle By_t, y - p \rangle, \end{aligned}$$

and hence

$$0 \leq f(y_t, y) + (1-t)\langle By_t, y - p \rangle.$$

Letting $t \to 0$, we have

$$0 \leq f(p, y) + \langle Bp, y - p \rangle, \quad \forall y \in C.$$

This implies that $p \in GEP(f, B)$.

(b) We show that $p \in \cap_{i=1}^{\infty} F(S_i)$. From (3.4) and (3.8), for $q \in F$, we have

$$\lim_{n\to\infty} \phi(q, y_n) = \phi(q, p). \tag{3.25}$$

We note that

$$\begin{aligned} \phi(u_n, z_n) &= \phi(T_{r_n} y_n, z_n) \\ &\leq \phi(q, x_n) - \phi(q, T_{r_n} y_n) \\ &\leq \phi(q, x_n) - \phi(q, y_n) \\ &\to 0 \quad \text{as } n \to \infty. \end{aligned}$$

From Lemma 2.3, we get

$$\lim_{n\to\infty} \| u_n - z_n \| = 0. \tag{3.26}$$

By using the triangle inequality, we have

$$\| x_n - z_n \| \leq \| x_n - u_n \| + \| u_n - z_n \| .$$

It follows from (3.19) and (3.26) that

$$\lim_{n\to\infty} \| x_n - z_n \| = 0, \tag{3.27}$$

and J is uniformly norm-to-norm continuous, we also have

$$\lim_{n\to\infty} \| Jz_n - Jx_n \| = 0. \tag{3.28}$$

By using the triangle inequality, we obtain

$$\| x_{n+1} - y_n \| \leq \| x_{n+1} - u_n \| + \| u_n - y_n \| . \tag{3.29}$$

By (3.18) and (3.22), we get

$$\lim_{n\to\infty} \| x_{n+1} - y_n \| = 0. \tag{3.30}$$

Since J is uniformly norm-to-norm continuous, we obtain

$$\lim_{n\to\infty} \| Jx_{n+1} - Jy_n \| = 0. \tag{3.31}$$

From (3.3), we note that

$$\begin{aligned}
\| Jx_{n+1} - Jy_n \| &= \left\| Jx_{n+1} - (\alpha_{n,0}Jx_n + \sum_{i=1}^{\infty} \alpha_{n,i} JS_i^n z_n) \right\| \\
&= \left\| \alpha_{n,0}Jx_{n+1} - \alpha_{n,0}Jx_n + \sum_{i=1}^{\infty} \alpha_{n,i}Jx_{n+1} - \sum_{i=1}^{\infty} \alpha_{n,i}JS_i^n z_n \right\| \\
&= \left\| \alpha_{n,0}(Jx_{n+1} - Jx_n) + \sum_{i=1}^{\infty} \alpha_{n,i}(Jx_{n+1} - JS_i^n z_n) \right\| \\
&= \left\| \sum_{i=1}^{\infty} \alpha_{n,i}(Jx_{n+1} - JS_i^n z_n) - \alpha_{n,0}(Jx_n - Jx_{n+1}) \right\| \\
&\geq \sum_{i=1}^{\infty} \alpha_{n,i} \| Jx_{n+1} - JS_i^n z_n \| - \alpha_{n,0} \| Jx_n - Jx_{n+1} \|,
\end{aligned}$$

and hence

$$\| Jx_{n+1} - JS_i^n z_n \| \leq \frac{1}{\sum_{i=1}^{\infty} \alpha_{n,i}} (\| Jx_{n+1} - Jy_n \| + \alpha_{n,0} \| Jx_n - Jx_{n+1} \|). \tag{3.32}$$

From (3.16), (3.31), and $\lim\ inf_{n\to\infty} \sum_{i=1}^{\infty} \alpha_{n,i} > 0$, for each $i \geq 1$, we obtain that

$$\lim_{n\to\infty} \| Jx_{n+1} - JS_i^n z_n \| = 0. \tag{3.33}$$

Since J^{-1} is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n\to\infty} \| x_{n+1} - S_i^n z_n \| = 0. \tag{3.34}$$

Again by using the triangle inequality, for each $i \geq 1$, we get

$$\| z_n - S_i^n z_n \| \leq \| z_n - x_n \| + \| x_n - x_{n+1} \| + \| x_{n+1} - S_i^n z_n \| .$$

From (3.15), (3.27), and (3.34), for each $i \geq 1$, it follows that

$$\lim_{n\to\infty} \| z_n - S_i^n z_n \| = 0. \tag{3.35}$$

Since $\lim_{n\to\infty} ||x_n - z_n|| = 0$ and $x_n \to p$ as $n \to \infty$, imply that $z_n \to p$ as $n \to \infty$. By using the triangle inequality, for each $i \geq 1$

$$\| S_i^n z_n - p \| \leq \| S_i^n z_n - z_n \| + \| z_n - p \| .$$

For each $i \geq 1$, we have

$$\lim_{n\to\infty} \| S_i^n z_n - p \| = 0. \tag{3.36}$$

Moreover, by the assumption that for each $i \geq 1$, S_i is uniformly L_i-Lipschitz continuous, hence we have

$$\begin{aligned} \| S_i^{n+1}z_n - S_i^n z_n \| &\leq \| S_i^{n+1}z_n - S_i^{n+1}z_{n+1} \| + \| S_i^{n+1}z_{n+1} - z_{n+1} \| + \| z_{n+1} - z_n \| + \| z_n - S_i^n z_n \| \\ &\leq (L_i + 1) \| z_{n+1} - z_n \| + \| S_i^{n+1}z_{n+1} - z_{n+1} \| + \| z_n - S_i^n z_n \| . \end{aligned} \tag{3.37}$$

By (3.15) and (3.35), it yields that $\| S_i^{n+1}z_n - S_i^n z_n \| \to 0$. From $S_i^n z_n \to p$, we have $S_i^{n+1}z_n \to p$, that is, $S_i S_i^n z_n \to p$. In view of closeness of S_i, we have $S_i p = p$, for all $i \geq 1$. This imply that $p \in \cap_{i=1}^{\infty} F(S_i)$

(c) We show that $p \in VI(A, C)$. Indeed, define $\mathcal{M} \subset E \times E^*$ by

$$\mathcal{M}v = \begin{cases} Av + N_C(v), & v \in C; \\ \emptyset, & v \notin C. \end{cases} \tag{3.38}$$

By Lemma 2.10, $\mathcal{M}$ is maximal monotone and $\mathcal{M}^{-1}0 = VI(A, C)$. Let $(v, w) \in G(\mathcal{M})$. Since $w \in \mathcal{M}v = Av + N_C(v)$, we get $w - Av \in N_C(v)$.

From $z_n \in C$, we have

$$\langle v - z_n, w - Av \rangle \geq 0. \tag{3.39}$$

On the other hand, since $z_n = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n)$. Then, by Lemma 2.4, we have

$$\langle v - z_n, Jz_n - (Jx_n - \lambda_n Ax_n) \rangle \geq 0,$$

and thus

$$\left\langle v - z_n, \frac{Jx_n - Jz_n}{\lambda_n} - Ax_n \right\rangle \leq 0. \tag{3.40}$$

It follows from (3.39) and (3.40) that

$$\begin{aligned} \langle v - z_n, w \rangle &\geq \langle v - z_n, Av \rangle \\ &\geq \langle v - z_n, Av \rangle + \left\langle v - z_n, \frac{Jx_n - Jz_n}{\lambda_n} - Ax_n \right\rangle \\ &= \langle v - z_n, Av - Ax_n \rangle + \left\langle v - z_n, \frac{Jx_n - Jz_n}{\lambda_n} \right\rangle \\ &= \langle v - z_n, Av - Az_n \rangle + \langle v - z_n, Az_n - Ax_n \rangle + \left\langle v - z_n, \frac{Jx_n - Jz_n}{\lambda_n} \right\rangle \\ &\geq - \| v - z_n \| \frac{\|z_n - x_n\|}{\alpha} - \| v - z_n \| \frac{\|Jx_n - Jz_n\|}{a} \\ &\geq -M \left(\frac{\|z_n - x_n\|}{\alpha} + \frac{\|Jx_n - Jz_n\|}{a} \right), \end{aligned}$$

where $M = \sup_{n \geq 1} ||v - z_n||$. Take the limit as $n \to \infty$ and (3.28), we obtain $\langle v - p, w \rangle \geq 0$. By the maximality of $\mathcal{M}$, we have $p \in \mathcal{M}^{-1}0$, that is, $p \in VI(A, C)$.

Finally, we show that $p = \Pi_F x_0$. From $x_n = \Pi_{C_n} x_0$, we have $\langle Jx_0 - Jx_n, x_n - z \rangle \geq 0$, $\forall z \in C_n$. Since $F \subset C_n$, we also have

$$\langle Jx_0 - Jx_n, x_n - y \rangle \geq 0, \quad \forall y \in F.$$

Taking limit $n \to \infty$, we obtain

$$\langle Jx_0 - Jp, p - y \rangle \geq 0, \quad \forall y \in F.$$

By Lemma 2.4, we can conclude that $p = \Pi_F x_0$ and $x_n \to p$ as $n \to \infty$. This completes the proof. □

If $S_i = S$ for each $i \in \mathbb{N}$, then Theorem 3.2 is reduced to the following corollary.

Corollary 3.3. *Let C be a nonempty closed and convex subset of a 2-uniformly convex and uniformly smooth Banach space E. Let A be an α-inverse-strongly monotone mapping of C into E^* satisfying $||Ay|| \leq ||Ay - Au||$, $\forall y \in C$ and $u \in VI(A, C) \neq \emptyset$. Let f*

be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4) and B be a continuous monotone mapping of C into E^*. Let $S : C \to C$ be a closed uniformly L-Lipschitz continuous and quasi-ϕ-asymptotically nonexpansive mappings with a sequence $\{k_n\} \subset [1, \infty)$, $k_n \to 1$ such that $F := F(S) \cap GEP(f, B) \cap VI(A, C)$ is a nonempty and bounded subset in C. For an initial point $x_0 \in E$ with $x_1 = \Pi_{C_1} x_0$ and $C_1 = C$, we define the sequence $\{x_n\}$ as follows:

$$\begin{cases} z_n = \Pi_C J^{-1}(Jx_n - \lambda_n A x_n), \\ y_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J S^n z_n), \\ f(u_n, y) + \langle B y_n, y - u_n \rangle + \frac{1}{r_n}\langle y - u_n, J u_n - J y_n \rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n) + \zeta_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \geq 1, \end{cases} \tag{3.41}$$

where $\zeta_n = \sup_{q \in F} (k_n - 1)\phi(q, x_n)$, $\{\alpha_n\}$ is sequence in $[0, 1]$, $\{r_n\} \subset [d, \infty)$ for some $d > 0$ and $\{\lambda_n\} \subset [a, b]$ for some a, b with $0 < a < b < c^2\alpha/2$, where $\frac{1}{c}$ is the 2-uniformly convexity constant of E. If $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$, then $\{x_n\}$ converges strongly to $p \in F$, where $p = \Pi_F x_0$.

For a special case that $i = 1, 2$, we can obtain the following results on a pair of quasi-ϕ-asymptotically nonexpansive mappings immediately from Theorem 3.2.

Corollary 3.4. *Let C be a nonempty closed and convex subset of a 2-uniformly convex and uniformly smooth Banach space E. Let A be an α-inverse-strongly monotone mapping of C into E^* satisfying $||Ay|| \leq ||Ay - Au||$, $\forall y \in C$ and $u \in VI(A, C) \neq \emptyset$. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4) and B be a continuous monotone mapping of C into E^*. Let $S, T : C \to C$ be two closed quasi-ϕ-asymptotically nonexpansive mappings and uniformly L_S, L_T -Lipschitz continuous, respectively, with a sequence $\{k_n\} \subset [1, \infty)$, $k_n \to 1$ such that $F := F(S) \cap F(T) \cap GEP(f, B) \cap VI(A, C)$ is a nonempty and bounded subset in C. For an initial point $x_0 \in E$ with $x_1 = \Pi_{C_1} x_0$ and $C_1 = C$, we define the sequence $\{x_n\}$ as follows:*

$$\begin{cases} z_n = \Pi_C J^{-1}(Jx_n - \lambda_n A x_n), \\ y_n = J^{-1}(\alpha_n J x_n + \beta_n J S^n z_n + \gamma_n J T^n z_n), \\ f(u_n, y) + \langle B y_n, y - u_n \rangle + \frac{1}{r_n}\langle y - u_n, J u_n - J y_n \rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n) + \zeta_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \geq 1, \end{cases} \tag{3.42}$$

where $\zeta_n = \sup_{q \in F} (k_n - 1)\phi(q, x_n)$, $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0, 1]$, $\{r_n\} \subset [d, \infty)$ for some $d > 0$ and $\{\lambda_n\} \subset [a, b]$ for some a, b with $0 < a < b < c^2\alpha/2$, where $\frac{1}{c}$ is the 2-uniformly convexity constant of E. If $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \geq 0$ and $\liminf_{n\to\infty} \alpha_n\beta_n > 0$ and $\liminf_{n\to\infty} \alpha_n\gamma_n > 0$, then $\{x_n\}$ converges strongly to $p \in F$, where $p = \Pi_F x_0$.

Corollary 3.5. *Let C be a nonempty closed and convex subset of a 2-uniformly convex and uniformly smooth Banach space E. Let A be an α-inverse-strongly monotone mapping of C into E^* satisfying $||Ay|| \leq ||Ay - Au||$, $\forall y \in C$ and $u \in VI(A, C) \neq \emptyset$. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4) and B be a continuous monotone mapping of C into E^*. Let $\{S_i\}_{i=1}^{\infty} : C \to C$ be an infinite family of closed quasi-ϕ-nonexpansive mappings such that $F := \cap_{i=1}^{\infty} F(S_i) \cap GEP(f, B) \cap VI(A, C) \neq \emptyset$. For an initial point $x_0 \in E$ with $x_1 = \Pi_{C_1} x_0$ and $C_1 = C$, we define the sequence $\{x_n\}$ as follows:*

$$\begin{cases} z_n = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n), \\ y_n = J^{-1}(\alpha_{n,0}Jx_n + \sum_{i=1}^{\infty} \alpha_{n,i} JS_i z_n), \\ f(u_n, y) + \langle By_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \geq 1, \end{cases} \tag{3.43}$$

where $\{\alpha_{n,i}\}$ *is sequence in* $[0, 1]$, $\{r_n\} \subset [d, \infty)$ *for some* $d > 0$ *and* $\{\lambda_n\} \subset [a, b]$ *for some a, b with* $0 < a < b < c^2\alpha/2$, *where* $\frac{1}{c}$*is the 2-uniformly convexity constant of E. If* $\sum_{i=0}^{\infty} \alpha_{n,i} = 1$*for all* $n \geq 0$ *and* $\lim\inf_{n\to\infty} \alpha_{n,0}\alpha_{n,i} > 0$ *for all* $i \geq 1$, *then* $\{x_n\}$ *converges strongly to* $p \in F$, *where* $p = \Pi_F x_0$.

Proof Since $\{S_i\}_{i=1}^{\infty} : C \to C$ is an infinite family of closed quasi-ϕ-nonexpansive mappings, it is an infinite family of closed and uniformly quasi-ϕ-asymptotically nonexpansive mappings with sequence $k_n = 1$. Hence, the conditions appearing in Theorem 3.2 F is a bounded subset in C and for each $i \geq 1$, S_i is uniformly L_i-Lipschitz continuous are of no use here. By virtue of the closeness of mapping S_i for each $i \geq 1$, it yields that $p \in F(S_i)$ for each $i \geq 1$, that is, $p \in \cap_{i=1}^{\infty} F(S_i)$. Therefore, all the conditions in Theorem 3.2 are satisfied. The conclusion of Corollary 3.5 is obtained from Theorem 3.2 immediately. □

Corollary 3.6. [39, Theorem 3.2] *Let C be a nonempty closed and convex subset of a 2-uniformly convex and uniformly smooth Banach space E. Let A be an* α*-inverse-strongly monotone mapping of C into* E^* *satisfying* $||Ay|| \leq ||Ay - Au||$, $\forall y \in C$ *and* $u \in VI(A, C) \neq \emptyset$. *Let f be a bifunction from* $C \times C$ *to* $\mathbb{R}$ *satisfying (A1)-(A4) and B be a continuous monotone mapping of C into* E^*. *Let* $\{S_i\}_{i=1}^{N} : C \to C$*be a finite family of closed quasi-*ϕ*-nonexpansive mappings such that* $F := \cap_{i=1}^{N} F(S_i) \cap GEP(f, B) \cap VI(A, C) \neq \emptyset$. *For an initial point* $x_0 \in E$ *with* $x_1 = \Pi_{C_1} x_0$*and* $C1 = C$, *we define the sequence* $\{x_n\}$ *as follows:*

$$\begin{cases} z_n = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n), \\ y_n = J^{-1}(\alpha_{n,0}Jx_n + \sum_{i=1}^{N} \alpha_{n,i} JS_i z_n), \\ f(u_n, y) + \langle By_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \geq 1, \end{cases} \tag{3.44}$$

where $\{\alpha_{n,i}\}$ *is sequence in* $[0, 1]$, $\{r_n\} \subset [d, \infty)$ *for some* $d > 0$ *and* $\{\lambda_n\} \subset [a, b]$ *for some a, b with* $0 < a < b < c^2\alpha/2$, *where* $\frac{1}{c}$*is the 2-uniformly convexity constant of E. If* $\alpha_i \in (0, 1)$ *such that* $\sum_{i=0}^{N} \alpha_i = 1$, *then* $\{x_n\}$ *converges strongly to* $p \in F$, *where* $p = \Pi_F x_0$.

Remark 3.7. Theorem 3.2, Corollaries 3.5 and 3.6 improve and extend the corresponding results in Wattanawitoon and Kumam [34] and Zegeye [39] in the following senses:

- from a solution of the classical equilibrium problem to the generalized equilibrium problem with an infinite family of quasi-ϕ-asymptotically mappings;
- for the mappings, we extend the mappings from nonexpansive mappings, relatively quasi-nonexpansive mappings or quasi-ϕ-nonexpansive mappings and a finite family of closed relatively quasi-nonexpansive mappings to an infinite family of quasi-ϕ-asymptotically nonexpansive mappings.

4 Deduced theorems

Corollary 4.1. *Let C be a nonempty closed and convex subset of a uniformly convex and uniformly smooth Banach space E. Let f be a bifunction from* $C \times C$ *to* $\mathbb{R}$ *satisfying (A1)-(A4). Let B be a continuous monotone mapping of C into* E^*. *Let* $\{S_i\}_{i=1}^{\infty} : C \to C$*be an infinite family of closed and uniformly quasi-*ϕ*-asymptotically nonexpansive mappings with a sequence* $\{k_n\} \subset [1, \infty)$, $k_n \to 1$ *and uniformly* L_i*-Lipschitz continuous such that* $F := \cap_{i=1}^{\infty} F(S_i) \cap GEP(f, B)$*is a nonempty and bounded subset in C. For an initial point* $x_0 \in E$ *with* $x_1 = \Pi_{C_1} x_0$*and* $C_1 = C$, *we define the sequence* $\{x_n\}$ *as follows:*

$$\begin{cases} y_n = J^{-1}(\alpha_{n,0} J x_n + \sum_{i=1}^{\infty} \alpha_{n,i} J S_i^n x_n), \\ f(u_n, y) + \langle B y_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, J u_n - J y_n \rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n) + \zeta_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \geq 0, \end{cases} \tag{4.1}$$

where $\zeta_n = \sup_{q \in F} (k_n - 1)\phi(q, x_n)$, $\{\alpha_{n,i}\}$ *is sequence in* $[0, 1]$, $\{r_n\} \subset [a, \infty)$ *for some* $a > 0$. *If* $\sum_{i=0}^{\infty} \alpha_{n,i} = 1$*for all* $n \geq 0$ *and* $\liminf_{n\to\infty} \alpha_{n,0}\alpha_{n,i} > 0$ *for all* $i \geq 1$, *then* $\{x_n\}$ *converges strongly to* $p \in F$, *where* $p = \Pi_F x_0$.

Proof Put $A \equiv 0$ in Theorem 3.2. Then, we get that $z_n = x_n$. Thus, the method of proof of Theorem 3.2 gives the required assertion without the requirement that E be 2-uniformly convex. □

If setting $B \equiv 0$ in Corollary 4.1, then we have the following corollary.

Corollary 4.2. *Let C be a nonempty closed and convex subset of a uniformly convex and uniformly smooth Banach space E. Let f be a bifunction from* $C \times C$ *to* $\mathbb{R}$ *satisfying (A1)-(A4). Let* $\{S_i\}_{i=1}^{\infty} : C \to C$*be an infinite family of closed and uniformly quasi-*ϕ*-asymptotically nonexpansive mappings with a sequence* $\{k_n\} \subset [1, \infty)$, $k_n \to 1$ *and uniformly* L_i*-Lipschitz continuous such that* $F := \cap_{i=1}^{\infty} F(S_i) \cap EP(f)$*is a nonempty and bounded subset in C. For an initial point* $x_0 \in E$ *with* $x_1 = \Pi_{C_1} x_0$*and* $C_1 = C$, *we define the sequence* $\{x_n\}$ *as follows:*

$$\begin{cases} y_n = J^{-1}(\alpha_{n,0} J x_n + \sum_{i=1}^{\infty} \alpha_{n,i} J S_i^n x_n), \\ f(u_n, y) + \frac{1}{r_n} \langle y - u_n, J u_n - J y_n \rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n) + \zeta_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \geq 0, \end{cases} \tag{4.2}$$

where $\zeta_n = \sup_{q \in F} (k_n - 1)\phi(q, x_n)$, $\{\alpha_{n,i}\}$ *is sequence in* $[0, 1]$, $\{r_n\} \subset [a, \infty)$ *for some* $a >0$. *If* $\sum_{i=0}^{\infty} \alpha_{n,i} = 1$*for all* $n \geq 0$ *and* $\liminf_{n\to\infty} \alpha_{n,0}\alpha_{n,i} > 0$ *for all* $i \geq 1$, *then* $\{x_n\}$ *converges strongly to* $p \in F$, *where* $p = \Pi_F x_0$.

If setting $f \equiv 0$ in Corollary 4.1, then we obtain the following corollary.

Corollary 4.3. *Let C be a nonempty closed and convex subset of a uniformly convex and uniformly smooth Banach space E. Let B be a continuous monotone mapping of C into* E^*. *Let* $\{S_i\}_{i=1}^{\infty} : C \to C$*be an infinite family of closed and uniformly quasi-*ϕ*-asymptotically nonexpansive mappings with a sequence* $\{k_n\} \subset [1, \infty)$, $k_n \to 1$ *and uniformly* L_i*-Lipschitz continuous such that* $F := \cap_{i=1}^{\infty} F(S_i) \cap VI(B, C)$*is a nonempty and bounded subset in C. For an initial point* $x_0 \in E$ *with* $x_1 = \Pi_{C_1} x_0$*and* $C_1 = C$, *we define the sequence* $\{x_n\}$ *as follows:*

$$\begin{cases} y_n = J^{-1}(\alpha_{n,0}Jx_n + \sum_{i=1}^{\infty} \alpha_{n,i}JS_i^n z_n), \\ \langle By_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n) + \zeta_n\}, \\ x_{n+1} = \Pi_{C_{n+1}}x_0, \quad \forall n \geq 0, \end{cases} \tag{4.3}$$

where $\zeta_n = \sup_{q\in F} (k_n - 1)\phi(q, x_n)$, $\{\alpha_{n,i}\}$ *is sequence in* $[0, 1]$, $\{r_n\} \subset [a, \infty)$ *for some* $a > 0$. *If* $\sum_{i=0}^{\infty} \alpha_{n,i} = 1$ *for all* $n \geq 0$ *and* $\liminf_{n\to\infty} \alpha_{n,0}\alpha_{n,i} > 0$ *for all* $i \geq 1$, *then* $\{x_n\}$ *converges strongly to* $p \in F$, *where* $p = \Pi_F x_0$.

Remark 4.4. Corollaries 4.1{4.3 improve and extend the corresponding results in Zegeye [39] and Wattanawitoon and Kumam [34] in the sense from a finite family of closed relatively quasi-nonexpansive mappings and closed relatively quasi-nonexpansive mappings to more general than an infinite family of closed and uniformly quasi-ϕ-asymptotically nonexpansive mappings.

5 Application to Hilbert spaces

If $E = H$, a Hilbert space, then E is 2-uniformly convex (we can choose $c = 1$) and uniformly smooth real Banach space and closed relatively quasi-nonexpansive map reduces to closed quasi-nonexpansive map. Moreover, $J = I$, identity operator on H and $\Pi_C = P_C$, projection mapping from H into C. Thus, the following corollaries hold.

Theorem 5.1. *Let C be a nonempty closed and convex subset of a Hilbert space H. Let f be a bifunction from* $C \times C$ *to* $\mathbb{R}$ *satisfying (A1)-(A4). Let A be an* α*-inverse-strongly monotone mapping of C into H satisfying* $||Ay|| \leq ||Ay - Au||$, $\forall y \in C$ *and* $u \in VI(A, C) \neq \emptyset$ *and B be a continuous monotone mapping of C into H. Let* $\{S_i\}_{i=1}^{\infty} : C \to C$ *be an infinite family of closed and uniformly quasi-*ϕ*-asymptotically nonexpansive mappings with a sequence* $\{k_n\} \subset [1, \infty)$, $k_n \to 1$ *and uniformly* L_i*-Lipschitz continuous such that* $F := \cap_{i=1}^{\infty} F(S_i) \cap GEP(f, B) \cap VI(A, C)$ *is a nonempty and bounded subset in C. For an initial point* $x_0 \in H$ *with* $x_1 = \Pi_{C_1}x_0$ *and* $C_1 = C$, *we define the sequence* $\{x_n\}$ *as follows:*

$$\begin{cases} z_n = P_C(x_n - \lambda_n Ax_n), \\ y_n = \alpha_{n,0}x_n + \sum_{i=1}^{\infty} \alpha_{n,i}S_i^n z_n, \\ f(u_n, y) + \langle By_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - y_n\rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \| z - u_n \| \leq \| z - x_n \| + \zeta_n\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \quad \forall n \geq 0, \end{cases} \tag{5.1}$$

where $\zeta_n = \sup_{q\in F} (k_n - 1)||q - x_n||$, $\{\alpha_{n,i}\}$ *is sequence in* $[0, 1]$, $\{r_n\} \subset [a, \infty)$ *for some* $a > 0$ *and* $\{\lambda_n\} \subset [a, b]$ *for some* a, b *with* $0 < a < b < \alpha/2$. *If* $\sum_{i=0}^{\infty} \alpha_{n,i} = 1$ *for all* $n \geq 0$ *and* $\liminf_{n\to\infty} \alpha_{n,0}\alpha_{n,i} > 0$ *for all* $i \geq 1$, *then* $\{x_n\}$ *converges strongly to* $p \in F$, *where* $p = \Pi_F x_0$.

Remark 5.2. Theorem 5.1 improve and extend the Corollary 3.7 in Zegeye [39] in the aspect for the mappings, we extend the mappings from a finite family of closed relatively quasi-nonexpansive mappings to more general an infinite family of closed and uniformly quasi-ϕ-asymptotically nonexpansive mappings.

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