Tian Journal of Inequalities and Applications 2011, 2011:77
http://www.journalofinequalitiesandapplications.com/content/2011/1/77

Journal of Inequalities and Applications
a SpringerOpen Journal

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Extension of Hu Ke's inequality and its applications

Jing-Feng Tian

Correspondence:
tianjfhxm_ncepu@yahoo.cn
College of Science and Technology, North China Electric Power University, Baoding, Hebei Province, 071051, People's Republic of China

Abstract

In this paper, we extend Hu Ke's inequality, which is a sharpness of Hölder's inequality. Moreover, the obtained results are used to improve Hao Z-C inequality and Beckenbach-type inequality that is due to Wang.

Mathematics Subject Classification (2000) Primary 26D15; Secondary 26D10

Keywords: integral inequality, Hölder's inequality, Hu Ke's inequality, Hao Z-C inequality, Beckenbach-type inequality, arithmetic-geometric mean inequality

1. Introduction

The classical Hölder's inequality states that if $a_k \geq 0$, $b_k \geq 0 (k = 1, 2, ..., n)$, $p > 0$, $q > 0$ and $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\sum_{k=1}^{n} a_k b_k \leq \left(\sum_{k=1}^{n} a_k^p \right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} b_k^q \right)^{\frac{1}{q}}. \tag{1}$$

The inequality (1) is reversed for $p < 1 (p \neq 0)$. (For $p < 0$, we assume that $a_k, b_k > 0$.)

The following generalization of (1) is given in [1]:

Theorem A. (Generalized Hölder inequality). *Let* $A_{nj} \geq 0$, $\sum_n A_{nj}^{\lambda_j} < \infty$, $\lambda_j > 0$ $(j = 1, 2, ..., k)$. *If* $\sum_{j=1}^{k} \frac{1}{\lambda_j} = 1$, *then*

$$\sum_n \prod_{j=1}^{k} A_{nj} \leq \prod_{j=1}^{k} \left(\sum_n A_{nj}^{\lambda_j} \right)^{1/\lambda_j}. \tag{2}$$

As is well known, Hölder's inequality plays a very important role in different branches of modern mathematics such as linear algebra, classical real and complex analysis, probability and statistics, qualitative theory of differential equations and their applications. A large number of papers dealing with refinements, generalizations and applications of inequalities (1) and (2) and their series analogues in different ares of mathematics have appeared (see e.g. [2-30] and the references therein).

Among various refinements of (1), Hu in [13] established the following interesting theorems.

Theorem B. *Let* $p \geq q > 0$, $\frac{1}{p} + \frac{1}{q} = 1$, *let* $a_n, b_n \geq 0$, $\sum_n a_n^p < \infty$, $\sum_n b_n^q < \infty$, *and let* $1 - e_n + e_m \geq 0$, $\sum_n |e_n| < \infty$. *Then*

$$\sum_n a_n b_n \leq \left(\sum_n b_n^q\right)^{\frac{1}{q}-\frac{1}{p}} \left\{\left[\left(\sum_n b_n^q\right)\left(\sum_n a_n^p\right)\right]^2 - \left[\left(\sum_n b_n^q e_n\right)\left(\sum_n a_n^p\right) - \left(\sum_n b_n^q\right)\left(\sum_n a_n^p e_n\right)\right]^2\right\}^{\frac{1}{2p}}. \tag{3}$$

The integral form is as follows:

Theorem C. *Let E be a measurable set, let* $f(x)$ *and* $g(x)$ *be nonnegative measurable functions with* $\int_E f^p(x)\mathrm{d}x < \infty$, $\int_E g^q(x)\mathrm{d}x < \infty$, *and let* $e(x)$ *be a measurable function with* $1 - e(x) + e(y) \geq 0$. *If* $p \geq q > 0$, $\frac{1}{p} + \frac{1}{q} = 1$, *then*

$$\int_E f(x)g(x)dx \leq \left(\int_E g^q(x)dx\right)^{\frac{1}{q}-\frac{1}{p}} \left[\left(\int_E f^p(x)dx \int_E g^q(x)dx\right)^2 - \left(\int_E f^p(x)e(x)dx \int_E g^q(x)dx - \int_E f^p(x)dx \int_E g^q(x)e(x)dx\right)^2\right]^{\frac{1}{2p}}. \tag{4}$$

The purpose of this work is to give extensions of inequalities (3) and (4) and establish their corresponding reversed versions. Moreover, the obtained results will be applied to improve Hao Z-C inequality [31] and Beckenbach-type inequality that is due to Wang [32]. The rest of this paper is organized as follows. In Section 2, we present extensions of (3) and (4) and establish their corresponding reversed versions. In Section 3, we apply the obtained results to improve Hao Z-C inequality and Beckenbach-type inequality that is due to Wang. Consequently, we obtain the refinement of arithmetic-geometric mean inequality. Finally, a brief summary is given in Section 4.

2. Extension of Hu Ke's Inequality

We begin this section with two lemmas, which will be used in the sequel.

Lemma 2.1. *(e.g.* [16], *p. 12). Let* $A_{kj} > 0$ $(j = 1, 2, ..., m, k = 1, 2, ..., n)$, $\sum_{j=1}^m \frac{1}{\lambda_j} = 1$. *If* $\lambda_1 > 0$, $\lambda_j < 0$ $(j = 2, 3, ..., m)$, *then*

$$\sum_{k=1}^n \prod_{j=1}^m A_{kj} \geq \prod_{j=1}^m \left(\sum_{k=1}^n A_{kj}^{\lambda_j}\right)^{\frac{1}{\lambda_j}}. \tag{5}$$

Lemma 2.2. [9]*If* $x > -1$, $\alpha > 1$ or $\alpha < 0$, then

$$(1+x)^\alpha \geq 1 + \alpha x. \tag{6}$$

The inequality is reversed for $0 < \alpha < 1$.

Next, we give an extension of Hu Ke's inequality, as follows.

Theorem 2.3. *Let* $A_{nj} \geq 0$, $\sum_n A_{nj}^{\lambda_j} < \infty$ $(j = 1, 2, \ldots, k)$, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$, $\sum_{j=1}^{k} \frac{1}{\lambda_j} = 1$, *and let* $1\text{-}e_n + e_m \geq 0$, $\sum_n |e_n| < \infty$. *If k is even, then*

$$\begin{aligned}\sum_n \prod_{j=1}^{k} A_{nj} &\leq \prod_{j=1}^{\frac{k}{2}} \Bigg\{ \left(\sum_n A_{n(2j-1)}^{\lambda_{2j-1}} \right)^{\frac{1}{\lambda_{2j-1}} - \frac{1}{\lambda_{2j}}} \times \Bigg[\left(\left(\sum_n A_{n(2j-1)}^{\lambda_{2j-1}} \right) \left(\sum_n A_{n(2j)}^{\lambda_{2j}} \right) \right)^2 \\ &\quad - \left(\left(\sum_n A_{n(2j-1)}^{\lambda_{2j-1}} e_n \right) \left(\sum_n A_{n(2j)}^{\lambda_{2j}} \right) \right. \\ &\quad \left. - \left(\sum_n A_{n(2j-1)}^{\lambda_{2j-1}} \right) \left(\sum_n A_{n(2j)}^{\lambda_{2j}} e_n \right) \right)^2 \Bigg]^{\frac{1}{2\lambda_{2j}}} \Bigg\}. \end{aligned} \tag{7}$$

If k is odd, then

$$\begin{aligned}\sum_n \prod_{j=1}^{k} A_{nj} &\leq \left(\sum_n A_{nk}^{\lambda_k} \right)^{\frac{1}{\lambda_k}} \times \prod_{j=1}^{\frac{k-1}{2}} \Bigg\{ \left(\sum_n A_{n(2j-1)}^{\lambda_{2j-1}} \right)^{\frac{1}{\lambda_{2j-1}} - \frac{1}{\lambda_{2j}}} \\ &\quad \times \Bigg[\left(\left(\sum_n A_{n(2j-1)}^{\lambda_{2j-1}} \right) \left(\sum_n A_{n(2j)}^{\lambda_{2j}} \right) \right)^2 - \left(\left(\sum_n A_{n(2j-1)}^{\lambda_{2j-1}} e_n \right) \left(\sum_n A_{n(2j)}^{\lambda_{2j}} \right) \right. \\ &\quad \left. - \left(\sum_n A_{n(2j-1)}^{\lambda_{2j-1}} \right) \left(\sum_n A_{n(2j)}^{\lambda_{2j}} e_n \right) \right)^2 \Bigg]^{\frac{1}{2\lambda_{2j}}} \Bigg\}. \end{aligned} \tag{8}$$

The integral form is as follows:

Theorem 2.4. *Let* $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$, $\sum_{j=1}^{k} \frac{1}{\lambda_j} = 1$, *let E be a measurable set,* $F_j(x)$ *be nonnegative measurable functions with* $\int_E F_j^{\lambda_j}(x)dx < \infty$, *and let* $e(x)$ *be a measurable function with* $1\text{-}e(x) + e(y) \geq 0$. *If k is even, then*

$$\begin{aligned}\int_E \prod_{j=1}^{k} F_j(x)\mathrm{d}x &\leq \prod_{j=1}^{\frac{k}{2}} \Bigg\{ \left(\int_E F_{2j-1}^{\lambda_{2j-1}}(x)\mathrm{d}x \right)^{\frac{1}{\lambda_{2j-1}} - \frac{1}{\lambda_{2j}}} \\ &\quad \times \Bigg[\left(\int_E F_{2j-1}^{\lambda_{2j-1}}(x)\mathrm{d}x \int_E F_{2j}^{\lambda_{2j}}(x)\mathrm{d}x \right)^2 \\ &\quad - \left(\int_E F_{2j-1}^{\lambda_{2j-1}}(x)e(x)\mathrm{d}x \int_E F_{2j}^{\lambda_{2j}}(x)\mathrm{d}x \right. \\ &\quad \left. - \int_E F_{2j-1}^{\lambda_{2j-1}}(x)\mathrm{d}x \int_E F_{2j}^{\lambda_{2j}}(x)e(x)\mathrm{d}x \right)^2 \Bigg]^{\frac{1}{2\lambda_{2j}}} \Bigg\}. \end{aligned} \tag{9}$$

If k is odd, then

$$\int_E \prod_{j=1}^{k} F_j(x)\mathrm{d}x \le \left(\int_E F_k^{\lambda_k}(x)\mathrm{d}x\right)^{\frac{1}{\lambda_k}} \times \prod_{j=1}^{\frac{k-1}{2}} \left\{\left(\int_E F_{2j-1}^{\lambda_{2j-1}}(x)\mathrm{d}x\right)^{\frac{1}{\lambda_{2j-1}}-\frac{1}{\lambda_{2j}}} \times \left[\left(\int_E F_{2j-1}^{\lambda_{2j-1}}(x)\mathrm{d}x \int_E F_{2j}^{\lambda_{2j}}(x)\mathrm{d}x\right)^2 - \left(\int_E F_{2j-1}^{\lambda_{2j-1}}(x)e(x)\mathrm{d}x \int_E F_{2j}^{\lambda_{2j}}(x)\mathrm{d}x - \int_E F_{2j-1}^{\lambda_{2j-1}}(x)\mathrm{d}x \int_E F_{2j}^{\lambda_{2j}}(x)e(x)\mathrm{d}x\right)^2\right]^{\frac{1}{2\lambda_{2j}}}\right\}. \tag{10}$$

Proof. We need to prove only Theorem 2.3. The proof of Theorem 2.4 is similar. A simple calculation gives

$$\begin{aligned} &\sum_n \left(\prod_{j=1}^{k} A_{nj}\right) \sum_m \left(\prod_{i=1}^{k} A_{mi}\right)(1-e_n+e_m) \\ &= \sum_n \sum_m \left(\prod_{j=1}^{k} A_{nj}\right)\left(\prod_{i=1}^{k} A_{mi}\right) - \sum_n \sum_m \left(\prod_{j=1}^{k} A_{nj}\right)\left(\prod_{i=1}^{k} A_{mi}\right)e_n \\ &\quad + \sum_n \sum_m \left(\prod_{j=1}^{k} A_{nj}\right)\left(\prod_{i=1}^{k} A_{mi}\right)e_m \\ &= \left(\sum_n \prod_{j=1}^{k} A_{nj}\right)^2. \end{aligned} \tag{11}$$

Case (I). When k is even, by the inequality (2), we have

$$\begin{aligned} &\sum_n \left(\prod_{j=1}^{k} A_{nj}\right) \sum_m \left(\prod_{i=1}^{k} A_{mi}\right)(1-e_n+e_m) \\ &= \sum_n \left(\prod_{j=1}^{k} A_{nj}\right) \sum_m \prod_{i=1}^{k} A_{mi}(1-e_n+e_m)^{\frac{1}{\lambda_i}} \\ &\le \sum_n \left(\prod_{j=1}^{k} A_{nj}\right)\left[\prod_{i=1}^{k}\left(\sum_m A_{mi}^{\lambda_i}(1-e_n+e_m)\right)^{\frac{1}{\lambda_i}}\right] \\ &= \sum_n \left\{\prod_{j=1}^{\frac{k}{2}}\left[\left(A_{n(2j-1)}^{\lambda_{2j-1}} \sum_m A_{m(2j-1)}^{\lambda_{2j-1}}(1-e_n+e_m)\right)^{\frac{1}{\lambda_{2j-1}}-\frac{1}{\lambda_{2j}}} \right.\right. \\ &\quad \times \left(A_{n(2j-1)}^{\lambda_{2j-1}} \sum_m A_{m(2j)}^{\lambda_{2j}}(1-e_n+e_m)\right)^{\frac{1}{\lambda_{2j}}} \\ &\quad \left.\left. \times \left(A_{n(2j)}^{\lambda_{2j}} \sum_m A_{m(2j-1)}^{\lambda_{2j-1}}(1-e_n+e_m)\right)^{\frac{1}{\lambda_{2j}}}\right]\right\}. \end{aligned} \tag{12}$$

Consequently, according to $(\frac{1}{\lambda_1}-\frac{1}{\lambda_2})+\frac{1}{\lambda_2}+\frac{1}{\lambda_2}+(\frac{1}{\lambda_3}-\frac{1}{\lambda_4})+\frac{1}{\lambda_4}+\frac{1}{\lambda_4}+\cdots+(\frac{1}{\lambda_{k-1}}-\frac{1}{\lambda_k})+\frac{1}{\lambda_k}+\frac{1}{\lambda_k}=1$, by using the inequality (2) on the right side of (12), we observe that

$$\begin{aligned}
&\sum_n\Big(\prod_{j=1}^k A_{nj}\Big)\sum_m\Big(\prod_{i=1}^k A_{mi}\Big)(1-e_n+e_m)\\
&\le \prod_{j=1}^{\frac{k}{2}}\Bigg[\Big(\sum_n A_{n(2j-1)}^{\lambda_{2j-1}}\sum_m A_{m(2j-1)}^{\lambda_{2j-1}}(1-e_n+e_m)\Big)^{\frac{1}{\lambda_{2j-1}}-\frac{1}{\lambda_{2j}}}\\
&\quad\times\Big(\sum_n A_{n(2j-1)}^{\lambda_{2j-1}}\sum_m A_{m(2j)}^{\lambda_{2j}}(1-e_n+e_m)\Big)^{\frac{1}{\lambda_{2j}}}\\
&\quad\times\Big(\sum_n A_{n(2j)}^{\lambda_{2j}}\sum_m A_{m(2j-1)}^{\lambda_{2j-1}}(1-e_n+e_m)\Big)^{\frac{1}{\lambda_{2j}}}\Bigg]\\
&=\prod_{j=1}^{\frac{k}{2}}\Bigg\{\Big(\sum_n A_{n(2j-1)}^{\lambda_{2j-1}}\Big)^{\frac{2}{\lambda_{2j-1}}-\frac{2}{\lambda_{2j}}}\times\Bigg[\Big(\sum_n\sum_m A_{n(2j-1)}^{\lambda_{2j-1}}A_{m(2j)}^{\lambda_{2j}}(1-e_n+e_m)\Big)\\
&\quad\times\Big(\sum_n\sum_m A_{n(2j)}^{\lambda_{2j}}A_{m(2j-1)}^{\lambda_{2j-1}}(1-e_n+e_m)\Big)\Bigg]^{\frac{1}{\lambda_{2j}}}\Bigg\}\\
&=\prod_{j=1}^{\frac{k}{2}}\Bigg\{\Big(\sum_n A_{n(2j-1)}^{\lambda_{2j-1}}\Big)^{\frac{2}{\lambda_{2j-1}}-\frac{2}{\lambda_{2j}}}\times\Bigg[\Big(\sum_n A_{n(2j-1)}^{\lambda_{2j-1}}\sum_m A_{m(2j)}^{\lambda_{2j}}\\
&\quad-\sum_n A_{n(2j-1)}^{\lambda_{2j-1}}e_n\sum_m A_{m(2j)}^{\lambda_{2j}}+\sum_n A_{n(2j-1)}^{\lambda_{2j-1}}\sum_m A_{m(2j)}^{\lambda_{2j}}e_m\Big)\\
&\quad\times\Big(\sum_n A_{n(2j)}^{\lambda_{2j}}\sum_m A_{m(2j-1)}^{\lambda_{2j-1}}-\sum_n A_{n(2j)}^{\lambda_{2j}}e_n\sum_m A_{m(2j-1)}^{\lambda_{2j-1}}\\
&\quad+\sum_n A_{n(2j)}^{\lambda_{2j}}\sum_m A_{m(2j-1)}^{\lambda_{2j-1}}e_m\Big)\Bigg]^{\frac{1}{\lambda_{2j}}}\Bigg\}\\
&=\prod_{j=1}^{\frac{k}{2}}\Bigg\{\Big(\sum_n A_{n(2j-1)}^{\lambda_{2j-1}}\Big)^{\frac{2}{\lambda_{2j-1}}-\frac{2}{\lambda_{2j}}}\times\Bigg[\Big(\Big(\sum_n A_{n(2j-1)}^{\lambda_{2j-1}}\Big)\Big(\sum_n A_{n(2j)}^{\lambda_{2j}}\Big)\Big)^2\\
&\quad-\Big(\Big(\sum_n A_{n(2j-1)}^{\lambda_{2j-1}}e_n\Big)\Big(\sum_n A_{n(2j)}^{\lambda_{2j}}\Big)\\
&\quad-\Big(\sum_n A_{n(2j-1)}^{\lambda_{2j-1}}\Big)\Big(\sum_n A_{n(2j)}^{\lambda_{2j}}e_n\Big)\Big)^2\Bigg]^{\frac{1}{\lambda_{2j}}}\Bigg\}.
\end{aligned}\tag{13}$$

Combining inequalities (11) and (13) leads to inequality (7) immediately.

Case (II). When k is odd, by the same method as in the above case (I), we have the inequality (8). The proof of Theorem 2.3 is complete. □

To illustrate the significance of the introduction of the sequence $(e_n)_{n=1}^{\infty}$, let us sketch an example as follows.

Example 2.5. Let $\lambda_j=\frac{1}{2N}$, $j = 1, 2, ..., 2N$, $n = 1, 2, ..., 2N$, $N \ge 2$, let $A_{nj}=\begin{cases}1 \text{ if } j=1, n=1,2,\dots,2N\\ 1 \quad \text{if } n=j\\ 0 \quad \text{otherwise}\end{cases}$, and let $e_n=\begin{cases}0 \text{ if } n \text{ even}\\ 1 \text{ if } n \text{ odd}\end{cases}$. Then from the generalized

Hölder inequality (2), we obtain $0 \le (2N)\frac{1}{2N}$. However, from Theorem 2.3, we obtain $0 \le 0$.

Corollary 2.6. *Let A_{nj}, λ_j, e_n be as in Theorem 2.3, and let $\sum_n A_{nj}^{\lambda_j} \ne 0$. Then, the following inequality holds:*

$$\sum_n \prod_{j=1}^{k} A_{nj} \le \left[\prod_{j=1}^{k}\left(\sum_n A_{nj}^{\lambda_j}\right)^{\frac{1}{\lambda_j}}\right] \times \left\{\prod_{j=1}^{\rho(k)}\left[1 - \frac{1}{2\lambda_{2j}}\left(\frac{\sum_n A_{n(2j-1)}^{\lambda_{2j-1}} e_n}{\sum_n A_{n(2j-1)}^{\lambda_{2j-1}}} - \frac{\sum_n A_{n(2j)}^{\lambda_{2j}} e_n}{\sum_n A_{n(2j)}^{\lambda_{2j}}}\right)^2\right]\right\}, \tag{14}$$

where $\rho(k) = \begin{cases} \frac{k}{2} & \text{if } k \text{ even} \\ \frac{k-1}{2} & \text{if } k \text{ odd} \end{cases}$.

Corollary 2.7. *Let $F_j(x)$, λ_j, $e(x)$ be as in Theorem 2.4, and let $\int_E F_j^{\lambda_j}(x)\mathrm{d}x \ne 0$. Then, the following inequality holds:*

$$\int_E \prod_{j=1}^{k} F_j(x)\mathrm{d}x \le \left[\prod_{j=1}^{k}\left(\int_E F_j^{\lambda_j}(x)\mathrm{d}x\right)^{\frac{1}{\lambda_j}}\right] \times \left\{\prod_{j=1}^{\rho(k)}\left[1 - \frac{1}{2\lambda_{2j}}\left(\frac{\int_E F_{2j-1}^{\lambda_{2j-1}}(x)e(x)\mathrm{d}x}{\int_E F_{2j-1}^{\lambda_{2j-1}}(x)\mathrm{d}x} - \frac{\int_E F_{2j}^{\lambda_{2j}}(x)e(x)\mathrm{d}x}{\int_E F_{2j}^{\lambda_{2j}}(x)\mathrm{d}x}\right)^2\right]\right\}, \tag{15}$$

where $\rho(k) = \begin{cases} \frac{k}{2} & \text{if } k \text{ even} \\ \frac{k-1}{2} & \text{if } k \text{ odd} \end{cases}$.

Proof. We need to prove only Corollary 2.6. The proof of Corollary 2.7 is similar. From inequalities (7) and (8), we obtain

$$\sum_n \prod_{j=1}^{k} A_{nj} \le \left[\prod_{j=1}^{k}\left(\sum_n A_{nj}^{\lambda_j}\right)^{\frac{1}{\lambda_j}}\right] \times \left\{\prod_{j=1}^{\rho(k)}\left[1 - \left(\frac{\sum_n A_{n(2j-1)}^{\lambda_{2j-1}} e_n}{\sum_n A_{n(2j-1)}^{\lambda_{2j-1}}} - \frac{\sum_n A_{n(2j)}^{\lambda_{2j}} e_n}{\sum_n A_{n(2j)}^{\lambda_{2j}}}\right)^2\right]^{\frac{1}{2\lambda_{2j}}}\right\}. \tag{16}$$

Furthermore, performing some simple computations, we have

$$\left|\frac{\sum_n A_{n(2j-1)}^{\lambda_{2j-1}} e_n}{\sum_n A_{n(2j-1)}^{\lambda_{2j-1}}} - \frac{\sum_n A_{n(2j)}^{\lambda_{2j}} e_n}{\sum_n A_{n(2j)}^{\lambda_{2j}}}\right| < 1. \tag{17}$$

Consequently, from Lemma 2.2 and the inequalities (16) and (17), we have the desired inequality (14). The proof of Corollary 2.6 is complete. □

It is clear that inequalities (7), (14) and (16) are sharper than the inequality (2).

Now, we present the following reversed versions of inequalities (7), (8), (9) and (10).

Theorem 2.8. *Let* $A_{rj} > 0$, $(r = 1, 2, ..., n, j = 1, 2, ..., m)$, $\sum_{j=1}^{m} \frac{1}{\lambda_j} = 1$, *and let* $1\text{-}e_r + e_s \geq 0$ $(s = 1, 2, ..., n)$. *If* $\lambda_1 > 0$, $\lambda_j < 0$ $(j = 2, 3, ..., m)$, *then*

$$\begin{aligned}\sum_{r=1}^{n}\prod_{j=1}^{m}A_{rj} \geq \left(\sum_{r=1}^{n}A_{r1}^{\lambda_1}\right)^{\frac{1}{\lambda_1}-\sum_{j=2}^{m}\frac{1}{\lambda_j}}\prod_{j=2}^{m}\Bigg\{\left[\left(\sum_{r=1}^{n}A_{r1}^{\lambda_1}\right)\left(\sum_{r=1}^{n}A_{rj}^{\lambda_j}\right)\right]^2 \\ -\left[\left(\sum_{r=1}^{n}A_{r1}^{\lambda_1}e_r\right)\left(\sum_{r=1}^{n}A_{rj}^{\lambda_j}\right)-\left(\sum_{r=1}^{n}A_{r1}^{\lambda_1}\right)\left(\sum_{r=1}^{n}A_{rj}^{\lambda_j}e_r\right)\right]^2\Bigg\}^{\frac{1}{2\lambda_j}}.\end{aligned} \tag{18}$$

The integral form is as follows:

Theorem 2.9. *Let* $F_j(x)$ *be nonnegative integrable functions on* $[a, b]$ *such that* $\int_a^b F_j^{\lambda_j}(x)\mathrm{d}x$*exist, let* $1\text{-}e(x) + e(y) \geq 0$ *for all* $x, y \in [a, b]$, *and* $\int_a^b e(x)\mathrm{d}x < \infty$, *and let* $\sum_{j=1}^{m}\frac{1}{\lambda_j} = 1$. *If* $\lambda_1 > 0$, $\lambda_j < 0$ $(j = 2, 3, ..., m)$, *then*

$$\begin{aligned}\int_a^b\prod_{j=1}^{m}F_j(x)\mathrm{d}x \geq \left(\int_a^b F_1^{\lambda_1}(x)\mathrm{d}x\right)^{\frac{1}{\lambda_1}-\sum_{j=2}^{m}\frac{1}{\lambda_j}} \\ \times\prod_{j=2}^{m}\Bigg[\left(\int_a^b F_1^{\lambda_1}(x)\mathrm{d}x\int_a^b F_j^{\lambda_j}(x)\mathrm{d}x\right)^2 \\ -\Bigg(\int_a^b F_1^{\lambda_1}(x)e(x)\mathrm{d}x\int_a^b F_j^{\lambda_j}(x)\mathrm{d}x \\ -\int_a^b F_1^{\lambda_1}(x)\mathrm{d}x\int_a^b F_j^{\lambda_j}(x)e(x)\mathrm{d}x\Bigg)^2\Bigg]^{\frac{1}{2\lambda_j}}.\end{aligned} \tag{19}$$

Proof. We need to prove only Theorem 2.8. The proof of Theorem 2.9 is similar. By the inequality (5), we have

$$\begin{aligned}&\sum_{s=1}^{n}\left(\prod_{i=1}^{m}A_{si}\right)\sum_{r=1}^{n}\left(\prod_{j=1}^{m}A_{rj}\right)(1-e_r+e_s) \\ &= \sum_{s=1}^{n}\left(\prod_{i=1}^{m}A_{si}\right)\sum_{r=1}^{n}\prod_{j=1}^{m}A_{rj}(1-e_r+e_s)^{\frac{1}{\lambda_j}} \\ &\geq \sum_{s=1}^{n}\left(\prod_{i=1}^{m}A_{si}\right)\left[\prod_{j=1}^{m}\left(\sum_{r=1}^{n}A_{rj}^{\lambda_j}(1-e_r+e_s)\right)^{\frac{1}{\lambda_j}}\right] \\ &= \sum_{s=1}^{n}\Bigg\{\left(A_{s1}^{\lambda_1}\sum_{r=1}^{n}A_{r1}^{\lambda_1}(1-e_r+e_s)\right)^{\frac{1}{\lambda_1}-\sum_{j=2}^{m}\frac{1}{\lambda_j}} \\ &\quad\times\left[\prod_{j=2}^{m}\left(A_{s1}^{\lambda_1}\sum_{r=1}^{n}A_{rj}^{\lambda_j}(1-e_r+e_s)\right)^{\frac{1}{\lambda_j}}\right] \\ &\quad\times\left[\prod_{j=2}^{m}\left(A_{sj}^{\lambda_j}\sum_{r=1}^{n}A_{r1}^{\lambda_1}(1-e_r+e_s)\right)^{\frac{1}{\lambda_j}}\right]\Bigg\}.\end{aligned} \tag{20}$$

Consequently, according to $\left(\frac{1}{\lambda_1} - \sum_{j=2}^{m} \frac{1}{\lambda_j}\right) + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} + \cdots + \frac{1}{\lambda_m} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} + \cdots + \frac{1}{\lambda_m} = 1$, by using the inequality (5) on the right side of (20), we observe that

$$\begin{aligned}
&\sum_{s=1}^{n}\left(\prod_{i=1}^{m} A_{si}\right)\sum_{r=1}^{n}\left(\prod_{j=1}^{m} A_{rj}\right)(1 - e_r + e_s) \\
&\geq \left(\sum_{s=1}^{n}\sum_{r=1}^{n} A_{s1}^{\lambda_1} A_{r1}^{\lambda_1}(1 - e_r + e_s)\right)^{\frac{1}{\lambda_1} - \sum_{j=2}^{m}\frac{1}{\lambda_j}} \\
&\quad \times \left[\prod_{j=2}^{m}\left(\sum_{s=1}^{n}\sum_{r=1}^{n} A_{s1}^{\lambda_1} A_{rj}^{\lambda_j}(1 - e_r + e_s)\right)^{\frac{1}{\lambda_j}}\right] \\
&\quad \times \left[\prod_{j=2}^{m}\left(\sum_{s=1}^{n}\sum_{r=1}^{n} A_{sj}^{\lambda_j} A_{r1}^{\lambda_1}(1 - e_r + e_s)\right)^{\frac{1}{\lambda_j}}\right] \\
&= \left(\sum_{r=1}^{n} A_{r1}^{\lambda_1}\right)^{\frac{2}{\lambda_1} - \sum_{j=2}^{m}\frac{2}{\lambda_j}} \times \left\{\prod_{j=2}^{m}\left[\left(\sum_{s=1}^{n}\sum_{r=1}^{n} A_{s1}^{\lambda_1} A_{rj}^{\lambda_j}(1 - e_r + e_s)\right)\right.\right. \\
&\quad \left.\left. \times \left(\sum_{s=1}^{n}\sum_{r=1}^{n} A_{sj}^{\lambda_j} A_{r1}^{\lambda_1}(1 - e_r + e_s)\right)\right]^{\frac{1}{\lambda_j}}\right\} \qquad (21) \\
&= \left(\sum_{r=1}^{n} A_{r1}^{\lambda_1}\right)^{\frac{2}{\lambda_1} - \sum_{j=2}^{m}\frac{2}{\lambda_j}} \times \left\{\prod_{j=2}^{m}\left[\left(\sum_{s=1}^{n} A_{s1}^{\lambda_1}\sum_{r=1}^{n} A_{rj}^{\lambda_j}\right.\right.\right. \\
&\quad \left. - \sum_{s=1}^{n} A_{s1}^{\lambda_1}\sum_{r=1}^{n} A_{rj}^{\lambda_j} e_r + \sum_{s=1}^{n} A_{s1}^{\lambda_1} e_s \sum_{r=1}^{n} A_{rj}^{\lambda_j}\right) \\
&\quad \left.\left. \times \left(\sum_{s=1}^{n} A_{sj}^{\lambda_j}\sum_{r=1}^{n} A_{r1}^{\lambda_1} - \sum_{s=1}^{n} A_{sj}^{\lambda_j}\sum_{r=1}^{n} A_{r1}^{\lambda_1} e_r + \sum_{s=1}^{n} A_{sj}^{\lambda_j} e_s \sum_{r=1}^{n} A_{r1}^{\lambda_1}\right)\right]^{\frac{1}{\lambda_j}}\right\} \\
&= \left(\sum_{r=1}^{n} A_{r1}^{\lambda_1}\right)^{\frac{2}{\lambda_1} - \sum_{j=2}^{m}\frac{2}{\lambda_j}} \times \left\{\prod_{j=2}^{m}\left[\left(\left(\sum_{r=1}^{n} A_{r1}^{\lambda_1}\right)\left(\sum_{r=1}^{n} A_{rj}^{\lambda_j}\right)\right)^2\right.\right. \\
&\quad \left.\left. - \left(\left(\sum_{r=1}^{n} A_{r1}^{\lambda_1}\right)\left(\sum_{r=1}^{n} A_{rj}^{\lambda_j} e_r\right) - \left(\sum_{r=1}^{n} A_{r1}^{\lambda_1} e_r\right)\left(\sum_{r=1}^{n} A_{rj}^{\lambda_j}\right)\right)^2\right]^{\frac{1}{\lambda_j}}\right\}.
\end{aligned}$$

Combining inequalities (11) and (21) leads to inequality (18) immediately. The proof of Theorem 2.8 is complete. □

Corollary 2.10. *Let* A_{rj}, λ_j, e_r *be as in Theorem 2.8, and let* $\sum_{r=1}^{n} A_{rj}^{\lambda_j} \neq 0$. *Then*

$$\begin{aligned}
&\sum_{r=1}^{n}\prod_{j=1}^{m} A_{rj} \\
&\geq \left[\prod_{j=1}^{m}\left(\sum_{r=1}^{n} A_{rj}^{\lambda_j}\right)^{\frac{1}{\lambda_j}}\right]\left\{\prod_{j=2}^{m}\left[1 - \frac{1}{2\lambda_j}\left(\frac{\sum_{r=1}^{n} A_{r1}^{\lambda_1} e_r}{\sum_{r=1}^{n} A_{r1}^{\lambda_1}} - \frac{\sum_{r=1}^{n} A_{rj}^{\lambda_j} e_r}{\sum_{r=1}^{n} A_{rj}^{\lambda_j}}\right)^2\right]\right\}.
\end{aligned} \qquad (22)$$

Corollary 2.11. *Let $F_j(x)$, λ_j, $e(x)$ be as in Theorem 2.9, and let* $\int_a^b F_j^{\lambda_j}(x)\mathrm{d}x \neq 0$. *Then*

$$\begin{aligned}\int_a^b \prod_{j=1}^m F_j(x)\mathrm{d}x \geq & \left[\prod_{j=1}^m \left(\int_a^b F_j^{\lambda_j}(x)\mathrm{d}x\right)^{\frac{1}{\lambda_j}}\right] \\ & \times\left\{\prod_{j=2}^m \left[1-\frac{1}{2\lambda_j}\left(\frac{\int_a^b F_1^{\lambda_1}(x)e(x)\mathrm{d}x}{\int_a^b F_1^{\lambda_1}(x)\mathrm{d}x}-\frac{\int_a^b F_j^{\lambda_j}(x)e(x)\mathrm{d}x}{\int_a^b F_j^{\lambda_j}(x)\mathrm{d}x}\right)^2\right]\right\}.\end{aligned} \tag{23}$$

Proof. Making similar arguments as in the proof of Corollary 2.6, we have the desired inequalities (22) and (23). □

It is clear that inequalities (18) and (22) are sharper than the generalized Hölder inequality (5).

Now, we give here some direct consequences from Theorem 2.8 and Theorem 2.9. Putting $m = 2$ in (18) and (19), respectively, we obtain the following corollaries.

Corollary 2.12. *Let A_{r1}, A_{r2}, λ_1, λ_2, e_r be as in Theorem 2.8. Then, the following reversed version of Hu Ke's inequality (3) holds:*

$$\begin{aligned}\sum_{r=1}^n A_{r1}A_{r2} \geq & \left(\sum_{r=1}^n A_{r1}^{\lambda_1}\right)^{\frac{1}{\lambda_1}-\frac{1}{\lambda_2}}\left[\left(\sum_{r=1}^n A_{r1}^{\lambda_1}\right)\left(\sum_{r=1}^n A_{r2}^{\lambda_2}\right)\right]^2 \\ & -\left[\left(\sum_{r=1}^n A_{r1}^{\lambda_1}e_r\right)\left(\sum_{r=1}^n A_{r2}^{\lambda_2}\right)-\left(\sum_{r=1}^n A_{r1}^{\lambda_1}\right)\left(\sum_{r=1}^n A_{r2}^{\lambda_2}e_r\right)\right]^2\Bigg\}^{\frac{1}{2\lambda_2}}.\end{aligned} \tag{24}$$

Corollary 2.13. *Let $F_1(x)$, $F_2(x)$, λ_1, λ_2, $e(x)$ be as in Theorem 2.9. Then, the following reversed version of Hu Ke's inequality (4) holds:*

$$\begin{aligned}& \int_a^b F_1(x)F_2(x)\mathrm{d}x \\ & \geq \left(\int_a^b F_1^{\lambda_1}(x)\mathrm{d}x\right)^{\frac{1}{\lambda_1}-\frac{1}{\lambda_2}} \times\left[\left(\int_a^b F_1^{\lambda_1}(x)\mathrm{d}x\int_a^b F_2^{\lambda_2}(x)\mathrm{d}x\right)^2\right. \\ & \qquad -\left(\int_a^b F_1^{\lambda_1}(x)e(x)\mathrm{d}x\int_a^b F_2^{\lambda_2}(x)\mathrm{d}x\right. \\ & \qquad\quad \left.\left.-\int_a^b F_1^{\lambda_1}(x)\mathrm{d}x\int_a^b F_2^{\lambda_2}(x)e(x)\mathrm{d}x\right)^2\right]^{\frac{1}{2\lambda_2}}.\end{aligned} \tag{25}$$

Example 2.14. Putting $e(x) = \dfrac{1}{2}\cos\dfrac{\pi(b-x)}{b-a}$ in (23), we obtain

$$\begin{aligned}\int_a^b \prod_{j=1}^m F_j(x)\mathrm{d}x \geq & \left[\prod_{j=1}^m \left(\int_a^b F_j^{\lambda_j}(x)\mathrm{d}x\right)^{\frac{1}{\lambda_j}}\right] \\ & \times\left\{\prod_{j=2}^m \left[1-\frac{1}{8\lambda_j}\left(\frac{\int_a^b F_1^{\lambda_1}(x)\cos\frac{\pi(b-x)}{b-a}\mathrm{d}x}{\int_a^b F_1^{\lambda_1}(x)\mathrm{d}x}-\frac{\int_a^b F_j^{\lambda_j}(x)\cos\frac{\pi(b-x)}{b-a}\mathrm{d}x}{\int_a^b F_j^{\lambda_j}(x)\mathrm{d}x}\right)^2\right]\right\},\end{aligned} \tag{26}$$

where $\lambda_1 > 0$, $\lambda_j < 0$ $(j = 2, 3, ..., m)$, $\sum_{j=1}^{m} \frac{1}{\lambda_j} = 1$.

3. Applications

In this section, we show some applications of our new inequalities. Firstly, we provide an application of the obtained results to improve Hao Z-C inequality, which is related to the generalized arithmetic-geometric mean inequality with weights. The generalized arithmetic-geometric mean inequality (e.g. [9]) states that if $a_j > 0$, $\lambda_j > 0 (j = 1, 2, ..., k)$, $p > 0$ and $\sum_{j=1}^{k} \frac{1}{\lambda_j} = 1$, then

$$\prod_{j=1}^{k} a_j^{\frac{1}{\lambda_j}} \le \sum_{j=1}^{k} \frac{a_j}{\lambda_j}. \tag{27}$$

The classical arithmetic-geometric mean inequality is one of the most important inequalities in analysis. This classical inequality has been widely studied by many authors, and it has motivated a large number of research papers involving different proofs, various generalizations and improvements (see e.g. [1,9,12,19,33] and references therein). In the year 1990, Hao Z-C in [31] established the following interesting inequality

$$\prod_{j=1}^{k} a_j^{\frac{1}{\lambda_j}} \le \left\{ p \int_0^{\infty} \left[\prod_{j=1}^{k} (x + a_j)^{\frac{1}{\lambda_j}} \right]^{-p-1} \mathrm{d}x \right\}^{-\frac{1}{p}} \le \sum_{j=1}^{k} \frac{a_j}{\lambda_j}, \tag{28}$$

where $a_j > 0$, $\lambda_j > 0 (j = 1, 2, ..., k)$, $p > 0$ and $\sum_{j=1}^{k} \frac{1}{\lambda_j} = 1$. The above Hao Z-C inequality is refined by using Corollary 2.7 as follows:

Theorem 3.1. *Let* $a_j > 0 (j = 1, 2, ..., k)$, $p > 0$, *let* $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_k > 0$, $\sum_{j=1}^{k} \frac{1}{\lambda_j} = 1$, *and let* $1\text{-}e(x) + e(y) \ge 0$, $\int_0^{\infty} e(x)\mathrm{d}x < \infty$. *Then*

$$\begin{aligned} \prod_{j=1}^{k} a_j^{\frac{1}{\lambda_j}} &\le \left(\prod_{j=1}^{k} a_j^{\frac{1}{\lambda_j}} \right) \times \left[\prod_{j=1}^{\rho(k)} \left(1 - \frac{1}{2\lambda_j} R^2(x, e; a_j, p) \right) \right]^{-\frac{1}{p}} \\ &\le \left\{ p \int_0^{\infty} \left[\prod_{j=1}^{k} (x + a_j)^{\frac{1}{\lambda_j}} \right]^{-p-1} dx \right\}^{-\frac{1}{p}} \le \sum_{j=1}^{k} \frac{a_j}{\lambda_j}, \end{aligned} \tag{29}$$

where $\rho(k) = \begin{cases} \frac{k}{2} & \text{if } k \text{ even} \\ \frac{k-1}{2} & \text{if } k \text{ odd} \end{cases}$,

$$R(x, e; a_j, p) = \frac{\int_0^{\infty} (x + a_{2j-1})^{-p-1} e(x)\mathrm{d}x}{\int_0^{\infty} (x + a_{2j-1})^{-p-1}\mathrm{d}x} - \frac{\int_0^{\infty} (x + a_{2j})^{-p-1} e(x)\mathrm{d}x}{\int_0^{\infty} (x + a_{2j})^{-p-1}\mathrm{d}x}.$$

Proof. For $x \geq 0$, with a substitution $a_j \to x + a_j$ in (27), we have

$$0 < \prod_{j=1}^{k} (x + a_j)^{\frac{1}{\lambda_j}} \leq \sum_{j=1}^{k} \frac{x + a_j}{\lambda_j} = x + \sum_{j=1}^{k} \frac{a_j}{\lambda_j}. \tag{30}$$

Now, integrating both sides of (30) from 0 to ∞, we observe that

$$\int_0^\infty \left(\prod_{j=1}^{k} (x + a_j)^{\frac{1}{\lambda_j}} \right)^{-p-1} \mathrm{d}x \geq \int_0^\infty \left[x + \sum_{j=1}^{k} \frac{a_j}{\lambda_j} \right]^{-p-1} \mathrm{d}x = \frac{1}{p} \left(\sum_{j=1}^{k} \frac{a_j}{\lambda_j} \right)^{-p}. \tag{31}$$

On the other hand, applying the inequality (15), we obtain

$$\begin{aligned} \int_0^\infty \left[\prod_{j=1}^{k} (x + a_j)^{\frac{1}{\lambda_j}} \right]^{-p-1} \mathrm{d}x &= \int_0^\infty \prod_{j=1}^{k} \left[(x + a_j)^{-p-1} \right]^{\frac{1}{\lambda_j}} \mathrm{d}x \\ &\leq \left[\prod_{j=1}^{k} \left(\int_0^\infty (x + a_j)^{-p-1} \mathrm{d}x \right)^{\frac{1}{\lambda_j}} \right] \times \left[\prod_{j=1}^{\rho(k)} \left(1 - \frac{1}{2\lambda_{2j}} R^2(x, e; a_j, p) \right) \right] \\ &= \left(\frac{1}{p} \prod_{j=1}^{k} a_j^{-\frac{p}{\lambda_j}} \right) \times \left[\prod_{j=1}^{\rho(k)} \left(1 - \frac{1}{2\lambda_{2j}} R^2(x, e; a_j, p) \right) \right]. \end{aligned} \tag{32}$$

Combining inequalities (32) and (31) yields inequality (29) immediately. The proof of Theorem 3.1 is complete. □

From Theorem 3.1, we have the following Corollary.

Corollary 3.2. *With notation as in Theorem 3.1, we have*

$$\prod_{j=1}^{k} a_j^{\frac{1}{\lambda_j}} \leq \left[\prod_{j=2}^{\rho(k)} \left(1 - \frac{1}{2\lambda_j} R^2(x, e; a_j, p) \right) \right]^{\frac{1}{p}} \left(\sum_{j=1}^{k} \frac{a_j}{\lambda_j} \right). \tag{33}$$

It is clear that inequality (33) is sharper than the inequality (27).

Now, we give a sharpness of Beckenbach-type inequality from Corollary 2.10. The famous Beckenbach inequality [8] has been generalized and extended in several directions; see, e.g., [16]. In 1983, Wang [32] established the following Beckenbach-type inequality.

Theorem D. *Let f*(x)*, g*(x) *be positive integrable functions defined on* [0, *T*], *and let* $\frac{1}{p} + \frac{1}{q} = 1$. *If* $0<p < 1$, *then, for any positive numbers a, b, c, the inequality*

$$\frac{\left(a + c \int_0^T h^p(x) \mathrm{d}x \right)^{\frac{1}{p}}}{b + c \int_0^T h(x) g(x) \mathrm{d}x} \geq \frac{\left(a + c \int_0^T f^p(x) \mathrm{d}x \right)^{\frac{1}{p}}}{b + c \int_0^T f(x) g(x) \mathrm{d}x} \tag{34}$$

holds, where $h(x) = (\frac{ag(x)}{b})^{\frac{q}{p}}$. *The sign of the inequality in (34) is reversed if p > 1.*

Theorem 3.3. *Let f(x), g(x), e(x) be integrable functions defined on* [0, *T*], *let f(x), g(x)* > 0, 1 - *e*(*x*) + *e*(*y*) ≥ 0 *for all x, y* ∈ [0, *T*], *and let* $\frac{1}{p}+\frac{1}{q}=1$. *If* 0 <*p* < 1, *then, for any positive numbers a, b, c, the inequality*

$$\frac{\left(a+c\int_0^T h^p(x)\mathrm{d}x\right)^{\frac{1}{p}}}{b+c\int_0^T h(x)g(x)\mathrm{d}x} \geq \frac{\left(a+c\int_0^T f^p(x)\mathrm{d}x\right)^{\frac{1}{p}}}{b+c\int_0^T f(x)g(x)\mathrm{d}x} \times\left[1-\frac{1}{2q}\left(\frac{c\int_0^T f^p(x)\mathrm{d}x}{a+c\int_0^T f^p(x)\mathrm{d}x}-\frac{c\int_0^T g^q(x)\mathrm{d}x}{a^{-\frac{q}{p}}b^q+c\int_0^T g^q(x)\mathrm{d}x}\right)^2\right] \tag{35}$$

holds, where $h(x)=\left(\frac{ag(x)}{b}\right)^{\frac{q}{p}}$.

Proof. Performing some simple computations, we have

$$\frac{\left(a+c\int_0^T h^p(x)\mathrm{d}x\right)^{\frac{1}{p}}}{b+c\int_0^T h(x)g(x)\mathrm{d}x} = \left(a^{-\frac{q}{p}}b^q+c\int_0^T g^q(x)\mathrm{d}x\right)^{-\frac{1}{q}}. \tag{36}$$

On the other hand, putting $e_1 = 0$, $e_2 = 1$, $m = 2$ in (22), from Corollary 2.10 we obtain

$$\begin{aligned} b+c\int_0^T f(x)g(x)\mathrm{d}x &\geq b+c\left(\int_0^T f^p(x)\mathrm{d}x\right)^{\frac{1}{p}}\left(\int_0^T g^q(x)\mathrm{d}x\right)^{\frac{1}{q}} \\ &= a^{\frac{1}{p}}\left(ba^{-\frac{1}{p}}\right)+\left(c\int_0^T f^p(x)\mathrm{d}x\right)^{\frac{1}{p}}\left(c\int_0^T g^q(x)\mathrm{d}x\right)^{\frac{1}{q}} \\ &\geq \left(a+c\int_0^T f^p(x)\mathrm{d}x\right)^{\frac{1}{p}}\left(a^{-\frac{q}{p}}b^q+c\int_0^T g^q(x)\mathrm{d}x\right)^{\frac{1}{q}} \\ &\quad\times\left[1-\frac{1}{2q}\left(\frac{c\int_0^T f^p(x)\mathrm{d}x}{a+c\int_0^T f^p(x)\mathrm{d}x}-\frac{c\int_0^T g^q(x)\mathrm{d}x}{a^{-\frac{q}{p}}b^q+c\int_0^T g^q(x)\mathrm{d}x}\right)^2\right], \end{aligned} \tag{37}$$

that is

$$\left(a^{-\frac{q}{p}}b^q+c\int_0^T g^q(x)\mathrm{d}x\right)^{-\frac{1}{q}} \geq \frac{\left(a+c\int_0^T f^p(x)\mathrm{d}x\right)^{\frac{1}{p}}}{b+c\int_0^T f(x)g(x)\mathrm{d}x} \times\left[1-\frac{1}{2q}\left(\frac{c\int_0^T f^p(x)\mathrm{d}x}{a+c\int_0^T f^p(x)\mathrm{d}x}-\frac{c\int_0^T g^q(x)\mathrm{d}x}{a^{-\frac{q}{p}}b^q+c\int_0^T g^q(x)\mathrm{d}x}\right)^2\right]. \tag{38}$$

Combining inequalities (36) and (38) yields inequality (35). The proof of Theorem 3.3 is complete. □

4. Conclusions

The classical Hölder's inequality plays a very important role in both theory and applications. In this paper, we have presented an extension of Hu Ke's inequality, which is a sharp Hölder's inequality, and established their corresponding reversed versions. Moreover, we have improved Hao Z-C inequality and Beckenbach-type inequality by using the obtained results. Finally, we have obtained the refinement of arithmetic-geometric mean inequality. We think that our results will be useful for those areas in which inequalities (2) and (5) play a role. In the future research, we will continue to explore other applications of the obtained inequalities.

Acknowledgements
The author would like to express his sincere thanks to the anonymous referees for their making great efforts to improve this paper. This work was supported by the NNSF of China (Grant No. 61073121), and the Fundamental Research Funds for the Central Universities (No. 11ML65).

Competing interests
The author declares that they have no competing interests.

Received: 30 April 2011 Accepted: 6 October 2011 Published: 6 October 2011

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doi:10.1186/1029-242X-2011-77
Cite this article as: Tian: **Extension of Hu Ke's inequality and its applications.** *Journal of Inequalities and Applications* 2011 **2011**:77.