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# Common fixed point theorems for generalized $\mathcal{JH}$-operator classes and invariant approximations

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## Abstract

In this article, we introduce two new different classes of noncommuting selfmaps. The first class is more general than $\mathcal{JH}$-operator class of Hussain et al. (Common fixed points for $\mathcal{JH}$-operators and occasionally weakly biased pairs under relaxed conditions. Nonlinear Anal. **74**(6), 2133-2140, 2011) and occasionally weakly compatible class. We establish the existence of common fixed point theorems for these classes. Several invariant approximation results are obtained as applications. Our results unify, extend, and complement several well-known results.

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**Keywords:** common fixed point, occasionally weakly compatible maps, Banach operator pair, P-operator pair, JH-operator pair, generalized JH-operator pair, invariant approximation

## 1. Introduction

The fixed point theorem, generally known as the Banach contraction principle, appeared in explicit form in Banach's thesis in 1922 [1], where it was used to establish the existence of a solution for an integral equation. Since its simplicity and usefulness, it has become a very popular tool in solving existence problems in many branches of mathematical analysis. Banach contraction principle has been extended in many different directions. Many authors established fixed point theorems involving more general contractive conditions.

In 1976, Jungck [2] extend the Banach contraction principle to a common fixed point theorem for commuting maps. Sessa [3] defined the notion of weakly commuting maps and established a common fixed point for this maps. Jungck [4] coined the term compatible mappings to generalize the concept of weak commutativity and showed that weakly commuting maps are compatible but the converse is not true. Afterward, many authors studied about common fixed point theorems for noncommuting maps (see [5-14]).

In 1996, Al-Thagafi [15] established some theorems on invariant approximations for commuting maps. Shahzad [16], Al-Thagafi and Shahzad [17,18], Hussain and Jungck [19], Hussain [20], Hussain and Rhoades [21], Jungck and Hussain [22], O'Regan and Hussain [23], and Pathak and Hussain [24] extended the result of Al-Thagafi [15] and Ciric [25] for pointwise $R$-subweakly commuting maps, compatible maps, $C_q$-commuting maps, and Banach operator pairs. Pathak and Hussain [26] introduced two new classes of noncommuting selfmaps, so-called $\mathcal{P}$-operator and $\mathcal{P}$-suboperator pair class. Recently, Hussain et al. [27] introduced $\mathcal{JH}$-operator and occasionally weakly $g$-biased class which are more general than above classes and established common fixed point theorems for these class.

In this article shall introduce two new classes of noncommuting selfmaps. First class, generalized $\mathcal{JH}$-operator class, contains $\mathcal{JH}$-operator classes of Hussain et al. [27] and occasionally weakly compatible classes. Second class is the so-called generalized $\mathcal{JH}$-suboperator class. We will be present some common fixed point theorems for these classes and the existence of the common fixed points for best approximation. Our results improve, extend, and complement all the results in literature.

## 2. Preliminaries

Let $M$ be a subset of a norm space $X$. We shall use $cl(A)$ and $wcl(A)$ to denote the closure and the weak closure of a set $A$, respectively, and $d(x, A)$ to denote $\inf\{||x-y|| : y \in A\}$ where $x \in X$ and $A \subseteq X$. Let $f$ and $T$ be selfmaps of $M$. A point $x \in M$ is called a *fixed point* of $f$ if $fx = x$. The set of all fixed points of $f$ is denoted by $F(f)$. A point $x \in M$ is called a *coincidence point* of $f$ and $T$ if $fx = Tx$. We shall call $w = fx = Tx$ a *point of coincidence* of $f$ and $T$. A point $x \in M$ is called a *common fixed point* of $f$ and $T$ if $x = fx = Tx$. Let $C(f, T)$, $PC(f, T)$, and $F(f, T)$ denote the sets of all coincidence points, points of coincidence, and common fixed points, respectively, of the pair $(f, T)$.

The map $T$ is called *contraction* [resp. *f-contraction*] on $M$ if $||Tx-Ty|| \leq k||x-y||$ [resp. $||Tx - Ty|| \leq k||fx - fy||$] for all $x, y \in M$ and for some $k \in [0, 1)$. The map $T$ is called *nonexpansive* [resp. *f-nonexpansive*] on $M$ if $||Tx - Ty|| \leq ||x - y||$ [resp. $||Tx - Ty|| \leq ||fx - fy||$] for all $x, y \in M$. The pair $(f, T)$ is called:

**(i):** *commuting* if $Tfx = fTx$ for all $x \in M$;

**(ii):** *R-weakly commuting* [8] if for all $x \in M$, there exists $R > 0$ such that

$$||fTx - Tfx|| \leq R||fx - Tx||.$$

If $R = 1$, then the maps are called *weakly commuting*;

**(iii):** *compatible* [28] if $\lim_{n\to\infty} ||Tfx_n - fTx_n|| = 0$ when $\{x_n\}$ is a sequence such that

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} fx_n = t$$

for some $t \in M$;

**(iv):** *weakly compatible* [29] if $Tfx = fTx$ for all $x \in C(f, T)$;

**(v):** *occasionally weakly compatible* [18,30] if $fTx = Tfx$ for some $x \in C(f, T)$;

**(vi):** *Banach operator pair* [31] if $f(F(T)) \subseteq F(T)$;

**(vii):** *$\mathcal{P}$-operator* [26] if $||u - Tu|| \leq \text{diam}\,(C(f, T))$ for some $u \in C(f, T)$;

**(viii):** *$\mathcal{JH}$-operator* [27] if there exist a point $w = fx = Tx$ in $PC(f, T)$ such that

$$||w - x|| \leq \text{diam}\,(PC(f, T)).$$

The set $M$ is called *convex* if $kx + (1 - k)y \in M$ for all $x, y \in M$ and all $k \in [0, 1]$; and *q-starshaped* with $q \in M$ if the segment $[q, x] = \{kx + (1 - k)q : k \in [0, 1]\}$ joining $q$ to $x$ is contained to $M$. The map $f : M \to M$ is called *affine* if $M$ is convex and $f(kx + (1 - k)y) = kfx + (1 - k)fy$ for all $x, y \in M$ and all $k \in [0, 1]$; and *q-affine* if $M$ is $q$-starshaped and $f(kx + (1 - k)q) = kfx + (1 - k)fq$ for all $x, y \in M$ and all $k \in [0, 1]$.

A map $T : M \to X$ is said to be *semicompact* if a sequence $\{x_n\}$ in $M$ such that $(x_n - Tx_n) \to 0$ has a subsequence $\{x_j\}$ in $M$ such that $x_j \to z$ for some $z \in M$. Clearly if $cl(T(M))$ is compact, then $T(M)$ is complete, $T(M)$ is bounded, and $T$ is semicompact. The map $T : M \to X$ is said to be *weakly semicompact* if a sequence $\{x_n\}$ in $M$ such that $(x_n - Tx_n) \to 0$ has a subsequence $\{x_j\}$ in $M$ such that $x_j \to z$ weakly for some $z \in M$. The map $T : M \to X$ is said to be *demiclosed* at 0 if, for every sequence $\{x_n\}$ in $M$ converging weakly to $x$ and $\{Tx_n\}$ converges to $0 \in X$, then $Tx = 0$.

## 3. Generalized $\mathcal{JH}$-operator classes

We begin this section by introduce a new noncommuting class.

**Definition 3.1**. Let $f$ and $T$ be selfmaps of a normed space $X$. The order pair $(f, T)$ is called a *generalized $\mathcal{JH}$-operator with order n* if there exists a point $w = fx = Tx$ in $PC(f, T)$ such that

$$||w - x|| \leq (\text{diam}\ (PC(f, T)))^n \tag{3.1}$$

for some $n \in \mathbb{N}$.

It is obvious that a $\mathcal{JH}$-operator pair $(f, T)$ is generalized $\mathcal{JH}$-operator with order $n$. But the converse is not true in general, see Example 3.2.

**Example 3.2**. Let $X = \mathbb{R}$ with usual norm and $M = [0, \infty)$. Define $f, T : M \to M$ by

$$fx = \begin{cases} 3, & x = 0; \\ 5, & x = 2; \\ 2x, & \text{another point,} \end{cases} \qquad Tx = \begin{cases} 3, & x = 0; \\ 5, & x = 2; \\ x^2, & \text{another point.} \end{cases}$$

Then $C(f, T) = \{0, 2\}$ and $PC(f, T) = \{3, 5\}$. Obvious $(f, T)$ is a generalized $\mathcal{JH}$-operator with order $n \geq 2$ but not a $\mathcal{JH}$-operator and so not a occasionally weakly compatible and not weakly compatible. Moreover, note that $F(T) = \{1\}$ and $f1 = 2 \notin F(T)$ which implies that $(f, T)$ is not a Banach operator pair.

**Theorem 3.3**. *Let f and T be selfmaps of a nonempty subset M of a normed space X and (f, T) be a generalized $\mathcal{JH}$-operator with order n on M. If f and T satisfying the following condition:*

$$||Tx - Ty|| \leq k \max\{||fx - fy||, ||fx - Tx||, ||fy - Ty||, ||fx - Ty||, ||fy - Tx||\}, \tag{3.2}$$

*for all $x, y \in M$ and $0 \leq k < 1$, then f and T have a unique common fixed point.*

*Proof.* By the notation of generalized $\mathcal{JH}$-operator, we get that there exists a point $w \in M$ such that $w = fx = Tx$ and

$$||w - x|| \leq (\text{diam}\ (PC(f, T)))^n \tag{3.3}$$

for some $n \in \mathbb{N}$. Suppose there exists another point $y \in M$ for which $z = fy = Ty$. Then from (3.2), we get

$$\begin{aligned} ||Tx - Ty|| &\leq k \max\{||fx - fy||, ||fx - Tx||, ||fy - Ty||, ||fx - Ty||, ||fy - Tx||\} \\ &= k \max\{||Tx - Ty||, 0, 0, ||Tx - Ty||, ||Ty - Tx||\} \\ &\leq k||Tx - Ty||. \end{aligned} \tag{3.4}$$

Since $0 \leq k < 1$, the inequality (3.4) implies that $||Tx - Ty|| = 0$, which, in turn implies that $w = fx = Tx = z$. Therefore, there exists a unique element $w$ in $M$ such that $w = fx = Tx$. So $\text{diam}(PC(f, T)) = 0$. Using (3.3), we have

$$d(w, x) \leq (\text{diam } (PC(f, T)))^n = 0.$$

Thus $w = x$, that is $x$ is a unique common fixed point of $f$ and $T$. □

**Definition 3.4**. Let $M$ be a $q$-starshaped subset of a normed space $X$ and $f$, $T$ selfmaps of a normed space $M$. The order pair $(f, T)$ is called a *generalized $\mathcal{JH}$-suboperator with order $n$* if for each $k \in [0, 1]$, $(f, T_k)$ is a generalized $\mathcal{JH}$-operator with order $n$ that is, for $k \in [0, 1]$ there exists a point $w = fx = T_k x$ in $PC(f, T_k)$ such that

$$d(w, x) \leq (\text{diam } (PC(f, T_k)))^n \tag{3.5}$$

for some $n \in \mathbb{N}$, where $T_k$ is selfmap of $M$ such that $T_k x = kTx + (1 - k)q$ for all $x \in M$.

Clearly, a generalized $\mathcal{JH}$-suboperator with order $n$ is generalized $\mathcal{JH}$-operator with order $n$ but the converse is not true in general, see Example 3.5.

**Example 3.5**. Let $X = \mathbb{R}$ with usual norm and $M = [0, \infty)$. Define $f, T : M \to M$ (see Example 3.2). Then $M$ is $q$-starshaped for $q = 0$ and $C(f, T) = \{0, 2\}$, $C(f, T_k) = \{\frac{2}{k}\}$, and $PC(f, T_k) = \{\frac{4}{k}\}$ for $k \in (0, 1)$. Obvious $(f, T)$ is a generalized $\mathcal{JH}$-operator with $n = 2$ but not a generalized $\mathcal{JH}$-suboperator for every $n \in \mathbb{N}$ as

$$\left\| \frac{2}{k} - T_k\left(\frac{2}{k}\right) \right\| = \left\| \frac{2}{k} - \frac{4}{k} \right\| = \frac{2}{k} > 0 = (\text{diam } (PC(f, T_k)))^n \tag{3.6}$$

for each $k \in (0, 1)$.

**Theorem 3.6**. *Let $f$ and $T$ be selfmaps on a $q$-starshaped subset $M$ of a normed space $X$. Assume that $f$ is $q$-affine, $(f, T)$ is a generalized $\mathcal{JH}$-suboperator with order $n_0$, and for all $x, y \in M$,*

$$||Tx - Ty|| \leq \max\{||fx - fy||, d(fx, [q, Tx]), d(fy, [q, Ty]), d(fx, [q, Ty]), d(fy, [q, Tx])\}. \tag{3.7}$$

*Then $F(f, T) \neq \emptyset$ if one of the following conditions holds:*

**(a):** *$cl(T(M))$ is compact and $f$ and $T$ are continuous;*
**(b):** *$wcl(T(M))$ is weakly compact, $f$ is weakly continuous and $(f - T)$ is demiclosed at* 0;
**(c):** *$T(M)$ is bounded, $T$ is semicompact and $f$ and $T$ are continuous;*
**(d):** *$T(M)$ is bounded, $T$ is weakly semicompact, $f$ is weakly continuous and $(f - T)$ is demiclosed at* 0.

*Proof.* Let $\{k_n\} \subseteq (0, 1)$ such that $k_n \to 1$ as $n \to \infty$. For $n \in \mathbb{N}$, we define $T_n : M \to M$ by $T_n x = k_n Tx + (1 - k_n)q$ for all $x \in M$. Since $(f, T)$ is a generalized $\mathcal{JH}$-suboperator with order $n_0$, $(f, T_n)$ is a generalized $\mathcal{JH}$-operator order $n_0$ for all $n \in \mathbb{N}$. Using inequality (3.7) it follows that

$$\begin{aligned} ||T_n x - T_n y|| &= k_n ||Tx - Ty|| \\ &\leq k_n \max\{||fx - fy||, d(fx, [q, Tx]), d(fy, [q, Ty]), d(fx, [q, Ty]), d(fy, [q, Tx])\} \\ &\leq k_n \max\{||fx - fy||, ||fx - T_n x||, ||fy - T_n y||, ||fx - T_n y||, ||fy - T_n x||\}, \end{aligned}$$

for all $x, y \in M$. By Theorem 3.3, there exists $x_n \in M$ such that $x_n = fx_n = T_n x_n$ for every $n \in \mathbb{N}$.

(a): As $cl(T(M))$ is compact, there exists a subsequence $\{Tx_m\}$ of $\{Tx_n\}$ such that $\lim_{m\to\infty} Tx_m = y$ for some $y \in M$. By the definition of $T_m$, we get

$$\lim_{m\to\infty} x_m = \lim_{m\to\infty} T_m x_m = \lim_{m\to\infty} (k_m Tx_m + (1-k_m)q) = \lim_{m\to\infty} Tx_m = y.$$

Since $f$ and $T$ are continuous, $y = fy = Ty$ that is $y \in F(f, T)$ and then $F(f, T) \neq \emptyset$.

(b): From weakly compact of $wcl(T(M))$ there exist a subsequence $\{x_m\}$ of $\{x_n\}$ in $M$ converging weakly to $y \in M$ as $m \to \infty$. Since $f$ is weakly continuous, $fy = y$ that is $\lim_{m\to\infty} (fx_m - Tx_m) = 0$. It follows from $(f - T)$ is demiclosed at 0 and $\lim_{m\to\infty} (fx_m - Tx_m) = 0$ that $fy - Ty = 0$. Therefore, $y = fy = Ty$ that is $F(f, T) \neq \emptyset$.

(c): Since $T(M)$ is bounded, $k_n \to 1$, and

$$\begin{aligned} ||x_n - Tx_n|| &= ||T_n x_n - Tx_n|| \\ &= ||k_n Tx_n + (1-k_n)q - Tx_n|| \\ &= ||(1-k_n)(q - Tx_n)|| \\ &\leq (1-k_n)(||q|| + ||Tx_n||) \end{aligned}$$

for all $n \in \mathbb{N}$, we get $\lim_{m\to\infty} (x_n - Tx_n) = 0$. As $T$ is semicompact, there exist a subsequence $\{x_m\}$ of $\{x_n\}$ in $M$ such that $\lim_{m\to\infty} x_m = y$ for some $y \in M$. By definition of $T_m$, we get

$$y = \lim_{m\to\infty} x_m = \lim_{m\to\infty} T_m x_m = \lim_{m\to\infty} (k_m Tx_m + (1-k_m)q) = \lim_{m\to\infty} Tx_m.$$

By the continuous of both $f$ and $T$, we have $y = fy = Ty$. Therefore $F(f, T) \neq \emptyset$.

(d): Similarly case (c), we have $\lim_{m\to\infty} (x_n - Tx_n) = 0$. Since $T$ is weakly semicompact, there exist a subsequence $\{x_m\}$ of $\{x_n\}$ in $M$ such that converging weakly to $y \in M$ as $m \to \infty$. By weak continuity of $f$, we get $fy = y$. It follows from $\lim_{m\to\infty} (fx_m - Tx_m) = \lim_{m\to\infty} (x_m - Tx_m) = 0$, $x_m$ converging weakly to $y$, and $f - T$ is demiclosed at 0 that $(f - T)(y) = 0$ which implies that $fy = Ty$. Therefore $y = fy = Ty$ and hence $y \in F(f, T)$.

□

**Remark 3.7**. We can replace assumption of $f$ being $q$-affine by $q \in F(f)$ and $f(M) = M$ in Theorem 3.6.

If $f$ is identity mapping in Theorem 3.6, then we get the following corollary.

**Corollary 3.8**. *Let $T$ be selfmaps on a $q$-starshaped subset $M$ of a normed space $X$. Assume that for all $x, y \in M$,*

$$||Tx - Ty|| \leq \max\{||x - y||, d(x, [q, Tx]), d(y, [q, Ty]), d(x, [q, Ty]), d(y, [q, Tx])\}. \quad (3.8)$$

*Then $F(T) \neq \emptyset$ if one of the following conditions holds:*

(a): *$cl(T(M))$ is compact and $T$ is continuous;*

(b): *$wcl(T(M))$ is weakly compact and $(I - T)$ is demiclosed at* 0, *where I is identity on M;*

(c): *$T(M)$ is bounded, T is semicompact and T is continuous;*

(d): *$T(M)$ is bounded, T is weakly semicompact and $(I - T)$ is demiclosed at* 0, *where I is identity on M.*

## 4. Invariant approximations

In 1999, invariant approximations for noncommuting maps were considered by Shahzad [32]. As $M$ is a subset of a normed space $X$ and $p \in X$, let

$$B_M(p) := \{x \in M : ||x - p|| = d(p, M)\},$$
$$C_M^f(p) := \{x \in M : fx \in B_M(p)\},$$
$$D_M^f(p) := B_M(p) \cap C_M^f(p),$$

and

$$M_p := \{x \in M : ||x|| \leq 2||p||\}.$$

The set $B_M(p)$ is called the set of best approximants to $p \in X$ out of $M$. Let $\mathcal{C}_0$ denote the class of closed convex subsets $M$ of $X$ containing 0. It is known that $B_M(p)$ is closed, convex, and contained in $M_p \in \mathcal{C}_0$.

**Theorem 4.1**. *Let M be a subset of a normed space X, f and T be selfmaps of X with $T(\partial M \cap M) \subseteq M$, $p \in F(f, T)$, $B_M(p)$ be a closed q-starshaped. Assume that $f(B_M(p)) = B_M(p)$, $q \in F(f)$, $(f, T)$ is a generalized $\mathcal{JH}$-suboperator with order $n_0$ on $B_M(p)$, and for all $x, y \in B_M(p) \cup \{p\}$,*

$$||Tx - Ty|| \leq \begin{cases} ||fx - fp|| & \text{if } y = p; \\ \max\{||fx - fy||, d(fx, [q, Tx]), d(fy, [q, Ty]), & \\ d(fx, [q, Ty]), d(fy, [q, Tx])\} & \text{if } y \in B_M(p). \end{cases} \tag{4.1}$$

*If $cl(T(B_M(p)))$ is compact, f and T are continuous on $B_M(p)$, then $F(f, T) \cap B_M(p) \neq \emptyset$.*

*Proof*. Let $x \in B_M(p)$. It follows from $||kx + (1 - k)p - p)|| = k||x - p|| < d(p, M)$ for all $k \in (0, 1)$ that $\{kx+(1 - k)p : k \in (0, 1)\} \cap M \neq \emptyset$ which implies that $x \in \partial M \cap M$. So $B_M(p) \subseteq \partial M \cap M$ and hence $T(B_M(p)) \subseteq T(\partial M \cap M)$. As $T(\partial M \cap M) \subseteq M$ that $T(B_M(p)) \subseteq M$. Now the result follows from Theorem 3.6 ($a$) with $M = B_M(p)$. Therefore, $F(f, T) \cap B_M(p) \neq \emptyset$. □

**Theorem 4.2**. *Let M be a subset of a normed space X, f and T be selfmaps of X with $T(\partial M \cap M) \subseteq M$, $p \in F(f, T)$, $C_M^f(p)$ be a closed q-starshaped. Assume that $f(C_M^f(p)) = C_M^f(p)$, $q \in F(f)$, $(f, T)$ is a generalized $\mathcal{JH}$-suboperator with order $n_0$ on $C_M^f(p)$, and for all $x, y \in C_M^f(p) \cup \{p\}$,*

$$||Tx - Ty|| \leq \begin{cases} ||fx - fp|| & \text{if } y = p; \\ \max\{||fx - fy||, d(fx, [q, Tx]), d(fy, [q, Ty]), & \\ d(fx, [q, Ty]), d(fy, [q, Tx])\} & \text{if } y \in C_M^f(p). \end{cases} \tag{4.2}$$

*If $cl(T(C_M^f(p)))$ is compact, f and T are continuous on $C_M^f(p)$, then $F(f, T) \cap B_M(p) \neq \emptyset$.*

*Proof*. Let $x \in C_M^f(p)$. By definition of $C_M^f(p)$ and $f(C_M^f(p)) = C_M^f(p)$, we have $C_M^f(p) \subseteq B_M(p)$. Using the same argument in the proof of Theorem 4.1 shows that there exists $x \in \partial M \cap M$. It follows from $T(\partial M \cap M) \subseteq f(M) \cap M$ that $Tx \in f(M)$.

Therefore, we can find a point $z \in M$ such that $Tx = fz$. Thus $z \in C_M^f(p)$ which implies that $T(C_M^f(p)) \subseteq f(C_M^f(p)) = C_M^f(p)$. Now the result follows from Theorem 3.6 (*a*) with $M = B_M^f(p)$. Therefore, we have $F(f, T) \cap B_M(p) \neq \emptyset$. □

**Theorem 4.3**. *Let $M$ be a subset of a normed space $X$, $f$ and $T$ be selfmaps of $X$ with $T(\partial M \cap M) \subseteq M$, $p \in F(f, T)$, $B_M(p)$ be a weakly closed and $q$-starshaped. Assume that $f(B_M(p)) = B_M(p)$, $q \in F(f)$, $(f, T)$ is a generalized $\mathcal{JH}$-suboperator with order $n_0$ on $B_M(p)$, and for all $x, y \in B_M(p) \cup \{p\}$,*

$$\|Tx - Ty\| \leq \begin{cases} \|fx - fp\| & \text{if } y = p; \\ \max\{\|fx - fy\|, d(fx, [q, Tx]), d(fy, [q, Ty]), \\ d(fx, [q, Ty]), d(fy, [q, Tx])\} & \text{if } y \in B_M(p). \end{cases} \tag{4.3}$$

*If $wcl(T(B_M(p)))$ is weakly compact, $f$ is weakly continuous on $B_M(p)$ and $(f - T)$ is demiclosed at 0, then $F(f, T) \cap B_M(p) \neq \emptyset$.*

*Proof*. We use an argument similar to that in Theorem 4.1 and apply Theorem 3.6 (*b*) instead of Theorem 3.6 (*a*). □

**Theorem 4.4**. *Let $M$ be a subset of a normed space $X$, $f$ and $T$ be selfmaps of $X$ with $T(\partial M \cap M) \subseteq M$, $p \in F(f, T)$, $C_M^f(p)$ be a weakly closed and $q$-starshaped. Assume that $f(C_M^f(p)) = C_M^f(p)$, $q \in F(f)$, $(f, T)$ is a generalized $\mathcal{JH}$-suboperator with order $n_0$ on $C_M^f(p)$, and for all $x, y \in C_M^f(p) \cup \{p\}$,*

$$\|Tx - Ty\| \leq \begin{cases} \|fx - fp\| & \text{if } y = p; \\ \max\{\|fx - fy\|, d(fx, [q, Tx]), d(fy, [q, Ty]), \\ d(fx, [q, Ty]), d(fy, [q, Tx])\} & \text{if } y \in C_M^f(p). \end{cases} \tag{4.4}$$

*If $wcl(T(C_M^f(p)))$ is weakly compact, $f$ is weakly continuous on $C_M^f(p)$ and $(f - T)$ is demiclosed at 0, then $F(f, T) \cap B_M(p) \neq \emptyset$.*

*Proof*. We use an argument similar to that in Theorem 4.2 and apply Theorem 3.6 (*b*) instead of Theorem 3.6 (*a*). □

**Theorem 4.5**. *Let $M$ be a subset of a normed space $X$, $f$ and $T$ be selfmaps of $X$, $p \in F(f, T)$, $M \in \mathcal{C}_0$ with $T(M_p) \subseteq f(M) \subseteq M$. Assume that $\|fx - p\| = \|x - p\|$ for all $x \in M$ and for all $x, y \in M_p \cup \{p\}$,*

$$\|Tx - Ty\| \leq \begin{cases} \|fx - fp\| & \text{if } y = p; \\ \max\{\|fx - fy\|, d(fx, [q, Tx]), d(fy, [q, Ty]), \\ d(fx, [q, Ty]), d(fy, [q, Tx])\} & \text{if } y \in M_p. \end{cases} \tag{4.5}$$

*If $cl(f(M_p))$ is compact, then $B_M(p)$ is nonempty, closed, and convex and $T(B_M(p)) \subseteq f(B_M(p)) \subseteq B_M(p)$. If in addition, for all $x, y \in BM(p)$,*

$$\|fx - fy\| \leq \max\{\|x - y\|, d(x, [q, fx]), d(y, [q, fy]), d(x, [q, fy]), d(y, [q, fx])\}, \tag{4.6}$$

*then $F(f) \cap B_M(p) \neq \emptyset$ and $F(T) \cap B_M(p) \neq \emptyset$. Moreover, $F(f, T) \cap B_M(p) \neq \emptyset$ if for some $q \in B_M(p)$, $f$ is $q$-affine and $(f, T)$ is a generalized $\mathcal{JH}$suboperator with order $n$ on $B_M(p)$.*

*Proof*. Assume that $p \notin M$. If $u \in M \backslash M_p$, then $\|u\| > 2\|p\|$. Since $0 \in M$, we get

$$\|x - p\| \geq \|x\| - \|p\| > \|p\| \geq d(p, M).$$

Thus $\alpha := d(p, M_p) = d(p, M)$. As $cl(f(M_p))$ is compact and the norm is continuous that there exists $z \in cl(f(M_p))$ such that $\beta := d(p, cl(f(M_p))) = ||z - p||$. So we have

$$d(p, cl(f(M_p))) \leq ||fy - p|| = ||y - p||.$$

for all $y \in M_p$. Therefore, $\alpha = \beta$ and $B_M(p)$ is nonempty closed and convex such that $f(B_M(p)) \subseteq B_M(p)$. Next step, we show that $T(B_M(p)) \subseteq f(B_M(p))$. Suppose that $w \in T(B_M(p))$. It follows from $T(B_M(p)) \subseteq T(M_p) \subseteq f(M)$ that there exists $w_1 \in M_p$ and $w_2 \in M$ such that $w = Tw_1 = fw_2$. Using the condition (4.5), we have

$$||w_2 - p|| = ||fw_2 - Tp|| = ||Tw_1 - Tp|| \leq ||fw_1 - fp|| = ||fw_1 - p|| = ||w_1 - p|| = d(p, M).$$

Thus, $w_2 \in B_M(p)$ and $w_1 \in f(B_M(p))$ which implies that $T(B_M(p)) \subseteq f(B_M(p)) \subseteq B_M(p)$. Now, suppose that $f$ satisfies inequality (4.6) on $B_M(p)$. Therefore, the condition (4.5) on $M_p \cup \{p\}$ implies that

$$||Tx - Ty|| \leq \max\{||x - y||, d(x, [q, Tx]), d(y, [q, Ty]), d(x, [q, Ty]), d(y, [q, Tx])\}, \quad (4.7)$$

for all $x, y \in B_M(p)$. Since $f(M_p)$ is compact, $f(B_M(p))$ and $T(B_M(p))$ are compact. Moreover, $f(B_M(p)) \subseteq B_M(p)$ and $T(B_M(p)) \subseteq B_M(p)$. It follows from Corollary 3.8 that $F(f) \cap B_M(p) \neq \emptyset$ and $F(T) \cap BM(p) \neq \emptyset$. Finally, we follow from Theorem 3.6 by replacing $M$ with $B_M(p)$. □

**Theorem 4.6**. *Let $M$ be a subset of a normed space $X$, $f$ and $T$ be selfmaps of $X$, $p \in F(f, T)$, $M \in \mathcal{C}_0$ with $T(M_p) \subseteq f(M) \subseteq M$. Assume that $||fx - p|| = ||x - p||$ for all $x \in M$ and for all $x, y \in M_p \cup \{p\}$,*

$$||Tx - Ty|| \leq \begin{cases} ||fx - fp|| & \text{if } y = p; \\ \max\{||fx - fy||, d(fx, [q, Tx]), d(fy, [q, Ty]), & \\ d(fx, [q, Ty]), d(fy, [q, Tx])\} & \text{if } y \in M_p. \end{cases} \quad (4.8)$$

*If $cl(T(M_p))$ is compact, then $B_M(p)$ is nonempty, closed, convex, and $T(B_M(p)) \subseteq f(B_M(p)) \subseteq B_M(p)$. If in addition, for all $x, y \in B_M(p)$,*

$$||fx - fy|| \leq \max\{||x - y||, d(x, [q, fx]), d(y, [q, fy]), d(x, [q, fy]), d(y, [q, fx])\}, \quad (4.9)$$

*then $F(T) \cap B_M(p) \neq \emptyset$. Moreover, $F(f, T) \cap B_M(p) \neq \emptyset$ if for some $q \in B_M(p)$, $f$ is $q$-affine and $(f, T)$ is a generalized $\mathcal{JH}$suboperator with order $n$ on $B_M(p)$.*

*Proof.* We can obtain the result by using an argument similar to that in Theorem 4.5. □

**Theorem 4.7**. *Let $M$ be a subset of a Banach space $X$, $f$ and $T$ be selfmaps of $X$, $p \in F(f, T)$, $M \in \mathcal{C}_0$ with $T(Mp) \subseteq f(M) \subseteq M$. Assume that $||fx - p|| = ||x - p||$ for all $x \in M$ and for all $x, y \in M_p \cup \{p\}$,*

$$||Tx - Ty|| \leq \begin{cases} ||fx - fp|| & \text{if } y = p; \\ \max\{||fx - fy||, d(fx, [q, Tx]), d(fy, [q, Ty]), & \\ d(fx, [q, Ty]), d(fy, [q, Tx])\} & \text{if } y \in M_p. \end{cases} \quad (4.10)$$

*If $wcl(f(M_p))$ is weakly compact and $(f - T)$ is demiclosed at 0, then $B_M(p)$ is nonempty, (weakly) closed, and convex and $T(B_M(p)) \subseteq f(B_M(p)) \subseteq B_M(p)$. If, in addition, for all $x, y \in B_M(p)$,*

$$||fx - fy|| \leq \max\{||x - y||, d(x, [q, fx]), d(y, [q, fy]), d(x, [q, fy]), d(y, [q, fx])\}, \quad (4.11)$$

*then* $F(f) \cap B_M(p) \neq \emptyset$ *and* $F(T) \cap B_M(p) \neq \emptyset$. *Moreover,* $F(f, T) \cap B_M(p) \neq \emptyset$ *if for some* $q \in B_M(p)$, *f is q-affine, weakly continuous on* $B_M(p)$ *and* $(f, T)$ *is a generalized* $\mathcal{JH}$*suboperator with order n on* $B_M(p)$.

*Proof.* To obtain the result, we use an argument similar to that in Theorem 4.5 and apply Theorem 3.6 (*b*) instead of Theorem 3.6(a), respectively. Finally, we use Lemma 5.5 of Singh et al. [33] with $f(x) = ||x - p||$ and $C = wcl(T(M_p))$ to show that there exists $z \in C$ such that $d(p, C) = ||z - p||$. □

**Theorem 4.8**. *Let M be a subset of a Banach space X, f and T be selfmaps of X,* $p \in F(f, T)$, $M \in \mathcal{C}_0$ *with* $T(M_p) \subseteq f(M) \subseteq M$. *Assume that* $||fx - p|| = ||x - p||$ *for all* $x \in M$ *and for all* $x, y \in M_p \cup \{p\}$,

$$||Tx - Ty|| \leq \begin{cases} ||fx - fp|| & \text{if } y = p; \\ \max\{||fx - fy||, d(fx, [q, Tx]), d(fy, [q, Ty]), & \\ d(fx, [q, Ty]), d(fy, [q, Tx])\} & \text{if } y \in M_p. \end{cases} \tag{4.12}$$

*If* $wcl(f(M_p))$ *is weakly compact and* $(f - T)$ *is demiclosed at* 0, *then* $B_M(p)$ *is nonempty, (weakly) closed, and convex and* $T(B_M(p)) \subseteq f(BM(p)) \subseteq B_M(p)$. *If in addition, for all* $x, y \in B_M(p)$,

$$||Tx - Ty|| \leq \max\{||x - y||, d(x, [q, Tx]), d(y, [q, Ty]), d(x, [q, Ty]), d(y, [q, Tx])\}, \tag{4.13}$$

*then* $F(T) \cap B_M(p) \neq \emptyset$. *Moreover,* $F(f, T) \cap B_M(p) \neq \emptyset$ *if for some* $q \in B_M(p)$, *f is q-affine, weakly continuous on* $B_M(p)$ *and* $(f, T)$ *is a generalized* $\mathcal{JH}$*suboperator with order n on* $B_M(p)$.

*Proof.* We can obtain the result using an argument similar to that in Theorem 4.7. □

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**Authors' contributions**
WS designed and performed all the steps of proof in this research and also wrote the paper. PK participated in the design of the study and suggest many good ideas that made this paper possible and helped to draft the first manuscript. All authors read and approved the final manuscript.

**Competing interests**
The authors declare that they have no competing interests.

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**References**
1. Banach, S: Sur les opérations dans les ensembles abstraits et leurs applications aux équations intégrales. Fund Math. **3**, 133–181 (1922)
2. Jungck, G: Commuting mappings and fixed points. Am Math Monthly. **83**, 261–263 (1976). doi:10.2307/2318216
3. Sessa, S: On a weak commutativity condition of mappings in fixed point considerations. Publ Inst Math (Beograd) (N.S.). **32**(46), 149–153 (1982)
4. Jungck, G: Compatible mappings and common fixed points. Int J Math Math Sci. **9**, 771–779 (1986). doi:10.1155/S0161171286000935
5. Kang, SM, Cho, CL, Jungck, G: Common fixed point of compatible mappings. Int J Math Math Sci. **13**, 61–66 (1990). doi:10.1155/S0161171290000096
6. Kang, SM, Ryu, JW: A common fixed point theorem for compatible mappings. Math Jpn. **35**, 153–157 (1990)
7. Mongkolkeha, C, Kumam, P: Fixed point and common fixed point theorems for generalized weak contraction mappings of integral type in modular spaces. Int J Math Math Sci **2011**, 12 (2011). Article ID 705943

8. Pant, RP: Common fixed points of noncommuting mappings. J Math Anal Appl. **188**, 436–440 (1994). doi:10.1006/jmaa.1994.1437
9. Pathak, HK, Cho, YJ, Kang, SM: Common fixed points of biased maps of type (A) and application. Int J Math Math Sci. **21**, 681–694 (1998). doi:10.1155/S0161171298000945
10. Sintunavart, W, Kumam, P: Coincidence and common fixed points for hybrid strict contractions without the weakly commuting condition. Appl Math Lett. **22**, 1877–1881 (2009). doi:10.1016/j.aml.2009.07.015
11. Sintunavart, W, Kumam, P: Weak condition for generalized multi-valued ($f$, $\alpha$, $\beta$)-weak contraction mappings. Appl Math Lett. **24**, 460–465 (2011). doi:10.1016/j.aml.2010.10.042
12. Sintunavart, W, Kumam, P: Coincidence and common fixed points for generalized contraction multi-valued mappings. J Comput Anal Appl. **13**(2), 362–367 (2011)
13. Sintunavart, W, Kumam, P: Gregus-type common fixed point theorems for tangential multivalued mappings of integral type in metric spaces. Int J Math Math Sci **2011**, 12 (2011). Article ID 923458
14. Sintunavart, W, Kumam, P: Gregus type fixed points for a tangential multi-valued mappings satisfying contractive conditions of integral type. J Inequal Appl. **2011**, 3 (2011). doi:10.1186/1029-242X-2011-3
15. Al-Thagafi, MA: Common fixed points and best approximation. J Approx Theory. **85**, 318–323 (1996). doi:10.1006/jath.1996.0045
16. Shahzad, N: Invariant approximations and R-subweakly commuting maps. J Math Anal Appl. **257**, 39–45 (2001). doi:10.1006/jmaa.2000.7274
17. Al-Thagafi, MA, Shahzad, N: Noncommuting selfmaps and invariant approximations. Nonlinear Anal. **64**, 2778–2786 (2006). doi:10.1016/j.na.2005.09.015
18. Al-Thagafi, MA, Shahzad, N: Generalized I-nonexpansive selfmaps and invariant approximations. Acta Math Sinica. **24**, 867–876 (2008). doi:10.1007/s10114-007-5598-x
19. Hussain, N, Jungck, G: Common fixed point and invariant approximation results for noncommuting generalized ($f$, $g$)-nonexpansive maps. J Math Anal Appl. **321**, 851–861 (2006). doi:10.1016/j.jmaa.2005.08.045
20. Hussain, N: Common fixed points in best approximation for Banach operator pairs with Ciric Type I-contractions. J Math Anal Appl. **338**, 1351–1363 (2008). doi:10.1016/j.jmaa.2007.06.008
21. Hussain, N, Rhoades, BE: $C_q$-commuting maps and invariant approximations. Fixed Point Theory Appl. **2006**, 9 (2006)
22. Jungck, G, Hussain, N: Compatible maps and invariant approximations. J Math Anal Appl. **325**, 1003–1012 (2007). doi:10.1016/j.jmaa.2006.02.058
23. O'Regan, D, Hussain, N: Generalized I-contractions and pointwise R-subweakly commuting maps. Acta Math Sinica. **23**, 1505–1508 (2007). doi:10.1007/s10114-007-0935-7
24. Pathak, HK, Hussain, N: Common fixed points for Banach operator pairs with applications. Nonlinear Anal. **69**, 2788–2802 (2008). doi:10.1016/j.na.2007.08.051
25. Ciric, LB: A generalization of Banachs contraction principle. Proc Am Math Soc. **45**, 267–273 (1974)
26. Pathak, HK, Hussain, N: Common fixed points for $\mathcal{P}$-operator pair with applications. Appl Math Comput. **217**, 3137–3143 (2010). doi:10.1016/j.amc.2010.08.046
27. Hussain, N, Khamsi, MA, Latif, A: Common fixed points for $\mathcal{JH}$-operators and occasionally weakly biased pairs under relaxed conditions. Nonlinear Anal. **74**(6), 2133–2140 (2011). doi:10.1016/j.na.2010.11.019
28. Jungck, G: Common fixed points for commuting and compatible maps on compacta. Proc Am Math Soc. **103**, 977–983 (1988). doi:10.1090/S0002-9939-1988-0947693-2
29. Jungck, G, Rhoades, BE: Fixed points for set valued functions without continuity. Indian J Pure Appl Math. **29**, 227–238 (1998)
30. Jungck, G, Rhoades, BE: Fixed point theorems for occasionally weakly compatible mappings. Fixed Point Theory. **7**, 287–296 (2006)
31. Chen, J, Li, Z: Common fixed points for Banach operator pairs in best approximation. J Math Anal Appl. **336**, 1466–1475 (2007). doi:10.1016/j.jmaa.2007.01.064
32. Shahzad, N: A result on best approximation. Tamkang J Math **29**, 223–226 (1998). corrections: Tamkang J Math **30**, 165 (1999)
33. Singh, SP, Watson, B, Srivastava, P: Fixed Point Theory and Best Approximation: The KKM-map Principle. Kluwer Academic Publishers, Dordrecht (1997)

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