Xiao Journal of Inequalities and Applications 2011, **2011**:132
http://www.journalofinequalitiesandapplications.com/content/2011/1/132

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Remarks on inequalities of Hardy-Sobolev Type

Ying-Xiong Xiao

Correspondence: yxxiao.math@gmail.com
School of Mathematics and Statistics, Xiaogan University, Xiaogan 432000 Hubei, People's Republic of China

Abstract

We obtain the sharp constants of some Hardy-Sobolev-type inequalities proved by Balinsky et al. (Banach J Math Anal 2(2):94-106).

2000 Mathematics Subject Classification: Primary 26D10; 46E35.

Keywords: Hardy inequality, Sobolev Inequality

1. Introduction

Hardy inequality in $\mathbb{R}^n$ reads, for all $f \in C_0^\infty(\mathbb{R}^n)$ and $n \geq 3$,

$$\int_{\mathbb{R}^n} |\nabla f|^2 \mathrm{d}x \geq \frac{(n-2)^2}{4} \int_{\mathbb{R}^n} \frac{f^2}{|x|^2} \mathrm{d}x. \tag{1.1}$$

The Sobolev inequality states that, for all $f \in C_0^\infty(\mathbb{R}^n)$ and $n \geq 3$,

$$\int_{\mathbb{R}^n} |\nabla f|^2 \mathrm{d}x \geq S_n \left(\int_{\mathbb{R}^n} |f|^{2^*} \mathrm{d}x \right)^{\frac{2}{2^*}}, \tag{1.2}$$

where $2^* = \dfrac{2n}{n-2}$ and $S_n = \pi n(n-2)(\Gamma(\frac{n}{2})/\Gamma(n))^{\frac{2}{n}}$ is the best constant (cf. [1,2]). A result of Stubbe [3] states that for $0 \leq \delta < \dfrac{(n-2)^2}{4}$,

$$\int_{\mathbb{R}^n} |\nabla f|^2 \mathrm{d}x - \delta \int_{\mathbb{R}^n} \frac{f^2}{|x|^2} \mathrm{d}x \geq \frac{\left(\frac{(n-2)^2}{4} - \delta \right)^{\frac{n-1}{n}}}{\left(\frac{(n-2)^2}{4} \right)^{\frac{n-1}{n}}} S_n \left(\int_{\mathbb{R}^n} |f|^{2^*} \mathrm{d}x \right)^{\frac{2}{2^*}} \tag{1.3}$$

and the constant in (1.3) is sharp. Recently, Balinsky et al. [4] prove analogous inequalities for the operator $\mathcal{L} := \mathbf{x} \cdot \nabla$. One of the results states that, for $0 \leq \delta < n^2/4$ and $f \in C_0^\infty(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} |\mathcal{L} f|^2 \mathrm{d}x - \delta \int_{\mathbb{R}^n} f^2 \mathrm{d}x \geq C \left(\frac{n^2}{4} - \delta \right)^{\frac{n-1}{n}} S_n \left(\int_{\mathbb{R}^n} |rF|^{2^*} \mathrm{d}x \right)^{\frac{2}{2^*}}, \tag{1.4}$$

where $F(r)$ is the integral mean of f over the unit sphere $\mathbb{S}^{n-1}$, i.e.,

$$F(r) = \frac{1}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}} f(r\omega)\mathrm{d}\omega,$$

and $|\mathbb{S}^{n-1}| = \int_{\mathbb{S}^{n-1}} d\omega = \frac{2\pi^{n/2}}{\Gamma(n/2)}$. Here, we use the polar coordinates $x = r\omega$. The aim of this note is to look for the sharp constant of inequality (1.4). To this end, we have:

Theorem 1.1. *Let* $f \in C_0^\infty(\mathbb{R}^n)$*and* $n \geq 3$. *There holds, for* $0 \leq \delta < \frac{n^2}{4}$,

$$\int_{\mathbb{R}^n} |\mathcal{L}f|^2\mathrm{d}x - \delta\int_{\mathbb{R}^n} f^2\mathrm{d}x \geq \frac{\left(\frac{n^2}{4}-\delta\right)^{\frac{n-1}{n}}}{\left(\frac{(n-2)^2}{4}\right)^{\frac{n-1}{n}}} S_n\left(\int_{\mathbb{R}^n} |rF(r)|^{2^*}\mathrm{d}x\right)^{\frac{2}{2^*}} \tag{1.5}$$

and the constant in (1.5) *is sharp.*

When $\delta = n^2/4$, we have the following Theorem, which generalize the results of [4], Corollary 4.6.

Theorem 1.2. *If* f *is supported in the annulus* $A_R := \{x \in \mathbb{R}^n : R^{-1} < |x| < R\}$, *then*

$$\int_{A_R} |\mathcal{L}f|^2\mathrm{d}x - \frac{n^2}{4}\int_{A_R} f^2\mathrm{d}x \geq [2(n-2)\ln R]^{-\frac{2(n-1)}{n}} S_n\left(\int_{A_R} |rF(r)|^{2^*}\mathrm{d}x\right)^{\frac{2}{2^*}}.$$

2. The proofs

We first recall the Bliss lemma [5]:

Lemma 2.1. *For* $s \geq 0$, $q > p > 1$ *and* $r = q/p - 1$,

$$\left(\int_0^\infty \left|\int_0^s g(t)\mathrm{d}t\right|^q s^{r-q}\mathrm{d}s\right)^{p/q} \leq C_{p,q}\int_0^\infty |g(t)|^p\mathrm{d}t,$$

where

$$C_{p,q} = (q-r-1)^{-p/q}\left(\frac{r\Gamma(q/r)}{\Gamma(1/r)\Gamma((q-1)/r)}\right)^{rp/q}$$

is the sharp constant. Equality is attained for functions of the form

$$g(t) = c_1(c_2 s^r + 1)^{-\frac{r+1}{r}}, \quad c_1 > 0, c_2 > 0.$$

Using the Bliss lemma, we can prove the Theorem 1.1 for the radial function f, i.e., $f(x) = \tilde{f}(|x|)$ for some $\tilde{f} \in C_0^\infty([0,\infty))$.

Lemma 2.2. *Let* $f(|x|) \in C_0^\infty(\mathbb{R}^n)$ *and* $n \geq 3$. *There holds, for* $0 \leq \delta < \frac{n^2}{4}$,

$$\int_{\mathbb{R}^n} |\mathcal{L}f|^2 \mathrm{d}x - \delta \int_{\mathbb{R}^n} f^2 \mathrm{d}x \geq \frac{\left(\frac{n^2}{4} - \delta\right)^{\frac{n-1}{n}}}{\left(\frac{(n-2)^2}{4}\right)^{\frac{n-1}{n}}} S_n \left(\int_{\mathbb{R}^n} |rF(r)|^{2^*} \mathrm{d}x\right)^{\frac{2}{2^*}} \tag{2.1}$$

and the constant in (2.1) *is sharp.*

Proof. We note if f is radial, then $F(r) = f(r)$ and $\mathcal{L}f = rf'(r)$. Therefore, inequality (2.1) is equivalent to

$$\begin{aligned} &\int_0^\infty |f'(r)|^2 r^{n+1} \mathrm{d}r - \delta \int_0^\infty |f(r)|^2 r^{n-1} \mathrm{d}r \\ &\geq \frac{\left(\frac{n^2}{4} - \delta\right)^{\frac{n-1}{n}}}{\left(\frac{(n-2)^2}{4}\right)^{\frac{n-1}{n}}} S_n \cdot |\mathbb{S}^{n-1}|^{-\frac{2}{n}} \left(\int_0^\infty |f(r)|^{2^*} r^{2^*+n-1} \mathrm{d}x\right)^{\frac{2}{2^*}}. \end{aligned} \tag{2.2}$$

Let $0 \leq \beta < n/2$ and set $g(r) = r^\beta f(r)$. Through integration by parts, we have that

$$\int_0^\infty |g'(r)|^2 r^{n+1-2\beta} \mathrm{d}r = \int_0^\infty |f'(r)|^2 r^{n+1} \mathrm{d}r - \beta(n-\beta) \int_0^\infty |f(r)|^2 r^{n-1} \mathrm{d}r. \tag{2.3}$$

Make the change of variables $s = r^{n-2\beta}$,

$$\int_0^\infty |g'(r)|^2 r^{n+1-2\beta} \mathrm{d}r = (n-2\beta) \int_0^\infty s^2 \left|\frac{\partial g}{\partial s}\right|^2 \mathrm{d}s. \tag{2.4}$$

On the other hand, set $h(s) = \frac{\partial g}{\partial s}$ so that $g = -\int_s^{+\infty} h(t)\mathrm{d}t$, we have

$$\int_0^\infty s^2 \left|\frac{\partial g}{\partial s}\right|^2 \mathrm{d}s = \int_0^\infty s^2 h^2 \mathrm{d}s = \int_0^\infty |w(s)|^2 \mathrm{d}s,$$

where $w(s) = s^{-2} h(s^{-1})$. By Bliss lemma,

$$\int_0^\infty |w(s)|^2 \mathrm{d}s \geq \left(\frac{n}{n-2}\right)^{\frac{n-2}{n}} \left(\frac{\Gamma(n/2)\Gamma(1+n/2)}{\Gamma(n)}\right)^{\frac{2}{n}} \left(\int_0^\infty \left|\int_0^s |w(t)|\mathrm{d}t\right|^{2^*} s^{\frac{2-2n}{n-2}} \mathrm{d}s\right)^{\frac{2}{2^*}},$$

i.e.,

$$
\begin{aligned}
&\int_0^\infty s^2\left|\frac{\partial g}{\partial s}\right|^2 \mathrm{d}s = \int_0^\infty s^2h^2\mathrm{d}s = \int_0^\infty |w(s)|^2\mathrm{d}s \\
&\geq \left(\frac{n}{n-2}\right)^{\frac{n-2}{n}}\left(\frac{\Gamma(n/2)\Gamma(1+n/2)}{\Gamma(n)}\right)^{\frac{2}{n}}\left(\int_0^\infty\left|\int_0^s |w(t)|\mathrm{d}t\right|^{2^*} s^{\frac{2-2n}{n-2}}\mathrm{d}s\right)^{\frac{2}{2^*}} \\
&= \left(\frac{n}{n-2}\right)^{\frac{n-2}{n}}\left(\frac{\Gamma(n/2)\Gamma(1+n/2)}{\Gamma(n)}\right)^{\frac{2}{n}}\left(\int_0^\infty\left|\int_s^{+\infty} |h(t)|\mathrm{d}t\right|^{2^*} s^{\frac{2}{n-2}}\mathrm{d}s\right)^{\frac{2}{2^*}} \qquad (2.5)\\
&\geq \left(\frac{n}{n-2}\right)^{\frac{n-2}{n}}\left(\frac{\Gamma(n/2)\Gamma(1+n/2)}{\Gamma(n)}\right)^{\frac{2}{n}}\left(\int_0^\infty |g|^{2^*}s^{\frac{2}{n-2}}\mathrm{d}s\right)^{\frac{2}{2^*}}.
\end{aligned}
$$

Recall that $s = r^{n-2\beta}$ and $g(r) = r^{\beta} f(r)$,

$$
\int_0^\infty g^{2^*}s^{\frac{2}{n-2}}\mathrm{d}s = (n-2\beta)\int_0^\infty \left(r^{1-\beta}g\right)^{2^*}r^{n-1}\mathrm{d}r = (n-2\beta)\int_0^\infty (rf)^{2^*}r^{n-1}\mathrm{d}r. \qquad (2.6)
$$

Therefore, by (2.3), (2.4), (2.5) and (2.6),

$$
\begin{aligned}
&\int_0^\infty |f'(r)|^2r^{n+1}\mathrm{d}r - \beta(n-\beta)\int_0^\infty |f(r)|^2r^{n-1}\mathrm{d}r \\
&= (n-2\beta)\int_0^\infty s^2\left|\frac{\partial g}{\partial s}\right|^2\mathrm{d}s \\
&\geq (n-2\beta)^{1+\frac{2}{2^*}}\left(\frac{n}{n-2}\right)^{\frac{n-2}{n}}\left(\frac{\Gamma(n/2)\Gamma(1+n/2)}{\Gamma(n)}\right)^{\frac{2}{n}}\left(\int_0^\infty (rf)^{2^*}r^{n-1}\mathrm{d}r\right)^{\frac{2}{2^*}} \\
&= (n-2\beta)^{\frac{2n-2}{n}}\left(\frac{n}{n-2}\right)^{\frac{n-2}{n}}\left(\frac{\Gamma(n/2)\Gamma(1+n/2)}{\Gamma(n)}\right)^{\frac{2}{n}}\left(\int_0^\infty (rf)^{2^*}r^{n-1}\mathrm{d}r\right)^{\frac{2}{2^*}}.
\end{aligned}
$$

Since $|\mathbb{S}^{n-1}| = \int_{\mathbb{S}^{n-1}}\mathrm{d}\omega = \dfrac{2\pi^{n/2}}{\Gamma(n/2)}$ and $S_n = \pi n(n-2)(\Gamma(\frac{n}{2})/\Gamma(n))^{\frac{2}{n}}$, we have

$$
\begin{aligned}
&\int_{\mathbb{R}^n}|\mathcal{L}f|^2\mathrm{d}x - \beta(n-\beta)\int_{\mathbb{R}^n}f^2\mathrm{d}x \\
&= |\mathbb{S}^{n-1}|\int_0^\infty |f'(r)|^2r^{n+1}\mathrm{d}r - \beta(n-\beta)|\mathbb{S}^{n-1}|\int_0^\infty |f(r)|^2r^{n-1}\mathrm{d}r \\
&\geq |\mathbb{S}^{n-1}|\cdot(n-2\beta)^{\frac{2n-2}{n}}\left(\frac{n}{n-2}\right)^{\frac{n-2}{n}}\left(\frac{\Gamma(n/2)\Gamma(1+n/2)}{\Gamma(n)}\right)^{\frac{2}{n}}\left(\int_0^\infty (rf)^{2^*}r^{n-1}\mathrm{d}r\right)^{\frac{2}{2^*}} \\
&= |\mathbb{S}^{n-1}|^{\frac{2}{n}}(n-2\beta)^{\frac{2n-2}{n}}\left(\frac{n}{n-2}\right)^{\frac{n-2}{n}}\left(\frac{\Gamma(n/2)\Gamma(1+n/2)}{\Gamma(n)}\right)^{\frac{2}{n}}\left(\int_{\mathbb{R}^n}|rf(r)|^{2^*}\mathrm{d}x\right)^{\frac{2}{2^*}} \\
&= \left(\frac{n-2\beta}{n-2}\right)^{\frac{2n-2}{n}}S_n\left(\int_{\mathbb{R}^n}|rf(r)|^{2^*}\mathrm{d}x\right)^{\frac{2}{2^*}}.
\end{aligned}
$$

Let $\beta = \dfrac{n - \sqrt{n^2 - 4\delta}}{2}$ when $0 \le \delta < n^2/4$. Then, $0 \le \beta < n/2$ and $\delta = \beta\ (n - \beta)$. Therefore,

$$\int_{\mathbb{R}^n} |\mathcal{L}f|^2 \mathrm{d}x - \delta \int_{\mathbb{R}^n} f^2 \mathrm{d}x \ge \left(\frac{n^2 - 4\delta}{(n-2)^2} \right)^{\frac{n-1}{n}} S_n \left(\int_{\mathbb{R}^n} |rf(r)|^{2^*} \mathrm{d}x \right)^{\frac{2}{2^*}}.$$

Inequality (2.1) follows.

Now we can prove Theorem 1.1.

Proof of Theorem 1.1. Decomposing f into spherical harmonics, we get (see e.g. [6])

$$f = \sum_{k=0}^{\infty} f_k := \sum_{k=0}^{\infty} g_k(r)\phi_k(\sigma),$$

where $\varphi_k(\sigma)$ are the orthonormal eigenfunctions of the Laplace-Beltrami operator with responding eigenvalues

$$c_k = k(N + k - 2), \quad k \ge 0.$$

The functions $g_k(r)$ belong to $C_0^\infty(\mathbb{R}^n)$, satisfying $g_k(r) = O(r^k)$ and $g_k'(r) = O(r^{k-1})$ as $r \to 0$. By orthogonality,

$$F(r) = \frac{1}{|\mathbb{S}^{n-1}|} \int_{\mathbb{S}^{n-1}} f(r\omega)\mathrm{d}\omega = g_0(r).$$

On the other hand,

$$\mathcal{L}f(x) = \sum_{k=0}^{\infty} r \frac{\partial (g_k(r)\phi_k)}{\partial r} = \sum_{k=0}^{\infty} r g_k'(r)\phi_k(\sigma).$$

Here, we use the radial derivative $\dfrac{\partial}{\partial r} = \dfrac{\mathbf{x} \cdot \nabla}{|x|} = \dfrac{\mathcal{L}}{|x|}$. Therefore,

$$\begin{aligned}\int_{\mathbb{R}^n} |\mathcal{L}f|^2 \mathrm{d}x - \delta \int_{\mathbb{R}^n} f^2 \mathrm{d}x &= \sum_{k=0}^{\infty} \left(\int_{\mathbb{R}^n} r^2 |{g'}_k(r)|^2 \mathrm{d}x - \delta \int_{\mathbb{R}^n} g_k^2 \mathrm{d}x \right) \\ &\ge \int_{\mathbb{R}^n} r^2 |{g'}_0(r)|^2 \mathrm{d}x - \delta \int_{\mathbb{R}^n} g_0^2 \mathrm{d}x = \int_{\mathbb{R}^n} r^2 |F'(r)|^2 \mathrm{d}x - \delta \int_{\mathbb{R}^n} F(r)^2 \mathrm{d}x\end{aligned}$$

since

$$\int_{\mathbb{R}^n} |\mathcal{L}u|^2 \mathrm{d}x \ge \frac{n^2}{4} \int_{\mathbb{R}^n} u^2 \mathrm{d}x$$

holds for all $u \in C_0^\infty(\mathbb{R}^n)$ and $\mathcal{L}u = ru'(r)$ if u is radial. By Lemma 2.2,

$$\int_{\mathbb{R}^n} r^2|F'(r)|^2\mathrm{d}x - \delta\int_{\mathbb{R}^n} F(r)^2\mathrm{d}x \geq \frac{\left(\frac{n^2}{4}-\delta\right)^{\frac{n-1}{n}}}{\left(\frac{(n-2)^2}{4}\right)^{\frac{n-1}{n}}} S_n\left(\int_{\mathbb{R}^n} |rF(r)|^{2^*}\mathrm{d}x\right)^{\frac{2}{2^*}}$$

Therefore,

$$\begin{aligned}
&\int_{\mathbb{R}^n} |\mathcal{L}f|^2\mathrm{d}x - \delta\int_{\mathbb{R}^n} f^2\mathrm{d}x \\
&\geq \int_{\mathbb{R}^n} r^2|F'(r)|^2\mathrm{d}x - \delta\int_{\mathbb{R}^n} F(r)^2\mathrm{d}x \\
&\geq \frac{\left(\frac{n^2}{4}-\delta\right)^{\frac{n-1}{n}}}{\left(\frac{(n-2)^2}{4}\right)^{\frac{n-1}{n}}} S_n\left(\int_{\mathbb{R}^n} |rF(r)|^{2^*}\mathrm{d}x\right)^{\frac{2}{2^*}}.
\end{aligned}$$

The proof of Theorem 1.1 is completed.

Proof of Theorem 1.2. We denote by $B_R \subset \mathbb{R}^N$ the unit ball centered at zero.
Step 1. Assume f is radial and $f \in C_0^\infty(B_R)$. Then,

$$\begin{aligned}
\int_{B_R} |\mathcal{L}f|^2\mathrm{d}x - \frac{n^2}{4}\int_{B_R} f^2\mathrm{d}x &= \int_{B_R} |rf'(r)|^2\mathrm{d}x - \frac{n^2}{4}\int_{B_R} f^2(r)\mathrm{d}x \\
&= \int_{B_R} |(rf(r))'|^2\mathrm{d}x - \frac{(n-2)^2}{4}\int_{B_R} \frac{(rf)^2}{|x|^2}\mathrm{d}x.
\end{aligned}$$

Therefore, by Theorem B in [7],

$$\begin{aligned}
&\int_{B_R} |(rf(r))'|^2\mathrm{d}x - \frac{(n-2)^2}{4}\int_{B_R} \frac{(rf)^2}{|x|^2}\mathrm{d}x \\
&\geq (n-2)^{-\frac{2(n-1)}{n}} S_n\left(\int_{B_R} X_1^{\frac{2(n-1)}{n-2}}\left(a, \frac{|x|}{R}\right)|rf|^{\frac{2n}{n-2}}\mathrm{d}x\right)^{\frac{n-2}{n}},
\end{aligned}$$

where

$$X_1(a,s) := (a - \ln s)^{-1}, \quad a > 0,\ 0 < s \leq 1.$$

Thus,

$$\int_{B_R} |\mathcal{L}f|^2 \mathrm{d}x - \frac{n^2}{4}\int_{B_R} f^2 \mathrm{d}x = \int_{B_R} |rf'(r)|^2 \mathrm{d}x - \frac{n^2}{4}\int_{B_R} f^2(r)\mathrm{d}x$$

$$\geq (n-2)^{-\frac{2(n-1)}{n}} S_n \left(\int_{B_R} X_1^{\frac{2(n-1)}{n-2}} \left(a, \frac{|x|}{R}\right) |rf|^{\frac{2n}{n-2}} \mathrm{d}x \right)^{\frac{n-2}{n}}.$$

Step 2. Assume f is not radial and $f \in C_0^\infty(B_R)$. We extend f as zero outside B_R. So $f \in C_0^\infty(\mathbb{R}^n)$. Decomposing f into spherical harmonics, we have

$$f = \sum_{k=0}^{\infty} f_k := \sum_{k=0}^{\infty} g_k(r)\phi_k(\sigma),$$

where $\varphi_k(\sigma)$ are the orthonormal eigenfunctions of the Laplace-Beltrami operator with responding eigenvalues

$$c_k = k(N+k-2), \quad k \geq 0.$$

The functions $f_k(r)$ belong to $C_0^\infty(B_R)$. By the proof of Theorem 1.1 and Step 1,

$$\int_{\mathbb{R}^n} |\mathcal{L}f|^2 \mathrm{d}x - \frac{n^2}{4}\int_{\mathbb{R}^n} f^2 \mathrm{d}x = \sum_{k=0}^{\infty} \left(\int_{\mathbb{R}^n} r^2 |g_k'(r)|^2 \mathrm{d}x - \frac{n^2}{4}\int_{\mathbb{R}^n} g_k^2 \mathrm{d}x \right)$$

$$\geq \int_{\mathbb{R}^n} r^2 |g_0'(r)|^2 \mathrm{d}x - \frac{n^2}{4}\int_{\mathbb{R}^n} g_0^2 \mathrm{d}x = \int_{\mathbb{R}^n} r^2 |F'(r)|^2 \mathrm{d}x - \frac{n^2}{4}\int_{\mathbb{R}^n} F(r)^2 \mathrm{d}x$$

$$\geq (n-2)^{-\frac{2(n-1)}{n}} S_n \left(\int_{B_R} X_1^{\frac{2(n-1)}{n-2}} \left(a, \frac{|x|}{R}\right) |rF|^{\frac{2n}{n-2}} \mathrm{d}x \right)^{\frac{n-2}{n}}.$$

Step 3. By Step 1 and Step 2, the following inequality holds for $f \in C_0^\infty(B_R)$

$$\int_{\mathbb{R}^n} |\mathcal{L}f|^2 \mathrm{d}x - \frac{n^2}{4}\int_{\mathbb{R}^n} f^2 \mathrm{d}x \geq (n-2)^{-\frac{2(n-1)}{n}} S_n \left(\int_{B_R} X_1^{\frac{2(n-1)}{n-2}} \left(a, \frac{|x|}{R}\right) |rF|^{\frac{2n}{n-2}} \mathrm{d}x \right)^{\frac{n-2}{n}}.$$

We note if $R^{-1} < |x| < R$, then

$$X_1^{\frac{2(N-1)}{N-2}} \left(a, \frac{|x|}{D}\right) = \left(\frac{1}{a - \ln \frac{|x|}{R}} \right)^{\frac{2(N-1)}{N-2}} \geq \left(\frac{1}{a + 2\ln R} \right)^{\frac{2(N-1)}{N-2}}.$$

Therefore, If f is supported in the annulus $A_R := \{x \in \mathbb{R}^n : R^{-1} < |x| < R\}$, then

$$\int_{A_R} |\mathcal{L}f|^2 \mathrm{d}x - \frac{n^2}{4}\int_{A_R} f^2 \mathrm{d}x \geq [(n-2)(2\ln R + a)]^{-\frac{2(n-1)}{n}} S_n \left(\int_{A_R} |rF(r)|^{2^*} \mathrm{d}x\right)^{\frac{2}{2^*}}.$$

Letting $a \to 0$, we have

$$\int_{A_R} |\mathcal{L}f|^2 \mathrm{d}x - \frac{n^2}{4}\int_{A_R} f^2 \mathrm{d}x \geq [2(n-2)\ln R]^{-\frac{2(n-1)}{n}} S_n \left(\int_{A_R} |rF(r)|^{2^*} \mathrm{d}x\right)^{\frac{2}{2^*}}.$$

The proof of Theorem 2 is completed.

Acknowledgements
The author thanks the referee for his/her careful reading and very useful comments that improved the final version of this paper.

Authors' contributions
YX designed and performed all the steps of proof in this research and also wrote the paper. All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

Received: 15 April 2011 Accepted: 5 December 2011 Published: 5 December 2011

References
1. Aubin, T: Probléme isopérimétric et espace de Sobolev. J Differ Geom. **11**, 573–598 (1976)
2. Talenti, G: Best constant in Sobolev inequality. Ann Matem Pura Appl. **110**(4), 353–372 (1976)
3. Stubbe, J: Bounds on the number of bound states for potentials with critical decay at infinity. J Math Phys. **31**(5), 1177–1180 (1990). doi:10.1063/1.528750
4. Balinsky, A, Evans, WD, Hundertmark, D, Lewis, RT: On inequalities of Hardy-Sobolev type. Banach J Math Anal. **2**(2), 94–106 (2008)
5. Bliss, G: An integral inequality. J Lond Math Soc. **5**, 40–46 (1930). doi:10.1112/jlms/s1-5.1.40
6. Tertikas, A, Zographopoulos, NB: Best constants in the Hardy-Rellich inequalities and related improvements. Adv Math. **209**, 407–459 (2007). doi:10.1016/j.aim.2006.05.011
7. Adimurthi, , Filippas, S, Tertikas, A: On the best constant of Hardy-Sobolev inequalities. Nonlinear Anal. **70**, 2826–2833 (2009). doi:10.1016/j.na.2008.12.019

doi:10.1186/1029-242X-2011-132
Cite this article as: Xiao: Remarks on inequalities of Hardy-Sobolev Type. *Journal of Inequalities and Applications* 2011 **2011**:132.